RELATIONAL ASPECTS OF RELATIONAL DATABASE DEPENDENCIES

Okuma, Hitomi
Department of Informatics, Kyushu University

Kawahara, Yasuo
Department of Informatics, Kyushu University

https://doi.org/10.5109/13495

出版情報：Bulletin of informatics and cybernetics. 32 (2), pp.91-104, 2000-12. 統計科学研究会
バージョン：
権利関係：

Bulletin of Informatics and Cybernetics, Vol. 32, No. 2, December, 2000

RELATIONAL ASPECTS OF RELATIONAL DATABASE DEPENDENCIES

By

Hitomi Okuma * and **Yasuo Kawahara**†

Abstract

This paper presents a relational treatment of inference rules for functional and multivalued dependencies in relational databases, to show the soundness and the completeness of the inference rules in Dedekind categories, which also cover the fuzzy case.

1. Introduction

It is well-known that relational database models were introduced by Codd (1970). Functional and multivalued dependencies are the most important examples of relationships (constraints) between attributes in database relations. The completeness theorems for functional and multivalued dependencies have been proved by Armstrong (1974) and Beeri et al. (1977), respectively.

Schmidt and Ströhlein (1993) explained a basic relational feature of functional dependency for relational models of databases, and Ounalli and Jaoua (1997) studied difunctional dependencies in relational databases. Orlowska (1987) proposed a relational formulation of functional, multivalued and other dependencies, and an axiomatic relational calculus for dependency theory was developed in Buszkowski and Orlowska (1998). MacCaull (2000a), (2000b) gave a relational formulation for functional, multivalued dependencies and association rules, and developed a Rasiowa/Sikorski-style tableaux method of proof which is sound and complete for the implication problem for these dependencies.

In this paper we will propose a foundation of a relational treatment for functional and multivalued dependencies in Dedekind categories.

This paper is organized as follows. In Section 2 we review the definitions of Dedekind categories (Olivier and Serrato (1980) and Kawahara (1998)) as one of relational categories, and relational products (Desharnais (1997)), that provide total spaces of database tuples. In section 3 we introduce a notion of database schemes to present a relational and algebraic framework to study database dependencies. In Sections 4 and 5 we will give the definitions of functional and multivalued dependencies in our database schemes, and show the inference rules, which implies the soundness theorems for the dependencies. Finally we will remark that database schemes with a weaker condition have sufficient basic properties to show the completeness theorems.

* Department of Informatics, Kyushu University 33 Fukuoka 812-8581 Japan. okuma@i.kyushu-u.ac.jp
† Department of Informatics, Kyushu University 33 Fukuoka 812-8581 Japan. kawahara@i.kyushu-u.ac.jp

2. Relational Products

In this section we recall the definition of Dedekind categories discussed in Olivier and Serrato (1980) and Kawahara (1998), and review relational products (Desharnais (1997)) in Dedekind categories.

Throughout this paper, a morphism α from an object X into an object Y in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoonup Y$, and the composite of a morphism $\alpha : X \rightharpoonup Y$ followed by a morphism $\beta : Y \rightharpoonup Z$ will be written as $\alpha\beta : X \rightharpoonup Z$. Also we will denote the identity morphism on X as id_X.

DEFINITION 2.1. A Dedekind category $\mathcal{D}$ is a category satisfying the following:
D1. [Complete Heyting Algebra] For all pairs of objects X and Y the hom-set $\mathcal{D}(X,Y)$ consisting of all morphisms of X into Y is a complete Heyting algebra (namely, a complete distributive lattice) with the least morphism 0_{XY} and the greatest morphism ∇_{XY}. Its algebraic structure will be denoted by

$$\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \sqsubseteq, \sqcup, \sqcap, 0_{XY}, \nabla_{XY}).$$

That is,
(a) $\sqsubseteq$ is a partial order on $\mathcal{D}(X,Y)$,
(b) $\forall\alpha \in \mathcal{D}(X,Y) :: 0_{XY} \sqsubseteq \alpha \sqsubseteq \nabla_{XY}$,
(c) $\sqcup_{i\in I}\alpha_i \sqsubseteq \alpha$ iff $\alpha_i \sqsubseteq \alpha$ for all $i \in I$,
(d) $\alpha \sqsubseteq \sqcap_{i\in I}\alpha_i$ iff $\alpha \sqsubseteq \alpha_i$ for all $i \in I$,
(e) $\alpha \sqcap (\sqcup_{i\in I}\alpha_i) = \sqcup_{i\in I}(\alpha \sqcap \alpha_i)$.
D2. [Converse] There is given a converse operation $\sharp : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$. That is, for all morphisms $\alpha, \alpha' : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$, the following involutive laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$,
(b) $(\alpha^\sharp)^\sharp = \alpha$,
(c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ and $\gamma : X \rightharpoonup Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residue] For all morphisms $\beta : Y \rightharpoonup Z$ and $\gamma : X \rightharpoonup Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoonup Y$ is a morphism such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoonup Y$. □

A morphism $f : X \rightharpoonup Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ is called a *partial function* (*univalent*) and may be introduced as $f : X \to Y$. A partial function $f : X \to Y$ such that $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function*. A morphism $\alpha : X \rightharpoonup Y$ such that $\alpha\alpha^\sharp\alpha \sqsubseteq \alpha$ is *difunctional*. In what follows the word *relation* is a synonym for morphism of a Dedekind category.

An object I of a Dedekind category $\mathcal{D}$ is called a *unit* if $0_{II} \neq \mathrm{id}_I = \nabla_{II}$. A unit I of $\mathcal{D}$ is called *strict* if $\nabla_{XI}\nabla_{IX} = \nabla_{XX}$ for all objects X of $\mathcal{D}$. An I-point x of X is a function $x : I \to X$, that is, a univalent and total relation.

For the basic properties of Dedekind categories, or relation categories, the reader is referred to Freyd and Ščedrov (1990), Schmidt and Ströhlein (1993).

LEMMA 2.2. *Let $\alpha : X \rightharpoonup Y$ be a relation, and let $x : I \to X$ and $y : I \to Y$ be I-points of X and Y, respectively. Then*

(a) *$x^\sharp y \sqsubseteq \alpha$ if and only if $x\alpha y^\sharp = \mathrm{id}_I$,*

(b) *If $x^\sharp y \sqcap \alpha = 0_{XY}$, then $x\alpha y^\sharp = 0_{II}$.*

PROOF. (a) Assume that $x^\sharp y \sqsubseteq \alpha$. Then $\mathrm{id}_I \sqsubseteq xx^\sharp yy^\sharp \sqsubseteq x\alpha y^\sharp$ by the totality of x and y. Hence $x\alpha y^\sharp = \mathrm{id}_I$. For the converse, suppose that $x\alpha y^\sharp = \mathrm{id}_I$. Then $x^\sharp y = x^\sharp x\alpha y^\sharp y \sqsubseteq \alpha$, since x and y are univalent.
(b) Assume that $x^\sharp y \sqcap \alpha = 0_{XY}$. Then $x\alpha y^\sharp = x\alpha y^\sharp \sqcap \mathrm{id}_I \sqsubseteq x(\alpha \sqcap x^\sharp y)y^\sharp = 0_{II}$. □

Let U be a finite set and let $\{D_a \mid a \in U\}$ be a U-indexed set of objects in a Dedekind category $\mathcal{D}$. We call U a set of attributes and D_a a domain of possible a-values for an attribute $a \in U$. In this section we assume that the Dedekind category $\mathcal{D}$ has finite relational products for all subsets of $\{D_a \mid a \in U\}$. That is, for each subset $X \subseteq U$ there exists an object $T[X]$ of $\mathcal{D}$ and an X-indexed set $\{p_{Xa} : T[X] \to D_a \mid a \in X\}$ of functions (called projections) in $\mathcal{D}$ satisfying the following three conditions:
(PD1) $\forall a \in X :: p^\sharp_{Xa} p_{Xa} = \mathrm{id}_{D_a}$,
(PD2) $\forall a \in X :: p^\sharp_{Xa}(\sqcap_{b \neq a} p_{Xb} p^\sharp_{Xb}) = \nabla_{D_a T[X]}$,
(PD3) $\sqcap_{a \in X} p_{Xa} p^\sharp_{Xa} = \mathrm{id}_{T[X]}$.

REMARK. (i) In the case that X is a singleton set, namely, $X = \{a\}$, we set $T[X] = D_a$ and $p_{\{a\}a} = \mathrm{id}_{D_a}$. Then the conditions (PD1), (PD2) and (PD3) hold under the convention that the infimum of the empty set of relations is the universal relation.
(ii) In the case that X is the empty set, we set $T[\emptyset] = I$. Then the above conditions are valid, too.

THEOREM 2.3. *(Sharpness) Let Y be a subset of U, and let $\{\alpha_y : A \rightharpoonup D_y \mid y \in Y\}$ and $\{\beta_y : B \rightharpoonup D_y \mid y \in Y\}$ be Y-indexed sets of relations in $\mathcal{D}$. If all α_y are difunctional, or all β_y are difunctional, then*

$$\sqcap_{y \in Y} \alpha_y \beta^\sharp_y = (\sqcap_{y \in Y} \alpha_y p^\sharp_{Yy})(\sqcap_{y \in Y} p_{Yy} \beta^\sharp_y).$$

For the proof of this theorem the reader is referred to Desharnais (1997). □

For a subset Y of X we define a relation $p_{XY} : T[X] \rightharpoonup T[Y]$ as follows:

$$p_{XY} = \sqcap_{a \in Y} p_{Xa} p^\sharp_{Ya}.$$

It is trivial that $p_{X\{a\}} = p_{Xa}$ for all a in X, and $p_{XX} = \mathrm{id}_{T[X]}$.

PROPOSITION 2.4. *Let X, Y and Z be subsets of U. Then the following hold:*

(a) *The relation $p_{XY} : T[X] \rightharpoonup T[Y]$ is the unique function such that $p_{XY} p_{Ya} = p_{Xa}$ for all $a \in Y$,*

(b) *$p_{XY} p_{YZ} = p_{XZ}$ if $Z \subseteq Y \subseteq X$.*

PROOF. (a) First we will see that p_{XY} is univalent and total. Univalence follows from

$$\begin{array}{rcll} p_{XY}^\sharp p_{XY} & = & (\sqcap_{a\in Y} p_{Ya} p_{Xa}^\sharp)(\sqcap_{a\in Y} p_{Xa} p_{Ya}^\sharp) & \\ & \sqsubseteq & \sqcap_{a\in Y} p_{Ya} p_{Xa}^\sharp p_{Xa} p_{Ya}^\sharp & \{ \text{Sub-distributivity} \} \\ & \sqsubseteq & \sqcap_{a\in Y} p_{Ya} p_{Ya}^\sharp & \{ p_{Xa}^\sharp p_{Xa} \sqsubseteq \mathrm{id}_{D_a} \} \\ & = & \mathrm{id}_{T[Y]} & \{ \text{(PD3)} \}, \end{array}$$

and totality from

$$\begin{array}{rcll} \mathrm{id}_{T[X]} & \sqsubseteq & \sqcap_{a\in Y} p_{Xa} p_{Xa}^\sharp & \{ \mathrm{id}_{T[X]} \sqsubseteq p_{Xa} p_{Xa}^\sharp \} \\ & = & (\sqcap_{a\in Y} p_{Xa} p_{Ya}^\sharp)(\sqcap_{a\in Y} p_{Ya} p_{Xa}^\sharp) & \{ \text{Theorem 2.3 (sharpness)} \} \\ & = & p_{XY} p_{XY}^\sharp, & \end{array}$$

since all p_{Xa} are difunctional. Finally assume that $f : T[X] \to T[Y]$ is a function such that $f p_{Ya} = p_{Xa}$ for all $a \in Y$. Then

$$f = f\mathrm{id}_{T[Y]} = f(\sqcap_{a\in Y} p_{Ya} p_{Ya}^\sharp) = \sqcap_{a\in Y} f p_{Ya} p_{Ya}^\sharp = \sqcap_{a\in Y} p_{Xa} p_{Ya}^\sharp = p_{XY},$$

which proves the uniqueness of p_{XY}.
(b) It is easy to see that $p_{XY} p_{YZ} p_{Za} = p_{XY} p_{Ya} = p_{Xa} = p_{XZ} p_{Za}$ for all $a \in Z$, and so $p_{XY} p_{YZ} = p_{XZ}$ by the uniqueness of p_{XZ}, proved in (a). □

For each subset X of U we define an equivalence relation $\theta[X] : T[U] \to T[U]$ by $\theta[X] = \sqcap_{a\in X} p_{Ua} p_{Ua}^\sharp$. When X is empty set, set $\theta[\emptyset] = \nabla_{T[U]T[U]}$.

PROPOSITION 2.5. *Let X and Y be subsets of U. Then*

(a) $\theta[U] = \mathrm{id}_{T[U]}$ *and* $\theta[X] = p_{UX} p_{UX}^\sharp$,

(b) $\theta[X \cup Y] = \theta[X] \sqcap \theta[Y]$,

(c) $\theta[X \cap Y] = \theta[X]\theta[Y]$.

PROOF. (a) First note that $\sqcap_{a\in X} p_{Xa} p_{Xa}^\sharp = \mathrm{id}_{T[X]}$ by (PD3), and that $p_{UX} p_{Xa} = p_{Ua}$ for all $a \in X$ by Proposition 2.4(a). We have

$$\begin{array}{rcll} p_{UX} p_{UX}^\sharp & = & p_{UX}(\sqcap_{a\in X} p_{Xa} p_{Xa}^\sharp) p_{UX}^\sharp & \{ \sqcap_{a\in X} p_{Xa} p_{Xa}^\sharp = \mathrm{id}_{T[X]} \} \\ & = & \sqcap_{a\in X} p_{UX} p_{Xa} p_{Xa}^\sharp p_{UX}^\sharp & \\ & = & \sqcap_{a\in X} p_{Ua} p_{Ua}^\sharp & \{ p_{UX} p_{Xa} = p_{Ua} \} \\ & = & \theta[X]. & \end{array}$$

(b) From the definition we have $\theta[X] \sqcap \theta[Y] = (\sqcap_{a\in X} p_{Ua} p_{Ua}^\sharp) \sqcap (\sqcap_{a\in Y} p_{Ua} p_{Ua}^\sharp) = \sqcap_{a\in X\cup Y} p_{Ua} p_{Ua}^\sharp = \theta[X \cup Y]$.
(c) For a subset X of U define a U-indexed set $\{\alpha_a^X \mid a \in U\}$ of relations by $\alpha_a^X = \mathrm{id}_{D_a}$ if $a \in X$ and $\alpha_a^X = \nabla_{D_a D_a}$ otherwise. As $p_{Ua} \alpha_a^X p_{Ua}^\sharp = p_{Ua} \nabla_{D_a D_a} p_{Ua}^\sharp = \nabla_{T[U]T[U]}$ for $a \notin X$ we have $\sqcap_{a\in U} p_{Ua} \alpha_a^X p_{Ua}^\sharp = \sqcap_{a\in X} p_{Ua} p_{Ua}^\sharp = \theta[X]$. Therefore, applying sharpness (Theorem 2.3) we obtain

$$\begin{array}{rcll} \theta[X]\theta[Y] & = & (\sqcap_{a\in U} p_{Ua} \alpha_a^X p_{Ua}^\sharp)(\sqcap_{a\in U} p_{Ua} \alpha_a^Y p_{Ua}^\sharp) & \\ & = & \sqcap_{a\in U} p_{Ua} \alpha_a^X \alpha_a^Y p_{Ua}^\sharp & \{ \text{Theorem 2.3 (sharpness)} \} \\ & = & \sqcap_{a\in U} p_{Ua} \alpha_a^{X\cap Y} p_{Ua}^\sharp & \{ \alpha_a^X \alpha_a^Y = \alpha_a^{X\cap Y} \} \\ & = & \theta[X \cap Y]. & \end{array}$$

(Note that all $p_{U_a}\alpha_a^X$ are difunctional.) □

Throughout this paper we will assume that $\mathcal{D}$ is a fixed Dedekind category with a strict unit I and relational products.

3. Attribute Schemes

To realize a relational treatment of some dependencies on relational databases we will introduce a notion of attribute schemes in the following:

DEFINITION 3.1. An attribute scheme in a Dedekind category $\mathcal{D}$ is a triple (U,T,θ) of a set U of attributes, an object T of $\mathcal{D}$ and a mapping $\theta : \wp(U) \to \mathcal{D}(T,T)$ assigning a relation $\theta[X]$ on T to each subset X of U such that

(a) $\theta[X]$ is an equivalence relation for each subset X of U,

(b) $\theta[U] = \mathrm{id}_T$ and $\theta[\emptyset] = \nabla_{TT}$,

(c) $\theta[X \cup Y] = \theta[X] \sqcap \theta[Y]$ and $\theta[X \cap Y] = \theta[X]\theta[Y]$ for all subsets X and Y of U. □

Note that $\theta[X]$ is an equivalence relation if and only if it is reflexive ($\mathrm{id}_T \sqsubseteq \theta[X]$), symmetric ($\theta[X]^\sharp \sqsubseteq \theta[X]$) and transitive ($\theta[X]\theta[X] \sqsubseteq \theta[X]$).

DEFINITION 3.2. A database relation on an attribute scheme (U,T,θ) is a relation $r : T \rightharpoonup T$ such that $r \sqsubseteq \mathrm{id}_T$. □

The following shows technical properties of database relations in attribute schemes.

LEMMA 3.3. *Let r be a database relation on an attribute scheme (U,T,θ). For all subsets X, Y and W of U the following hold:*

(a) *If $Y \subseteq X$, then $\theta[X] \sqsubseteq \theta[Y]$,*

(b) *If $X \cup Y = U$, then $\theta[X]r\theta[Y] \sqcap \theta[W] = \theta[X \cup W]r\theta[Y \cup W] \sqcap \theta[W]$.*

PROOF. (a) Assume that $Y \subseteq X$. Then $\theta[X] = \theta[X \cup Y] = \theta[X] \sqcap \theta[Y] \sqsubseteq \theta[Y]$ (and $\theta[X]\theta[Y] = \theta[X \cap Y] = \theta[Y]$).
(b) It is clear from (a) that $\theta[X \cup W]r\theta[Y \cup W] \sqcap \theta[W] \sqsubseteq \theta[X]r\theta[Y] \sqcap \theta[W]$. Conversely we have

$$
\begin{array}{lll}
 & \theta[X]r\theta[Y] \sqcap \theta[W] & \\
\sqsubseteq & \{\theta[X]r \sqcap \theta[W](\theta[Y])^\sharp\}\{\theta[Y] \sqcap (\theta[X]r)^\sharp\theta[W]\} & \{\text{ Dedekind Formula }\} \\
\sqsubseteq & \{\theta[X] \sqcap \theta[W](\theta[Y])^\sharp r^\sharp\}r\{\theta[Y] \sqcap (\theta[X]r)^\sharp\theta[W]\} & \{\text{ Dedekind Formula }\} \\
\sqsubseteq & (\theta[X] \sqcap \theta[W]\theta[Y])r(\theta[Y] \sqcap \theta[X]\theta[W]) & \{\ r \sqsubseteq \mathrm{id}_T,\ \theta[X]^\sharp = \theta[X]\ \} \\
= & \theta[X \cup (W \cap Y)]r\theta[Y \cup (X \cap W)] & \{\text{ Definition 3.1(c) }\} \\
= & \theta[X \cup W]r\theta[Y \cup W] & \\
 & \multicolumn{2}{r}{\{\ X \cup (W \cap Y) = X \cup W,\ Y \cup (X \cap W) = Y \cup W \text{ by } X \cup Y = U\ \}.}
\end{array}
$$

□

PROPOSITION 3.4. *Let r be a database relation on an attribute scheme (U,T,θ). For all subsets X and Y of U the following hold:*

(a) $r\theta[X]r \sqsubseteq \theta[Y]$ *if and only if* $r\theta[X]r \sqsubseteq r\theta[Y]r$.

(b) $\theta[\neg Y]r\theta[Y] \sqcap \theta[X] = \theta[X \cup \neg Y]r\theta[X \cup Y] \sqcap \theta[X] = \theta[\neg(X \cup Y)]r\theta[X \cup Y] \sqcap \theta[X]$.

PROOF. (a) It is obvious from $r \sqsubseteq \mathrm{id}_T$.
(b) From Lemma 3.3(b) we have

$$\begin{aligned} \theta[\neg(X \cup Y)]r\theta[X \cup Y] \sqcap \theta[X] &= \theta[X \cup \neg(X \cup Y)]r\theta[X \cup Y] \sqcap \theta[X] \\ &= \theta[X \cup \neg Y]r\theta[X \cup Y] \sqcap \theta[X]. \end{aligned}$$

□

4. Functional Dependencies

Let r be a database relation on an attribute scheme $(U, T[U], \theta)$ discussed in Section 2. A relation $\alpha_{XY} = p_{UX}^{\sharp} r p_{UY} : T[X] \rightarrow T[Y]$ is functional (or, univalent) if and only if :

$$\begin{array}{lll} & \alpha_{XY}^{\sharp}\alpha_{XY} \sqsubseteq \mathrm{id}_{T[Y]} & \\ \Longleftrightarrow & (p_{UX}^{\sharp} r p_{UY})^{\sharp}(p_{UX}^{\sharp} r p_{UY}) \sqsubseteq \mathrm{id}_{T[Y]} & \\ \Longleftrightarrow & p_{UY}p_{UY}^{\sharp} r p_{UX} p_{UX}^{\sharp} r p_{UY} p_{UY}^{\sharp} \sqsubseteq p_{UY}p_{UY}^{\sharp} & \{ \, p_{UY} \text{ is a function. } \} \\ \Longleftrightarrow & \theta[Y]r\theta[X]r\theta[Y] \sqsubseteq \theta[Y] & \{ \, \theta[Y] = p_{UY}p_{UY}^{\sharp} \, \} \\ \Longleftrightarrow & r\theta[X]r \sqsubseteq \theta[Y]. & \{ \text{ Definition 3.1(a) } \} \end{array}$$

Following the above observation we will define the functional dependencies on attribute schemes in Dedekind categories, as follows:

DEFINITION 4.1. Let r be a database relation on an attribute scheme (U, T, θ). We say that there is a functional dependency of a subset Y of U on a subset X of U, denoted by $r \models X \rightarrow Y$, if and only if $r\theta[X]r \sqsubseteq \theta[Y]$. □

We now show the basic laws on functional dependencies.

FD0. If $X \supseteq Y$, then $r \models X \rightarrow Y$. (Reflexive law)

PROOF. Assume that $X \supseteq Y$. Then $r\theta[X]r \sqsubseteq \theta[X] \sqsubseteq \theta[Y]$, since $r \sqsubseteq \mathrm{id}_T$. □

FD1. If $r \models X \rightarrow Y$ and $Z \supseteq W$, then $r \models X \cup Z \rightarrow Y \cup W$. (Augmentation law)

PROOF. Assume that $r\theta[X]r \sqsubseteq \theta[Y]$ and $Z \supseteq W$. Then

$$\begin{array}{rcll} r\theta[X \cup Z]r &=& r(\theta[X] \sqcap \theta[Z])r & \{ \text{ Definition 3.1(c) } \} \\ &\sqsubseteq& r\theta[X]r \sqcap \theta[Z] & \{ \, r \sqsubseteq \mathrm{id}_T \, \} \\ &\sqsubseteq& \theta[Y] \sqcap \theta[W] & \{ \, r\theta[X]r \sqsubseteq \theta[Y] \text{ and } Z \supseteq W \, \} \\ &=& \theta[Y \cup W] & \{ \text{ Definition 3.1(c) } \}. \end{array}$$

□

FD2. If $r \models X \rightarrow Y$ and $r \models Y \rightarrow Z$, then $r \models X \rightarrow Z$. (Transitive law)

PROOF. Assume that $r\theta[X]r \sqsubseteq \theta[Y]$ and $r\theta[Y]r \sqsubseteq \theta[Z]$. Then we have

$$r\theta[X]r \sqsubseteq r\theta[Y]r \sqsubseteq \theta[Z].$$

□

Let (U, T, θ) be an attribute scheme in a Dedekind category $\mathcal{D}$. A formal expression $X \to Y$, namely, an ordered pair of subsets X and Y of U, joined by an arrow, is called a functional dependency. Inference rules for functional dependencies are the following three rules:

$$\text{[FD0]} \quad \frac{}{X \to Y}\ \{X \supseteq Y\}$$

$$\text{[FD1]} \quad \frac{X \to Y}{X \cup Z \to Y \cup W}\ \{Z \supseteq W\}$$

$$\text{[FD2]} \quad \frac{X \to Y \quad Y \to Z}{X \to Z}$$

Let F be a set of functional dependencies (FD's). A derivation from F is a nonempty sequence

$$\{X_0 \to Y_0, X_1 \to Y_1, \ldots, X_n \to Y_n\}$$

of FD's such that, for all $k = 0, 1, \ldots, n$, one of the following holds:
(0) $Y_k \subseteq X_k$ ([FD0]) or $X_k \to Y_k$ is in F,
(i) $\exists i :: i < k$ such that

$$\text{[FD1]} \quad \frac{X_i \to Y_i}{X_k \to Y_k},$$

(ii) $\exists i, j :: i, j < k$ such that

$$\text{[FD2]} \quad \frac{X_i \to Y_i \quad X_j \to Y_j}{X_k \to Y_k}.$$

For example, the union rule

$$\text{[FD3]} \quad \frac{X \to Y \quad X \to Z}{X \to Y \cup Z}$$

is proved from [FD1] and [FD2] as follows:

$$\frac{\dfrac{X \to Y}{X \to X \cup Y}\ \{X \supseteq X\} \qquad \dfrac{X \to Z}{X \cup Y \to Y \cup Z}\ \{Y \supseteq Y\}}{X \to Y \cup Z}$$

A functional dependency $X \to Y$ is provable from F, written $F \vdash X \to Y$, if there is a derivation $\{X_0 \to Y_0, X_1 \to Y_1, \ldots, X_n \to Y_n\}$ from F such that $X = X_n$ and $Y = Y_n$. A database relation r is valid for F, written $r \models F$, if $r \models X \to Y$ is valid for all $X \to Y \in F$. Also we define a subset $F^+(X)$ of U by $F^+(X) = \{a \in U \mid F \vdash X \to a\}$.

LEMMA 4.2. *$F \vdash X \to Y$ iff $Y \subseteq F^+(X)$.*

PROOF. ($\Rightarrow$) Assume that $F \vdash X \to Y$ and let $a \in Y$. Then $\vdash Y \to a$ by [FD0] and so $F \vdash X \to a$ by [FD2]. Hence $a \in F^+(X)$.
($\Leftarrow$) Assume that $Y \subseteq F^+(X)$. Then, for all $a \in Y$, we have $F \vdash X \to a$ by the definition of $F^+(X)$ and hence $F \vdash X \to Y$ by the union rule [FD3], because U is a finite set. □

A family $X_0, W_1, \ldots, W_m$ of subsets of U is a *partition* of U if the family is disjoint and covers U. The set $\{1, \ldots, m\}$ will be denoted by M, and a union $\cup_{j \in J} W_j$ by W_J for all subsets J of M. When $J = \emptyset$, set $W_J = \emptyset$. Also for subsets J, J' of M a set $(J \cap J') \cup (J \cup J')^-$ will be written as $J \odot J'$.

DEFINITION 4.3. An attribute scheme (U, T, θ) in a Dedekind category $\mathcal{D}$ is *effective* if for all partitions $X_0, W_1, \ldots, W_m$ $(m \geq 1)$ of U, there exists a family $\{t_J : I \to T \mid J \subseteq M\}$ of 2^m I-points of T indexed by all subsets of M satisfying the following conditions:
(a_m) $t_J^\sharp t_{J'} \sqsubseteq \theta[X_0 \cup W_{J \odot J'}]$ for all subsets J and J' of M,
(b_m) $t_J^\sharp t_{J'} \sqcap \theta[\{a\}] = 0_{TT}$ for all attributes $a \notin X_0 \cup W_{J \odot J'}$. □

REMARK. The condition (b_m) in the above definition is equivalent to a condition (b'_m) If $Z \not\subseteq X_0 \cup W_{J \odot J'}$, then $t_J^\sharp t_{J'} \sqcap \theta[Z] = 0_{TT}$.

For example, if an attribute scheme (U, T, θ) is effective, then for a subset X_0 of U there is a pair of I-points $s, t : I \to T$ satisfying the following conditions:
(a_1) $s^\sharp t \sqsubseteq \theta[X_0]$,
(b_1) $s^\sharp t \sqcap \theta[\{a\}] = 0_{TT}$ for all attributes $a \notin X_0$.

PROPOSITION 4.4. *Assume that (U, T, θ) is effective and X_0 is a subset of U. and a given subset X_0 of U, a pair $s, t : I \to T$ of I-points of T satisfies the above conditions (a_1) and (b_1). Define a database relation $r_0 \sqsubseteq \mathrm{id}_T$ by $r_0 = s^\sharp s \sqcup t^\sharp t$. Then for all subsets Z of U the following hold:*

(a) *$Z \subseteq X_0$ if and only if $r_0 \models \emptyset \to Z$,*

(b) *If $Z \not\subseteq X_0$, then $r_0 \models Z \to U$.*

PROOF. (a) Note that $r_0 \nabla_{TI} = (s^\sharp s \sqcup t^\sharp t) \nabla_{TI} = s^\sharp \sqcup t^\sharp$ by the totality $s \nabla_{TI} = \nabla_{II} = \mathrm{id}_I$ and that

$$r_0 \nabla_{TT} r_0 = r_0 \nabla_{TI} \nabla_{IT} r_0 = (s^\sharp \sqcup t^\sharp)(s \sqcup t) = s^\sharp s \sqcup s^\sharp t \sqcup t^\sharp s \sqcup t^\sharp t.$$

Hence it is easy to see that $s^\sharp t \sqsubseteq \theta[Z]$ iff $r_0 \nabla_{TT} r_0 \sqsubseteq \theta[Z]$, which is equivalent to $r_0 \models \emptyset \to Z$. Thus it is enough to show that $s^\sharp t \sqsubseteq \theta[Z]$ iff $Z \subseteq X_0$. Assume that $Z \subseteq X_0$. Then $s^\sharp t \sqsubseteq \theta[X_0] \sqsubseteq \theta[Z]$ by the condition (a_1) and Lemma 3.3(a). Conversely assume that $s^\sharp t \sqsubseteq \theta[Z]$. Then for all $a \in Z$ we have $s^\sharp t \sqsubseteq \theta[\{a\}]$ by Lemma 3.3(a) and so $a \in X_0$, unless $s^\sharp t \sqcap \theta[\{a\}] = 0_{TT}$ by the property of the pair (s, t). (Note that $s^\sharp t \neq 0_{TT}$ because $s s^\sharp t t^\sharp = \mathrm{id}_I \neq 0_{II}$.)
(b) Assume that $Z \not\subseteq X_0$. Then there is at least one attribute $a \in Z$ such that $a \notin X_0$. It follows from $s^\sharp t \sqcap \theta[\{a\}] = 0_{TT}$ and $\theta[Z] \sqsubseteq \theta[\{a\}]$ that $s \theta[Z] t^\sharp = 0_{II}$ by Lemma 2.2(b) and so

$$r_0 \theta[Z] r_0 = (s^\sharp s \sqcup t^\sharp t) \theta[Z] (s^\sharp s \sqcup t^\sharp t) = s^\sharp s \theta[Z] s^\sharp s \sqcup t^\sharp t \theta[Z] t^\sharp t = s^\sharp s \sqcup t^\sharp t \sqsubseteq \mathrm{id}_T = \theta[U],$$

since $s^\sharp s \sqsubseteq \mathrm{id}_T \sqsubseteq \theta[Z]$. □

Now we will state the soundness and the completeness theorems of functional dependencies for database relations in Dedekind categories.

THEOREM 4.5. *If a relational scheme (U, T, θ) in a Dedekind category $\mathcal{D}$ is effective, then it is sound and complete, that is, for all sets F of FD's*

$$F \vdash X \to Y \iff [\forall r : r \models F \Rightarrow r \models X \to Y].$$

PROOF. (1) First we show the soundness

$$F \vdash X \to Y \Longrightarrow [\forall r : r \models F \Rightarrow r \models X \to Y].$$

Assume that $F \vdash X \to Y$ and $r \models F$. Then it is trivial $r \models X \to Y$, since the relational database model satisfies the basic laws FD0, FD1 and FD2.
(2) Next we show the completeness

$$[\forall r : r \models F \Rightarrow r \models X \to Y] \Longrightarrow F \vdash X \to Y$$

Assume that $[\forall r : r \models F \Rightarrow r \models X \to Y]$. By the effectivity of the relational scheme (U, T, θ), one can choose a pair of I-points $s, t : I \to T$ of T such that $s^\sharp t \sqsubseteq \theta[F^+(X)]$ and $s^\sharp t \sqcap \theta[\{a\}] = 0_{TT}$ for $a \notin F^+(X)$. Set $r_0 = s^\sharp s \sqcup t^\sharp t$ ($\sqsubseteq \mathrm{id}_T$). Then we will show $r_0 \models F$, that is, $r_0 \models Z \to W$ for all $Z \to W \in F$.
(i) In the case of $Z \subseteq F^+(X)$. Then $F \vdash X \to Z$ by Lemma 4.2. As $Z \to W \in F$ we have $F \vdash X \to W$ using [FD2] and so $W \subseteq F^+(X)$. Hence $r_0 \models \emptyset \to W$ by Proposition 4.4(a). Recall that $r_0 \models Z \to \emptyset$ always holds by the reflexive law FD0. Therefore $r_0 \models Z \to W$ by FD2.
(ii) In the case of $Z \not\subseteq F^+(X)$. Then $r_0 \models Z \to U$ holds by Proposition 4.4(b). Since $r_0 \models U \to W$ always holds by FD0, we have $r_0 \models Z \to W$ using FD2.
By the assumption, we have $r_0 \models X \to Y$. Then we will see $F \vdash X \to Y$. Since $X \subseteq F^+(X)$, we have $r_0 \models \emptyset \to X$ by Proposition 4.4(a). Hence $r_0 \models \emptyset \to Y$ by FD2, which is equivalent to $Y \subseteq F^+(X)$ by Proposition 4.4(a). We conclude $F \vdash X \to Y$ by Lemma 4.2. □

5. Multivalued Dependencies

Let r be a database relation on an attribute scheme $(U, T[U], \theta)$ discussed in Section 2. The original definition of the multivalued dependency in Beeri et al. (1977) will be transformed as follows (Set $Z = \neg(X \cup Y)$):

$$\begin{array}{lll}
& rp_{UX}p_{UX}^\sharp rp_{UY} = rp_{UX\cup Z}p_{UX\cup Z}^\sharp rp_{UY} & \\
\iff & rp_{UX}p_{UX}^\sharp rp_{UY}p_{UY}^\sharp = rp_{UX\cup Z}p_{UX\cup Z}^\sharp rp_{UY}p_{UY}^\sharp & \{\ p_{UY}p_{UY}^\sharp p_{UY} = p_{UY}\ \} \\
\iff & r\theta[X]r\theta[Y] = r\theta[X \cup Z]r\theta[Y] & \{\ \theta[X] = p_{UX}p_{UX}^\sharp\ \} \\
\iff & r\theta[X]r\theta[Y] \sqsubseteq r\theta[X \cup Z]r\theta[Y] & \{\ \theta[X \cup Z] \sqsubseteq \theta[X]\ \} \\
\iff & r\theta[X]r \sqsubseteq \theta[X \cup Z]r\theta[Y] & \{\ r \sqsubseteq \mathrm{id}_T \text{ and Definition 3.1(a) } \} \\
\iff & r\theta[X]r \sqsubseteq \theta[X \cup \neg Y]r\theta[Y] & \{\ X \cup Z = X \cup \neg Y\ \} \\
\iff & r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y] & \{\ \theta[X \cup \neg Y] \sqsubseteq \theta[\neg Y] \text{ and Proposition 3.4(b) } \}.
\end{array}$$

We now define multivalued dependencies on attribute schemes in Dedekind categories, as follows:

DEFINITION 5.1. Let r be a database relation on an attribute scheme (U, T, θ). We say that there is a multivalued dependency of a subset Y of U on a subset X of U, denoted by $r \models X \twoheadrightarrow Y$, if and only if $r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y]$. □

First of all we show the basic laws for multivalued dependencies.

MVD0. If $X \supseteq Y$, then $r \models X \twoheadrightarrow Y$. (Reflexive law)

PROOF. Assume that $X \supseteq Y$. Then $r\theta[X]r \sqsubseteq r\theta[X] \sqsubseteq r\theta[Y] \sqsubseteq \theta[\neg Y]r\theta[Y]$, since $r \sqsubseteq \mathrm{id}_T$ and $\mathrm{id}_T \sqsubseteq \theta[\neg Y]$. □

MVD1. If $r \models X \twoheadrightarrow Y$ and $Z \supseteq W$, then $r \models X \cup Z \twoheadrightarrow Y \cup W$. (Augmentation law)

PROOF. Assume that $r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y]$ and $Z \supseteq W$. Then

$$\begin{array}{rcll} r\theta[X \cup Z]r & = & r(\theta[X] \sqcap \theta[Z])r & \{ \text{Definition 3.1(c)} \} \\ & \sqsubseteq & r\theta[X]r \sqcap \theta[Z] & \{ r \sqsubseteq \mathrm{id}_T \} \\ & \sqsubseteq & \theta[\neg Y]r\theta[Y] \sqcap \theta[Z] & \{ r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y] \} \\ & \sqsubseteq & \theta[\neg Y]r\theta[Y \cup Z] & \{ \text{Proposition 3.4(b)} \} \\ & \sqsubseteq & \theta[\neg Y \cap \neg W]r\theta[Y \cup W] & \{ \neg Y \supseteq \neg Y \cap \neg W \text{ and } Z \supseteq W \}. \end{array}$$

□

MVD2. If $r \models X \twoheadrightarrow Y$ and $r \models Y \twoheadrightarrow Z$, then $r \models X \twoheadrightarrow Z \cap \neg Y$. (Transitive law)

PROOF. Assume that $r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y]$ and $r\theta[Y]r \sqsubseteq \theta[\neg Z]r\theta[Z]$. First note that $r\theta[X]r \sqsubseteq r\theta[Y]r\theta[\neg Y]$ and $r\theta[Y]r \sqsubseteq r\theta[\neg Z \cup Y]r\theta[Z]$. Then

$$\begin{array}{rcl} r\theta[X]r & \sqsubseteq & r\theta[Y]r\theta[\neg Y] \\ & \sqsubseteq & \theta[\neg Z \cup Y]r\theta[Z]\theta[\neg Y] \\ & = & \theta[\neg Z \cup Y]r\theta[Z \cap \neg Y]. \end{array}$$

□

MVD3. $r \models X \twoheadrightarrow Y$ if and only if $r \models X \twoheadrightarrow \neg Y$. (Complement law)

PROOF. Assume that $r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y]$. Since $r^\sharp = r$ and $\theta[Y]^\sharp = \theta[Y]$, we have $r\theta[X]r = (r\theta[X]r)^\sharp \sqsubseteq (\theta[\neg Y]r\theta[Y])^\sharp = \theta[Y]r\theta[\neg Y]$. □

FD-MVD1. If $r \models X \rightarrow Y$, then $r \models X \twoheadrightarrow Y$.

PROOF. Assume that $r\theta[X]r \sqsubseteq \theta[Y]$. Then $r\theta[X]r \sqsubseteq r\theta[Y] \sqsubseteq \theta[\neg Y]r\theta[Y]$ by $\mathrm{id}_T \sqsubseteq \theta[\neg Y]$. □

FD-MVD2. Let $Y \cap Z = \emptyset$ and $Y \supseteq W$. If $r \models X \twoheadrightarrow Y$ and $r \models Z \rightarrow W$, then $r \models X \rightarrow W$.

PROOF. Assume that $Y \cap Z = \emptyset$, $Y \supseteq W$, $r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y]$ and $r\theta[Z]r \sqsubseteq \theta[W]$. Then

$$\begin{array}{rcll} r\theta[X]r & \sqsubseteq & r\theta[\neg Y]r\theta[Y] & \{ r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y] \} \\ & \sqsubseteq & r\theta[Z]r\theta[Y] & \{ \theta[\neg Y] \sqsubseteq \theta[Z] \text{ by } Y \cap Z = \emptyset \} \\ & \sqsubseteq & \theta[W]\theta[Y] & \{ r\theta[Z]r \sqsubseteq \theta[W] \} \\ & \sqsubseteq & \theta[W]\theta[W] & \{ Y \supseteq W \} \\ & \sqsubseteq & \theta[W]. & \end{array}$$

□

The next proposition indicates the well-known fact that multivalued dependency is a special case of join dependency. The proof follows from Lemma 3.3(b), and is omitted.

PROPOSITION 5.2. *Let r be a database relation on an attribute scheme (U,T,θ). Then for subsets X and Y of U the following two conditions are equivalent:*

(a) $r\theta[X]r \sqsubseteq \theta[\neg Y]r\theta[Y]$,

(b) $\theta[X \cup Y]r\theta[X \cup Y] \sqcap \theta[X \cup \neg Y]r\theta[X \cup \neg Y] \sqsubseteq r$.

Let (U,T,θ) be an attribute scheme in a Dedekind category $\mathcal{D}$. A formal expression $X \twoheadrightarrow Y$, namely, an ordered pair of subsets X and Y of U, joined by a two-head arrow, is called a multivalued dependency. The inference system for multivalued dependency consists of the following six rules:

$$[\text{MVD0}] \quad \frac{}{X \twoheadrightarrow Y}\ \{X \supseteq Y\}$$

$$[\text{MVD1}] \quad \frac{X \twoheadrightarrow Y}{X \cup W \twoheadrightarrow Y \cup Z}\ \{W \supseteq Z\}$$

$$[\text{MVD2}] \quad \frac{X \twoheadrightarrow Y \quad Y \twoheadrightarrow Z}{X \twoheadrightarrow Z \cap \neg Y}$$

$$[\text{MVD3}] \quad \frac{X \twoheadrightarrow Y}{X \twoheadrightarrow \neg Y}$$

$$[\text{FD-MVD1}] \quad \frac{X \rightarrow Y}{X \twoheadrightarrow Y}$$

$$[\text{FD-MVD2}] \quad \frac{X \twoheadrightarrow Y \quad Z \rightarrow W}{X \rightarrow W}\ \{Y \cap Z = \emptyset,\ Y \supseteq W\}$$

The following inference rules [MVD4] and [MVD5], called the union and the intersection rules respectively, were derived from the inference rules [MVD0]-[MVD3] in Mendelzon (1979).

$$[\text{MVD4}] \quad \frac{X \twoheadrightarrow Y \quad X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cup Z}$$

$$[\text{MVD5}] \quad \frac{X \twoheadrightarrow Y \quad X \twoheadrightarrow Z}{X \twoheadrightarrow Y \cap Z}$$

In the rest of this section we will prove some properties of a particular database relation, which will be needed for formalizing the proof originally given in Beeri et al. (1977) of completeness theorem for the inference rules [FD0]-[FD2], [MVD0]-[MVD3] and [FD-MVD1]-[FD-MVD2].

Let $X_0, W_1, \ldots, W_m$ $(m \geq 1)$ be a partition of U, and assume that a family $\{t_J : I \rightarrow T \mid J \subseteq M\}$ of 2^m I-points of T indexed by all subsets of M satisfies the following conditions:

(a_m) $t_J^\sharp t_{J'} \sqsubseteq \theta[X_0 \cup W_{J \odot J'}]$ for all subsets J and J' of M,

(b_m) $t_J^\sharp t_{J'} \sqcap \theta[\{a\}] = 0_{TT}$ for all attributes $a \notin X_0 \cup W_{J \odot J'}$.

Then define a particular database relation $r_0 : T \rightharpoonup T$ by $r_0 = \sqcup_{J \subseteq M} t_J^\sharp t_J$.

REMARK. If $t_J^\sharp t_{J'} \sqsubseteq \theta[Z]$, then $t_J^\sharp t_{J'} \sqsubseteq r_0\theta[Z]r_0$ as follows: Assume that $t_J^\sharp t_{J'} \sqsubseteq \theta[Z]$. Then $t_J\theta[Z]t_{J'}^\sharp = \mathrm{id}_I$ by Lemma 2.2(a), and so $t_J^\sharp t_{J'} = t_J^\sharp t_J\theta[Z]t_{J'}^\sharp t_{J'} \sqsubseteq r_0\theta[Z]r_0$.

LEMMA 5.3. (a) $r_0 \models Z \to X_0$ *for all subsets* Z *of* U,

(b) $r_0 \models \{a\} \to W_i$ *for all* $a \in W_i$,

(c) $r_0 \not\models \neg W_i \to W_i$,

(d) $r_0 \models Z \twoheadrightarrow W_i$ *for all subsets* Z *of* U,

(e) $r_0 \not\models \neg W_i \twoheadrightarrow W$ *for all nonempty proper subsets* W *of* W_i.

PROOF. (a) It is trivial by (a_m) that $r_0\nabla_{TT}r_0 = \sqcup_{J,J'\subseteq M} t_J^\sharp t_{J'} \sqsubseteq \theta[X_0]$.
(b) Assume that $a \in W_i$. If $i \in J \odot J'$, then $t_J^\sharp t_{J'} \sqsubseteq \theta[X_0 \cup W_{J\odot J'}] \sqsubseteq \theta[\{a\}]$ by (a_m) and so $t_J\theta[\{a\}]t_{J'}^\sharp = \mathrm{id}_I$ by Lemma 2.2(a). Moreover, if $i \notin J \odot J'$, then $a \notin X_0 \cup W_{J\odot J'}$ and so $t_J\theta[\{a\}]t_{J'}^\sharp = 0_{II}$ by (b_m) and Lemma 2.2(b). Hence

$$r_0\theta[\{a\}]r_0 = \sqcup_{i\in J\odot J'} t_J^\sharp t_{J'} \sqsubseteq \theta[W_i].$$

(c) First we recall that the condition (b_m) is equivalent to a condition $(\mathrm{b'}_m)$ If $Z \not\subseteq X_0 \cup W_{J\odot J'}$, then $t_J^\sharp t_{J'} \sqcap \theta[Z] = 0_{TT}$. Assume that $r_0 \models \neg W_i \to W_i$, that is, $r_0\theta[\neg W_i]r_0 \sqsubseteq \theta[W_i]$. Now choose $J = M$ and $J' = M - \{i\}$. (Note that $M \odot (M - \{i\}) = M - \{i\}$.) Then we have $t_M^\sharp t_{M-\{i\}} \sqsubseteq r_0\theta[\neg W_i]r_0 \sqsubseteq \theta[W_i]$, since $t_M^\sharp t_{M-\{i\}} \sqsubseteq \theta[X_0 \cup W_{M-\{i\}}] = \theta[\neg W_i]$ by (a_m). On the other hand it follows from $(\mathrm{b'}_m)$ that $t_M^\sharp t_{M-\{i\}} \sqcap \theta[W_i] = 0_{TT}$, which is a contradiction.
(d) Since $r_0\nabla_{TI} = \sqcup_{J\subseteq M} t_J^\sharp$ by the totality of t_J's, we have

$$r_0\nabla_{TT}r_0 = r_0\nabla_{TI}\nabla_{IT}r_0 = (\sqcup_{J\subseteq M} t_J^\sharp)(\sqcup_{J\subseteq M} t_J).$$

Hence to prove that $r_0\nabla_{TT}r_0 \sqsubseteq r_0\theta[\neg W_i]r_0\theta[W_i]r_0$ it suffices to see that $t_J^\sharp t_{J'} \sqsubseteq r_0\theta[\neg W_i]r_0\theta[W_i]r_0$ for all $J, J' \subseteq M$. First assume that $i \in J \odot J'$. Then $t_J^\sharp t_{J'} \sqsubseteq \theta[X_0 \cup W_{J\odot J'}] \sqsubseteq \theta[W_i]$. Recall that it is trivial that $t_J^\sharp t_J \sqsubseteq \theta[\neg W_i]$. Hence we have

$$t_J^\sharp t_{J'} = (t_J^\sharp t_J)(t_J^\sharp t_{J'}) \sqsubseteq r_0\theta[\neg W_i]r_0\theta[W_i]r_0.$$

Next assume that $i \notin J$ and $i \in J'$. (Note that $i \notin J \odot J'$ iff $i \notin J$ and $i \in J'$, or $i \in J$ and $i \notin J'$.) Set $J'' = J \cup \{i\}$. Then it is easy to see that $J \odot J'' = M - \{i\}$ and $J'' \odot J' = (J \odot J') \cup \{i\}$. Hence using (a_m) we have

$$t_J^\sharp t_{J''} \sqsubseteq \theta[X_0 \cup W_{M-\{i\}}] = \theta[\neg W_i]$$

and

$$t_{J''}^\sharp t_{J'} \sqsubseteq \theta[X_0 \cup W_{J\odot J'} \cup W_i] \sqsubseteq \theta[W_i]$$

which shows that $t_J^\sharp t_{J'} = t_J^\sharp t_{J''} t_{J''}^\sharp t_{J'} \sqsubseteq r_0\theta[\neg W_i]r_0\theta[W_i]r_0$.
(e) Assume that $r_0 \models \neg W_i \twoheadrightarrow W$ for a nonempty proper subsets W of W_i, that is, $r_0\theta[\neg W_i]r_0 \sqsubseteq \theta[\neg W]r_0\theta[W]$. Now choose $J = M$ and $J' = M - \{i\}$. Then, since $t_M^\sharp t_{M-\{i\}} \sqsubseteq \theta[\neg W_i]$ in the same way as in (c), we have

$$t_M^\sharp t_{M-\{i\}} \sqsubseteq r_0\theta[\neg W_i]r_0 \sqsubseteq \theta[\neg W]r_0\theta[W]$$

and so $t_M\theta[\neg W]r_0\theta[W]t^\sharp_{M-\{i\}} = \mathrm{id}_I$ by Lemma 2.2(a). To show that the last fact is a contradiction we will see that $t_M\theta[\neg W]r_0\theta[W]t^\sharp_{M-\{i\}} = 0_{II}$. Assume that J is a subset of M containing i. Since $J \odot (M - \{i\}) = J - \{i\}$ and $W \subset W_i \subseteq \neg(X_0 \cup W_{J-\{i\}})$, it follows from (b'$_m$) that $t_J\theta[W]t^\sharp_{M-\{i\}} = 0_{II}$. Next assume that $i \notin J$. Since $M \odot J = J$ and $\emptyset \neq \neg W \cap W_i \subseteq \neg W \cap \neg(X_0 \cup W_J)$, it follows from (b'$_m$) that $t_M\theta[\neg W]t^\sharp_J = 0_{II}$. Therefore

$$\begin{aligned} t_M\theta[\neg W]r_0\theta[W]t^\sharp_{M-\{i\}} &= \sqcup_{J\subseteq M} t_M\theta[\neg W]t^\sharp_J t_J\theta[W]t^\sharp_{M-\{i\}} \\ &= 0_{II}. \end{aligned}$$

□

COROLLARY 5.4. (a) *If* $\emptyset \neq W \subseteq W_i$, *then* $r_0 \models Z \to W$ *if and only if* $Z \cap W_i \neq \emptyset$,

(b) *If* $\emptyset \neq W \subset W_i$, *then* $r_0 \models Z \twoheadrightarrow W$ *if and only if* $Z \cap W_i \neq \emptyset$.

PROOF. (a) Let $\emptyset \neq W \subseteq W_i$. First assume that $Z \cap W_i \neq \emptyset$, that is, there exists $a \in Z \cap W_i$. Then, recalling $r_0 \models \{a\} \to W_i$ in Lemma 5.3(b) and applying the laws FD0, FD1 and FD2 a dependency $r_0 \models Z \to W$ follows. Conversely assume that $Z \cap W_i = \emptyset$, that is, $Z \subseteq \neg W_i$. Notice that $r_0 \models W \to W_i$ is valid by Lemma 5.3(b). If $r_0 \models Z \to W$ is valid, we immediately have $r_0 \models \neg W_i \to W_i$ again using FD2 and FD1. However this contradicts a fact $r_0 \not\models \neg W_i \to W_i$ already seen in Lemma 5.3(c). Hence $r_0 \not\models Z \to W$.
(b) Let $\emptyset \neq W \subset W_i$. First assume that $Z \cap W_i \neq \emptyset$. Then $r_0 \models Z \to W$ is valid by (a) and so $r_0 \models Z \twoheadrightarrow W$ by using the law FD-MVD1. Conversely assume that $Z \cap W_i = \emptyset$, that is, $Z \subseteq \neg W_i$. If $r_0 \models Z \twoheadrightarrow W$ is valid, a dependency $r_0 \models \neg W_i \twoheadrightarrow W$ follows from the augmentation law MVD1. But this is impossible by Lemma 5.3(e). Therefore $r_0 \not\models Z \twoheadrightarrow W$. □

References

Armstrong, W.W. (1974). Dependency structures of data base relationships, *Proc. of IFIP '74*, North Holland, 580-583.

Beeri, C., Fagin, R. and Howard, J.H. (1977). A complete axiomatization for functional and multivalued dependencies in database relations, *in: Proc. ACM SIGMOD Internat. Conf. on Management of Data*, Toronto, 1977 (ACM, New York, 1983), 47-61.

Bhattacharjee, T.K. and Mazumdar, A.K. (1998). Axiomatisation of fuzzy multivalued dependencies in fuzzy relational data model, *Fuzzy Sets and Systems*, **96**, 343-352.

Biskup, J. (1987). On the complementation rule for multivalued dependencies in database relations, *Acta Informatica*, **10**, 297-305.

Biskup, J. (1980). Inferences of multivalued dependencies in fixed and undetermined universes, *Theoretical Computer Science*, **10**, 93-105.

Buszkowski, W. and Orlowska, E. (1998). Indiscernibilitiy-based formalization of dependencies in information systems, *in: Orlowska,E.(ed.): Incomplete information: Rough set analysis*, Physica Verlag, 293-315.

Codd, E.F. (1970). A relational model for large shared data base relations, *CACM*, **13**, 377-387.

Desharnais, J. (1997). Monomorphic characterization of n-ary direct products, *Proc. 3rd RelMiCS in Hammamet*, Tunis, January 1997.

Freyd, P.J. and Ščedrov, A. (1990). *Categories, Allegories*, North-Holland.

Grant, J. and Jacobs, B.E. (1982). On the family of generalized dependency constraints, *J. ACM*, **29**(4), 986-997.

Kawahara, Y. (1998). Lattices in Dedekind categories, *Proc. 4th RelMiCS in Warsaw*, Poland.

MacCaull, W. (2000a). A proof system for dependencies for information relations, *Fundamenta Informaticae*, **42**, 1-27.

MacCaull, W. (2000b). A tableaux procedure for the implication problem for association rules, to appear in *Relational Methods for Computer Science Applications*, Orlowska,E. and Szalas,A.(eds.), Springer-Physica Verlag, 73-91.

Mendelzon, A.O. (1979). On axiomatizing multivalued dependencies in relational databases, *J. ACM*, **26**(1), 37-44.

Olivier, J.-P. and Serrato, D. (1980). Catégories de Dedekind. Morphismes dans les Catégories de Schröder, *C. R. Acad. Sci. Paris*, **260**, 939-941.

Orlowska, E. (1987). Algebraic approach to database constraints, *Fundamenta Informaticae*, **10**, 57-68.

Ounalli, H. and Jaoua, A. (1997). Fuzzy difunctional dependencies, *Proc. 3rd RelMiCS in Hammamet*, Tunis, January 1997.

Schmidt, G. and Ströhlein, T. (1993). *Relations and graphs – Discrete Mathematics for Computer Science* – (Springer-Verlag, Berlin).

Reveived November 10, 2000
Revised Feburuary 16, 2001