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AN APPLICATION OF A COMBINATORIAL THEOREM TO A SEPARATION PROBLEM: I

By

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Abstract

We consider functionals which arise naturally from game theoretical consideration and are also generalizations of the functionals in Hisano and Maruyama (1989). In the study of the natural partial order on the set of these functionals, we establish a separation theorem, making use of the theorem of Hales-Jewett.

1. Introduction

A sequence of operations of minimizing or maximizing a function subject to the variables can be considered an optimal play of some game. For example, for a real-valued function F defined on the Cartesian product of finite sets Z^1, Z^2, Z^3, Z^4,

$$\min_{z^2\in Z^2}\max_{z^1\in Z^1}\min_{z^4\in Z^4}\max_{z^3\in Z^3} F(z^1,z^2,z^3,z^4)$$

is the value of some zero-sum game between minimizer and maximizer where F(resp. $-F$) is the payoff function of maximizer (resp. minimizer). Since this game is a finite zero-sum game of perfect information, in normal-form representation, this game has a saddle point and the players have optimal pure strategies. And so there exist $z_0^2 \in Z^2, \varphi_0^1 \in {}^{Z^2}Z^1$ (where ${}^{Z^2}Z^1$ denotes the set of all functions from Z^2 into Z^1), $\varphi_0^3 \in {}^{(Z^2\times Z^4)}Z^3, \varphi_0^4 \in {}^{Z^1}Z^4$ such that (z_0^2, φ_0^4) and $(\varphi_0^1, \varphi_0^3)$ are optimal pure strategies of minimizer and maximizer respectively. Then

$$\begin{aligned}&\min_{z^2\in Z^2}\max_{z^1\in Z^1}\min_{z^4\in Z^4}\max_{z^3\in Z^3} F(z^1,z^2,z^3,z^4)\\ &\quad= F\big(\varphi_0^1(z_0^2), z_0^2, \varphi_0^3(z_0^2, \varphi_0^4(\varphi_0^1(z_0^2))), \varphi_0^4(\varphi_0^1(z_0^2))\big).\end{aligned}$$

Using such response functions we can also consider this game as a two-stage game, whose value is equal to

$$\begin{aligned}&\min_{(z^2,\varphi^4)\in Z^2\times {}^{Z^1}Z^4}\ \max_{(z^1,z^3)\in Z^1\times Z^3} F\big(z^1,z^2,z^3,\varphi^4(z^1)\big)\\ &= \max_{(\varphi^1,\varphi^3)\in {}^{Z^2}Z^1\times {}^{(Z^2\times Z^4)}Z^3}\ \min_{(z^2,z^4)\in Z^2\times Z^4} F\big(\varphi^1(z^2), z^2, \varphi^3(z^2,z^4), z^4\big).\end{aligned}$$

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Then, for instance, it is quite natural to consider two-stage games as follows.
(1) Minimizer chooses $\varphi^2 \in {}^{Z^1}Z^2$ and $\varphi^4 \in {}^{Z^3}Z^4$ in the first stage, maximizer observes minimizer's choice and chooses $z^1 \in Z^1$ and $z^3 \in Z^3$ in the second.
(2) Maximizer chooses $\varphi^1 \in {}^{Z^2}Z^1$ and $\varphi^3 \in {}^{Z^4}Z^3$ in the first stage, minimizer observes maximizer's choice and chooses $z^2 \in Z^2$ and $z^4 \in Z^4$ in the second.
The values of these games are

$$\min_{(\varphi^2,\varphi^4)\in {}^{Z^1}Z^2\times {}^{Z^3}Z^4}\ \max_{(z^1,z^3)\in Z^1\times Z^3} F\big(z^1,\varphi^2(z^1),z^3,\varphi^4(z^3)\big)$$

and

$$\max_{(\varphi^1,\varphi^3)\in {}^{Z^2}Z^1\times {}^{Z^4}Z^3}\ \min_{(z^2,z^4)\in Z^2\times Z^4} F\big(\varphi^1(z^2),z^2,\varphi^3(z^4),z^4\big)$$

respectively. The functionals

$$F \in {}^{(Z^1\times Z^2\times Z^3\times Z^4)}\boldsymbol{R} \longrightarrow \min_{(\varphi^2,\varphi^4)\in {}^{Z^1}Z^2\times {}^{Z^3}Z^4}\ \max_{(z^1,z^3)\in Z^1\times Z^3} F\big(z^1,\varphi^2(z^1),z^3,\varphi^4(z^3)\big)$$

and

$$F \in {}^{(Z^1\times Z^2\times Z^3\times Z^4)}\boldsymbol{R} \longrightarrow \max_{(\varphi^1,\varphi^3)\in {}^{Z^2}Z^1\times {}^{Z^4}Z^3}\ \min_{(z^2,z^4)\in Z^2\times Z^4} F\big(\varphi^1(z^2),z^2,\varphi^3(z^4),z^4\big)$$

cannot be represented by iteration of minimization and maximization.

The set of functionals corresponding to these kind of two-stage games is strictly wider than the set of functionals represented by iteration of minimization and maximization. We consider the natural pointwise order on the set of such functionals.

By the result in Hisano and Maruyama (1989), the order structure on the set of functionals of sequential minimization or maximization is clear. The purpose of this paper is to study the order structure on thus widened set of functionals. We construct a separating function in this paper and an ordering function in the follow-up paper. In Maruyama (1992) the games related to the functionals in this paper are described.

In Section 2, we give a precise definition of the set of functionals we are concerned with in this paper. In Section 3, we study the pairwise order relations between the functionals defined in Section 2. In Section 4, for functionals $\tau, \tau_1, \tau_2, \dots, \tau_t$ with $\tau_i \not\leqslant \tau$ (resp. $\tau_i \not\geqslant \tau$) $(1 \leqslant i \leqslant t)$, we prove the existence of a separating function F such that $\tau(F) < \min\limits_{1\leqslant i\leqslant t} \tau_i(F)$ (resp. $\tau(F) > \max\limits_{1\leqslant i\leqslant t} \tau_i(F)$) when the domain of each variable has sufficiently many elements.

In the case of propositional functions, this generalization of functionals corresponds to the generalization of linear quantifier prefixes to nonlinear quantifier prefixes (Henkin quantifier prefixes Krynicki and Mostowski (1995)) and thier dual quantifier prefixes. So in the present investigation we also obtain logical results relating to Henkin quantifier prefixes and their dual quantifier prefixes.

2. Notation

We denote the cardinality of a set U by $\sharp U$. For a set U and a cardinal k we write $\binom{U}{k} = \{V \subset U \mid \sharp V = k\}$. Denote by ${}^U V$ the set of all mappings from a set U into a set V. We note that ${}^\phi V = \{\phi\}$. Denote by $\mathfrak{D}(f)$ and $\mathfrak{R}(f)$ the domain and the range of a mapping f respectively.

For integers k and l, $[k,l]$ denotes the set of integers larger than $k-1$ and less than $l+1$. We denote by $\boldsymbol{N}, \boldsymbol{R}$, respectively, the set of all positive integers, and the set of all real numbers. We write $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, \infty\}$.

Let $n \in \boldsymbol{N}$ be fixed. For each $k \in [1,n]$ let Z^k be a set which contains at least two elements. We set

$$\begin{aligned}
&\Pi_k = \left\{\eta \in {}^{[1,k]}[1,n] \mid \sharp\eta([1,k]) = k\right\} \ (1 \leqslant k \leqslant n),\ \Pi = \Pi_n,\ \Pi_0 = \{\phi\},\\
&\Gamma(\pi,l) = \left\{\gamma = (\gamma_1, \dots, \gamma_l) \in \left(\bigcup_{k=0}^{n-l} \Pi_k\right)^l \mid \bigcup_{k=1}^{l} \mathfrak{R}(\gamma_k) \subset \pi([l+1,n])\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqslant l \leqslant n,\ \pi \in \Pi),\\
&\Gamma(\pi,0) = \{\phi\},\\
&\Phi(l,\eta) = {}^{(Z^{\eta(1)} \times \cdots \times Z^{\eta(k)})} Z^l \quad (k,l \in [1,n],\ \eta \in \Pi_k),\\
&\Phi(l,\phi) = Z^l \quad (1 \leqslant l \leqslant n).
\end{aligned}$$

For $F \in {}^{(Z^1 \times \cdots \times Z^n)} \boldsymbol{R}$, $\pi \in \Pi$, $l \in [1,n]$, $\gamma \in \Gamma(\pi,l)$ define

$$\pi * F : Z^{\pi(1)} \times \cdots \times Z^{\pi(n)} \longrightarrow \boldsymbol{R}$$

and

$$\gamma * \pi * F : \Phi(\pi(1),\gamma_1) \times \cdots \times \Phi(\pi(l),\gamma_l) \times Z^{\pi(l+1)} \times \cdots \times Z^{\pi(n)} \longrightarrow \boldsymbol{R}$$

by

$$(\pi * F)(z^{\pi(1)}, \dots, z^{\pi(n)}) = F(z^1, \dots, z^n)$$

and

$$\begin{aligned}
&(\gamma * \pi * F)(\varphi^1, \dots, \varphi^l, z^{\pi(l+1)}, \dots, z^{\pi(n)})\\
&= (\pi * F)\big(\varphi^1(z^{\gamma_1(1)}, \dots, z^{\gamma_1(h_1)}), \dots, \varphi^l(z^{\gamma_l(1)}, \dots, z^{\gamma_l(h_l)}), z^{\pi(l+1)}, \dots, z^{\pi(n)}\big)
\end{aligned}$$

respectively for all $\varphi^1 \in \Phi(\pi(1),\gamma_1), \dots, \varphi^l \in \Phi(\pi(l),\gamma_l)$ and $z^{\pi(l+1)} \in Z^{\pi(l+1)}, \dots,$ $z^{\pi(n)} \in Z^{\pi(n)}$ where $\mathfrak{D}(\gamma_k) = [1,h_k]$ $(1 \leqslant k \leqslant l)$. We set $\phi * \pi * F = \pi * F$.

For $l \in [0,n]$, $\pi \in \Pi$, $\gamma \in \Gamma(\pi,l)$ define

$$\sigma_+(l,\pi,\gamma), \sigma_-(l,\pi,\gamma) : {}^{(Z^1 \times \cdots \times Z^n)}\boldsymbol{R} \longrightarrow \overline{\boldsymbol{R}}$$

by

$$\sigma_+(l,\pi,\gamma)(F) = \sup_{\varphi^1\in\Phi(\pi(1),\gamma_1)} \cdots \sup_{\varphi^l\in\Phi(\pi(l),\gamma_l)} \inf_{z^{\pi(l+1)}\in Z^{\pi(l+1)}} \cdots \inf_{z^{\pi(n)}\in Z^{\pi(n)}} (\gamma * \pi * F)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}),$$

$$\sigma_-(l,\pi,\gamma)(F) = \inf_{\varphi^1\in\Phi(\pi(1),\gamma_1)} \cdots \inf_{\varphi^l\in\Phi(\pi(l),\gamma_l)} \sup_{z^{\pi(l+1)}\in Z^{\pi(l+1)}} \cdots \sup_{z^{\pi(n)}\in Z^{\pi(n)}} (\gamma * \pi * F)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}).$$

We set

$$\Sigma_+ = \left\{\sigma_+(l,\pi,\gamma) \in {}^{(Z^1\times\cdots\times Z^n)\boldsymbol{R}}\,\overline{\boldsymbol{R}} \mid 0 \leqslant l \leqslant n,\ \pi\in\Pi,\ \gamma\in\Gamma(\pi,l)\right\},$$
$$\Sigma_- = \left\{\sigma_-(l,\pi,\gamma) \in {}^{(Z^1\times\cdots\times Z^n)\boldsymbol{R}}\,\overline{\boldsymbol{R}} \mid 0 \leqslant l \leqslant n,\ \pi\in\Pi,\ \gamma\in\Gamma(\pi,l)\right\},$$
$$\Sigma = \Sigma_+ \cup \Sigma_-.$$

For $\nu = \sigma_+(l,\pi,\gamma), \mu = \sigma_-(l,\pi,\gamma) \in \Sigma$ we set $\overline{\nu} = \underline{\mu} = \pi([1,l])$, $\underline{\nu} = \overline{\mu} = \pi([l+1,n])$,

$$\nu_+ = \mu_- = \bigcup_{k=1}^{l} \left\{ \left(u,\pi(k)\right) \in [1,n]^2 \mid u \in \mathfrak{R}(\gamma_k) \right\},$$
$$\nu_- = \mu_+ = \bigcup_{k=1}^{l} \left\{ \left(\pi(k),u\right) \in [1,n]^2 \mid u \in \pi([l+1,n]) \setminus \mathfrak{R}(\gamma_k) \right\}.$$

For $\tau,\tau' \in \Sigma$ and $p \in \big[1,[n/2]\big]$ (where $[n/2]$ denotes the largest integer not exceeding $n/2$) define

$$C(\tau,\tau',p) = \Big\{\eta \in \Pi_{2p} \mid \big(\eta(p+k),\eta(k)\big) \in \tau'_+ \quad (1 \leqslant k \leqslant p),$$
$$\big(\eta(k),\eta(p+k+1)\big) \in \tau_- \quad (1 \leqslant k \leqslant p-1), \big(\eta(p),\eta(p+1)\big) \in \tau_-\Big\}.$$

We set $C(\tau,\tau',0) = \underline{\tau} \cap \overline{\tau'}$.

3. Order on Σ

We define the order $\leqslant$ on Σ by, for $\tau,\tau' \in \Sigma$, $\tau \leqslant \tau'$ if and only if $\tau(F) \leqslant \tau'(F)$ for all $F : Z^1 \times \cdots \times Z^n \longrightarrow \boldsymbol{R}$. We will give a necessary and sufficient condition of $\tau \leqslant \tau'$.

THEOREM 3.1. *Suppose that $\nu,\nu' \in \Sigma_+$ and $\mu,\mu' \in \Sigma_-$. Then*

(1) $\nu \nleqslant \nu' \iff C(\nu',\nu,0) \cup C(\nu',\nu,1) \neq \phi$,

(2) $\mu \nleqslant \mu' \iff C(\mu',\mu,0) \cup C(\mu',\mu,1) \neq \phi$.

PROOF. We prove only (1). The proof of (2) is similar. Without loss of generality we can assume that $\{0,1\} \subset Z^i$ $(1 \leqslant i \leqslant n)$.

$(\Longleftarrow)$: If $C(\nu', \nu, 0) \neq \phi$, then we take $k \in C(\nu', \nu, 0)$ and define $F : Z^1 \times \cdots \times Z^n \longrightarrow \boldsymbol{R}$ by

$$F(z^1, \ldots, z^n) = \begin{cases} 0 & (z^k = 0), \\ 1 & (z^k \neq 0). \end{cases}$$

If $C(\nu', \nu, 0) = \phi$ and $C(\nu', \nu, 1) \neq \phi$, then we take $\eta \in C(\nu', \nu, 1)$ and define F by

$$F(z^1, \ldots, z^n) = \begin{cases} 0 & (z^{\eta(1)} = 0 \Longleftrightarrow z^{\eta(2)} = 0), \\ 1 & (\text{otherwise}). \end{cases}$$

In both cases $\nu'(F) = 0$, $\nu(F) = 1$ and therefore $\nu \not\leqslant \nu'$.

$(\Longrightarrow)$: The condition $C(\nu', \nu, 0) \cup C(\nu', \nu, 1) = \phi$ implies that there exist $\pi \in \Pi$, $h, l \in [0, n]$, $\gamma \in \Gamma(\pi, h)$, $\delta \in \Gamma(\pi, l)$ such that $h \leqslant l$, $\mathfrak{R}(\gamma_k) \cap \pi([l+1, n]) \subset \mathfrak{R}(\delta_k)$ $(1 \leqslant k \leqslant h)$, $\nu = \sigma_+(h, \pi, \gamma)$, $\nu' = \sigma_+(l, \pi, \delta)$. Then for $F : Z^1 \times \cdots \times Z^n \longrightarrow \boldsymbol{R}$

$$\begin{aligned}
\nu(F) = {} & \sup_{\varphi^1 \in \Phi(\pi(1), \gamma_1)} \cdots \sup_{\varphi^h \in \Phi(\pi(h), \gamma_h)} \inf_{z^{\pi(h+1)} \in Z^{\pi(h+1)}} \cdots \inf_{z^{\pi(n)} \in Z^{\pi(n)}} \\
& \qquad (\gamma * \pi * F)(\varphi^1, \ldots, \varphi^h, z^{\pi(h+1)}, \ldots, z^{\pi(n)}) \\
\leqslant {} & \sup_{\varphi^1 \in \Phi(\pi(1), \gamma_1)} \cdots \sup_{\varphi^h \in \Phi(\pi(h), \gamma_h)} \inf_{z^{\pi(l+1)} \in Z^{\pi(l+1)}} \cdots \inf_{z^{\pi(n)} \in Z^{\pi(n)}} \\
& \qquad (\gamma * \pi * F)(\varphi^1, \ldots, \varphi^h, 0, \ldots, 0, z^{\pi(l+1)}, \ldots, z^{\pi(n)}) \\
\leqslant {} & \sup_{\varphi^1 \in \Phi(\pi(1), \delta_1)} \cdots \sup_{\varphi^h \in \Phi(\pi(h), \delta_h)} \inf_{z^{\pi(l+1)} \in Z^{\pi(l+1)}} \cdots \inf_{z^{\pi(n)} \in Z^{\pi(n)}} \\
& \qquad (\delta' * \pi * F)(\varphi^1, \ldots, \varphi^h, 0, \ldots, 0, z^{\pi(l+1)}, \ldots, z^{\pi(n)}) \\
& \qquad \big(\text{where } \delta' = (\delta_1, \ldots, \delta_h) \in \Gamma(\pi, h)\big) \\
\leqslant {} & \sup_{\varphi^1 \in \Phi(\pi(1), \delta_1)} \cdots \sup_{\varphi^l \in \Phi(\pi(l), \delta_l)} \inf_{z^{\pi(l+1)} \in Z^{\pi(l+1)}} \cdots \inf_{z^{\pi(n)} \in Z^{\pi(n)}} \\
& \qquad (\delta * \pi * F)(\varphi^1, \ldots, \varphi^l, z^{\pi(l+1)}, \ldots, z^{\pi(n)}) \\
= {} & \nu'(F).
\end{aligned}$$

Therefore $\nu \leqslant \nu'$. □

LEMMA 3.2. *Let $\mu \in \Sigma_-, \nu \in \Sigma_+, p \in \big[1, [n/2]\big], \eta \in C(\mu, \nu, p)$ and let $F : Z^1 \times \cdots \times Z^n \longrightarrow \boldsymbol{R}$ be defined by*

$$F(z^1, \ldots, z^n) = \begin{cases} 1 & \big(z^{\eta(k)} = 0 \Longleftrightarrow z^{\eta(p+k)} = 0 \quad (1 \leqslant k \leqslant p)\big), \\ 0 & (\textit{otherwise}). \end{cases}$$

Then $\mu(F) = 0$ and $\nu(F) = 1$.

PROOF. By the condition $\eta \in C(\mu, \nu, p)$, we can write $\mu = \sigma_-(l, \pi, \gamma)$ where $l \in [p, n]$, $\pi \in \Pi$ and $\gamma \in \Gamma(\pi, l)$ are such that $\pi(k) = \eta(p+k)$ $(1 \leqslant k \leqslant p)$, $\eta(k) \in$

$\mathfrak{R}(\gamma_{k+1})$ $(1 \leqslant k \leqslant p-1)$ and $\eta(p) \in \mathfrak{R}(\gamma_1)$.
Define $\varphi_0^k \in \Phi(\pi(k), \gamma_k)$ $(1 \leqslant k \leqslant p)$ by

$$\varphi_0^{k+1}(z^{\gamma_{k+1}(1)}, \dots, z^{\gamma_{k+1}(l_{k+1})}) = \begin{cases} 0 & (z^{\eta(k)} = 0), \\ 1 & (z^{\eta(k)} \neq 0) \end{cases} \quad (1 \leqslant k \leqslant p-1),$$

$$\varphi_0^1(z^{\gamma_1(1)}, \dots, z^{\gamma_1(l_1)}) = \begin{cases} 1 & (z^{\eta(p)} = 0), \\ 0 & (z^{\eta(p)} \neq 0) \end{cases}$$

where $\mathfrak{D}(\gamma_k) = [1, l_k]$ $(1 \leqslant k \leqslant p)$. Then

$$\begin{aligned} \mu(F) = & \inf_{\varphi^1 \in \Phi(\pi(1),\gamma_1)} \cdots \inf_{\varphi^l \in \Phi(\pi(l),\gamma_l)} \sup_{z^{\pi(l+1)} \in Z^{\pi(l+1)}} \cdots \sup_{z^{\pi(n)} \in Z^{\pi(n)}} \\ & \qquad (\gamma * \pi * F)(\varphi^1, \dots, \varphi^l, z^{\pi(l+1)}, \dots, z^{\pi(n)}) \\ \leqslant & \sup_{z^{\pi(l+1)} \in Z^{\pi(l+1)}} \cdots \sup_{z^{\pi(n)} \in Z^{\pi(n)}} \\ & \qquad (\gamma' * \pi * F)(\varphi_0^1, \dots, \varphi_0^p, 0, \dots, 0, z^{\pi(l+1)}, \dots, z^{\pi(n)}) \\ = & \, 0 \end{aligned}$$

where $\gamma' = (\gamma_1, \dots, \gamma_p) \in \Gamma(\pi, p)$. Clearly $\nu(F) = 1$. $\square$

LEMMA 3.3. *Let* $\nu \in \Sigma_+, \mu \in \Sigma_-, p \in [1, [n/2]], \eta \in C(\nu, \mu, p)$. *Let* T_k $(0 \leqslant k \leqslant p+1)$ *be defined by*

$$\begin{aligned} T_0 &= [\{0,1\}^p \times Z^{\eta(p+1)} \times \cdots \times Z^{\eta(2p)}] \setminus \{0,1\}^{2p}, \\ T_k &= \{0\}^{k-1} \times \{1\} \times \{0\}^{p-k} \times \{0,1\}^{k-1} \times \{0\} \times \{0,1\}^{p-k} \quad (1 \leqslant k \leqslant p), \\ T_{p+1} &= \{0\}^p \times \{1\}^p \end{aligned}$$

and let $T = \bigcup_{k=0}^{p+1} T_k$. *Define* $F : Z^1 \times \cdots \times Z^n \longrightarrow \boldsymbol{R}$ *by*

$$F(z^1, \dots, z^n) = \begin{cases} 0 & ((z^{\eta(1)}, \dots, z^{\eta(2p)}) \notin T), \\ 1 & ((z^{\eta(1)}, \dots, z^{\eta(2p)}) \in T). \end{cases}$$

Then $\nu(F) = 0$ *and* $\mu(F) = 1$.

PROOF. $\nu(F) = 0$: By the condition $\eta \in C(\nu, \mu, p)$, we can write $\nu = \sigma_+(l, \pi_0, \gamma^0)$ where $l \in [p, n], \pi_0 \in \Pi$ and $\gamma^0 \in \Gamma(\pi_0, l)$ are such that $\pi_0(k) = \eta(k)$ $(1 \leqslant k \leqslant p)$, $\eta(p+k+1) \notin \mathfrak{R}(\gamma_k^0)$ $(1 \leqslant k \leqslant p-1)$ and $\eta(p+1) \notin \mathfrak{R}(\gamma_p^0)$. Let $\rho_0 \in \Pi, \delta^0 \in \Gamma(\rho_0, p)$ be such that for each $k \in [1, 2p]$ $\rho_0(k) = \eta(k)$ and for each $k \in [1, p-1]$ $\mathfrak{D}(\delta_k^0) = [1, p-1]$,

$$\delta_k^0(j) = \begin{cases} \eta(p+j) & (1 \leqslant j \leqslant k), \\ \eta(p+j+1) & (k+1 \leqslant j \leqslant p-1) \end{cases}$$

and $\mathfrak{D}(\delta_p^0) = [1, p-1]$, $\delta_p^0(j) = \eta(p+j+1)$ $(1 \leqslant j \leqslant p-1)$.
We claim that

$$\begin{aligned} & \sup_{\varphi^1 \in \Phi(\rho_0(1), \delta_1^0)} \cdots \sup_{\varphi^p \in \Phi(\rho_0(p), \delta_p^0)} \inf_{z^{\eta(p+1)} \in Z^{\eta(p+1)}} \cdots \inf_{z^{\eta(2p)} \in Z^{\eta(2p)}} \\ & \qquad (\delta^0 * \rho_0 * F)(\varphi^1, \dots, \varphi^p, z^{\eta(p+1)}, \dots, z^{\eta(2p)}, 0, \dots, 0) \\ & = 0. \end{aligned}$$

Assume to the contrary that there exist $\varphi^1 \in \Phi(\rho_0(1), \delta_1^0), \dots, \varphi^p \in \Phi(\rho_0(p), \delta_p^0)$ such that

$$\begin{aligned}&\forall z^{\eta(p+1)} \in Z^{\eta(p+1)} \dots \forall z^{\eta(2p)} \in Z^{\eta(2p)}\\ &\quad\Big(\varphi^1(z^{\eta(p+1)}, z^{\eta(p+3)}, \dots, z^{\eta(2p)}), \dots, \varphi^{p-1}(z^{\eta(p+1)}, \dots, z^{\eta(2p-1)}),\\ &\qquad\qquad \varphi^p(z^{\eta(p+2)}, \dots, z^{\eta(2p)}), z^{\eta(p+1)}, \dots, z^{\eta(2p)}\Big) \in T.\end{aligned}$$

Let

$$\begin{aligned}S_k &= \Big\{(z^{\eta(p+1)}, \dots, z^{\eta(2p)}) \in \{0,1\}^p \mid\\ &\qquad \varphi^k(z^{\eta(p+1)}, \dots, z^{\eta(p+k)}, z^{\eta(p+k+2)}, \dots, z^{\eta(2p)}) = 1\Big\} \quad (1 \leqslant k \leqslant p-1),\\ S_p &= \Big\{(z^{\eta(p+1)}, \dots, z^{\eta(2p)}) \in \{0,1\}^p \mid \varphi^p(z^{\eta(p+2)}, \dots, z^{\eta(2p)}) = 1\Big\}.\end{aligned}$$

The assumption implies that $S_k \cap S_j = \phi$ $(1 \leqslant k, j \leqslant p,\ k \neq j)$ and

$$\{0,1\}^p \setminus \{1\}^p = \bigcup_{k=1}^{p} S_k.$$

Since $\natural S_k \equiv 0 \pmod 2$ $(1 \leqslant k \leqslant p)$ it follows that $2^p - 1 \equiv 0 \pmod 2$, a contradiction.

Combining with $\mathfrak{R}(\gamma_k^0) \cap \mathfrak{R}(\eta) \subset \mathfrak{R}(\delta_k^0)$ $(1 \leqslant k \leqslant p)$, the claim implies that

$$\begin{aligned}\nu(F) &= \sup_{\varphi^1 \in \Phi(\pi_0(1), \gamma_1^0)} \cdots \sup_{\varphi^l \in \Phi(\pi_0(l), \gamma_l^0)} \inf_{z^{\pi_0(l+1)} \in Z^{\pi_0(l+1)}} \cdots \inf_{z^{\pi_0(n)} \in Z^{\pi_0(n)}}\\ &\qquad (\gamma^0 * \pi_0 * F)(\varphi^1, \dots, \varphi^l, z^{\pi_0(l+1)}, \dots, z^{\pi_0(n)})\\ &\leqslant \sup_{\varphi^1 \in \Phi(\rho_0(1), \delta_1^0)} \cdots \sup_{\varphi^p \in \Phi(\rho_0(p), \delta_p^0)} \inf_{z^{\eta(p+1)} \in Z^{\eta(p+1)}} \cdots \inf_{z^{\eta(2p)} \in Z^{\eta(2p)}}\\ &\qquad (\delta^0 * \rho_0 * F)(\varphi^1, \dots, \varphi^p, z^{\eta(p+1)}, \dots, z^{\eta(2p)}, 0, \dots, 0)\\ &= 0.\end{aligned}$$

$\mu(F) = 1$: By the condition $\eta \in C(\nu, \mu, p)$, we can write $\mu = \sigma_-(h, \pi_1, \gamma^1)$ where $h \in [p, n], \pi_1 \in \Pi$ and $\gamma^1 \in \Gamma(\pi_1, h)$ are such that $\pi_1(k) = \eta(p+k), \eta(k) \notin \mathfrak{R}(\gamma_k^1)$ $(1 \leqslant k \leqslant p)$. Let $\rho_1 \in \Pi, \delta^1 \in \Gamma(\rho_1, p)$ be such that for $k \in [1,p], \rho_1(k) = \eta(p+k), \rho_1(p+k) = \eta(k)$,

$$\mathfrak{D}(\delta_k^1) = [1, p-1], \qquad \delta_k^1(j) = \begin{cases} \eta(j) & (1 \leqslant j \leqslant k-1),\\ \eta(j+1) & (k \leqslant j \leqslant p-1).\end{cases}$$

Let $\varphi^1 \in \Phi(\rho_1(1), \delta_1^1), \dots, \varphi^p \in \Phi(\rho_1(p), \delta_p^1)$. If for some $k \in [1,p]$ $\varphi^k(0, \dots, 0) \neq 1$, then for $(z^{\eta(1)}, \dots, z^{\eta(p)}) \in \{0\}^{k-1} \times \{1\} \times \{0\}^{p-k}$,

$$\big(z^{\eta(1)}, \dots, z^{\eta(p)}, \varphi^1(z^{\eta(2)}, \dots, z^{\eta(p)}), \dots, \varphi^p(z^{\eta(1)}, \dots, z^{\eta(p-1)})\big) \in T_0 \cup T_k \subset T.$$

If for all $k \in [1,p]$ $\varphi^k(0, \dots, 0) = 1$ then

$$\big(0, \dots, 0, \varphi^1(0, \dots, 0), \dots, \varphi^p(0, \dots, 0)\big) \in T_{p+1} \subset T.$$

Therefore, since $\mathfrak{R}(\gamma_k^1) \cap \mathfrak{R}(\eta) \subset \mathfrak{R}(\delta_k^1)$ $(1 \leqslant k \leqslant p)$,

$$\begin{aligned}\mu(F) = & \inf_{\varphi^1 \in \Phi(\pi_1(1),\gamma_1^1)} \cdots \inf_{\varphi^h \in \Phi(\pi_1(h),\gamma_h^1)} \sup_{z^{\pi_1(h+1)} \in Z^{\pi_1(h+1)}} \cdots \sup_{z^{\pi_1(n)} \in Z^{\pi_1(n)}} \\ & \qquad (\gamma^1 * \pi_1 * F)(\varphi^1, \ldots, \varphi^h, z^{\pi(h+1)}, \ldots, z^{\pi(n)}) \\ \geqslant & \inf_{\varphi^1 \in \Phi(\rho_1,\delta_1^1)} \cdots \inf_{\varphi^p \in \Phi(\rho_1(p),\delta_p^1)} \sup_{z^{\eta(1)} \in Z^{\eta(1)}} \cdots \sup_{z^{\eta(p)} \in Z^{\eta(p)}} \\ & \qquad (\delta^1 * \rho_1 * F)(\varphi^1, \ldots, \varphi^p, z^{\eta(1)}, \ldots, z^{\eta(p)}, 0, \ldots, 0) \\ = & 1. \quad \square\end{aligned}$$

THEOREM 3.4. *For any $\tau, \tau' \in \Sigma$, it follows that*

$$\tau \not\leqslant \tau' \Longleftrightarrow \bigcup_{i=0}^{[n/2]} C(\tau', \tau, i) \neq \phi.$$

PROOF. Let $\nu \in \Sigma_+, \mu \in \Sigma_-$. By Theorem 3.1 it is enough to prove

$$(1)\ \nu \not\leqslant \mu \Longleftrightarrow \bigcup_{i=0}^{[n/2]} C(\mu, \nu, i) \neq \phi,$$

$$(2)\ \mu \not\leqslant \nu \Longleftrightarrow \bigcup_{i=0}^{[n/2]} C(\nu, \mu, i) \neq \phi.$$

Proof of (1): If $C(\mu,\nu,0) \neq \phi$, then $\nu \not\leqslant \mu$. If $\bigcup_{i=1}^{[n/2]} C(\mu,\nu,i) \neq \phi$ then $\nu \not\leqslant \mu$ follows from Lemma 3.2.

Suppose that $\bigcup_{i=0}^{[n/2]} C(\mu,\nu,i) = \phi$ and let $k = \sharp \underline{\mu}$. Then there exists $\rho \in \Pi_k$ such that $\underline{\mu} = \rho([1,k])$ and

$$(i,\rho(j)) \in \mu_- \Longrightarrow \big((\rho(j),i) \notin \nu_+, (\rho(j+1),i) \notin \nu_+, \ldots, (\rho(k),i) \notin \nu_+\big) \qquad (i \in \overline{\mu},\ 1 \leqslant j \leqslant k).$$

We show this first. If there exists nonempty $S \subset \underline{\mu}$ such that

$$\forall p \in S\ \exists q \in \overline{\mu}\ \exists r \in S\ \big((p,q) \in \nu_+, (q,r) \in \mu_-\big)$$

then there exist $i \in [1,[n/2]]$, $p_1, p_2, \ldots, p_i \in \underline{\mu}$, $q_1, q_2, \ldots, q_i \in \overline{\mu}$ such that $(p_j,q_j) \in \nu_+$ $(1 \leqslant j \leqslant i)$, $(q_j, p_{j+1}) \in \mu_-$ $(1 \leqslant j \leqslant i-1)$, $(q_i, p_1) \in \mu_-$. Letting $\eta \in \Pi_{2i}$ be such that $\eta(j) = q_j, \eta(i+j) = p_j$ $(1 \leqslant j \leqslant i)$ we have $\eta \in C(\mu,\nu,i)$, which contradicts the assumption. Therefore

$$\phi \neq \forall S \subset \underline{\mu}\ \exists p \in S\ \forall q \in \overline{\mu}\ \forall r \in S\ \big((q,r) \in \mu_- \Longrightarrow (p,q) \notin \nu_+\big).$$

Using this we can define $\rho(k), \rho(k-1), \ldots, \rho(1) \in \underline{\mu}$ so that

$$\forall i \in \overline{\mu}\ \forall l \in \underline{\mu} \setminus \rho([j+1,k])\ \big((i,l) \in \mu_- \Longrightarrow (\rho(j),i) \notin \nu_+\big).$$

For $j \in [1,k]$ let

$$A_j = \{ i \in \overline{\mu} \mid (\rho(j), i) \notin \nu_+, (\rho(j+1), i) \notin \nu_+, \dots, (\rho(k), i) \notin \nu_+ \}.$$

Then, since $A_1 \subset A_2 \subset \cdots \subset A_k$, there exist $\eta \in \Pi_{n-k}, h_1, h_2, \dots, h_k \in [0, n-k]$ so that $h_1 \leqslant h_2 \leqslant \cdots \leqslant h_k$, $\eta([1,h_j]) = A_j$ $(1 \leqslant j \leqslant k)$.
Let

$$\begin{aligned}\tau = {} & \sup_{z^{\eta(1)} \in Z^{\eta(1)}} \cdots \sup_{z^{\eta(h_1)} \in Z^{\eta(h_1)}} \inf_{z^{\rho(1)} \in Z^{\rho(1)}} \sup_{z^{\eta(h_1+1)} \in Z^{\eta(h_1+1)}} \cdots \sup_{z^{\eta(h_2)} \in Z^{\eta(h_2)}} \inf_{z^{\rho(2)} \in Z^{\rho(2)}} \\ & \cdots \sup_{z^{\eta(h_{k-1}+1)} \in Z^{\eta(h_{k-1}+1)}} \cdots \sup_{z^{\eta(h_k)} \in Z^{\eta(h_k)}} \inf_{z^{\rho(k)} \in Z^{\rho(k)}} \sup_{z^{\eta(h_k+1)} \in Z^{\eta(h_k+1)}} \cdots \sup_{z^{\eta(n-k)} \in Z^{\eta(n-k)}} \\ & \in \Sigma_+ \cap \Sigma_-.\end{aligned}$$

If $(i, \rho(j)) \in \mu_-$ then $i \in A_j$ and so $(i, \rho(j)) \in \tau_-$ $(i \in \overline{\mu}, 1 \leqslant j \leqslant k)$. Since $\underline{\mu} = \underline{\tau}$ we have $\mu_- \subset \tau_-$. Therefore $C(\mu, \tau, 0) = \phi, C(\mu, \tau, 1) = \phi$ and, by Theorem 3.1, $\tau \leqslant \mu$. If $(\rho(j), i) \in \nu_+$ then $i \notin A_j$ and so $(\rho(j), i) \in \tau_+$ $(i \in \overline{\mu}, 1 \leqslant j \leqslant k)$. Thus $(\underline{\tau} \times \overline{\tau}) \cap \nu_+ \subset \tau_+$. Clearly $\overline{\nu} \subset \overline{\tau}$. Therefore $C(\tau, \nu, 0) = \phi, C(\tau, \nu, 1) = \phi$ and, by Theorem 3.1, $\nu \leqslant \tau$. Hence $\nu \leqslant \mu$. This completes the proof of (1).

Proof of (2): $(\Longleftarrow)$ follows from Lemma 3.3. The proof of $(\Longrightarrow)$ is similar to that of (1). $\square$

4. Separation Theorem

In this section we show the existence of $F, G : Z^1 \times \cdots \times Z^n \longrightarrow \boldsymbol{R}$ such that $\tau(F) < \min_{\tau' \nleqslant \tau} \tau'(F)$, $\max_{\tau \nleqslant \tau'} \tau'(G) < \tau(G)$ for each $\tau \in \Sigma$ when $\sharp Z^1, \dots, \sharp Z^n$ are sufficiently large.

THEOREM 4.1. (Erdös (1964)). *For all $p, q, r \in \boldsymbol{N}$ there exists $N = N(p,q,r) \in \boldsymbol{N}$ such that if $m \geqslant N$ and $S \subset [1,m]^p, \sharp S \geqslant m^p/r$ then there exist $S_1, \dots, S_p \subset [1,m]$ satisfying $\sharp S_1 = q, \dots, \sharp S_p = q, S_1 \times \cdots \times S_p \subset S$.*

The following theorem is one of the most famous and fundamental results in Ramsey theory.

THEOREM 4.2. (Hales-Jewett (1963)). *For all positive integers m, c there exists an integer k such that whenever $l \geqslant k$*

$$\forall F : [1,m]^l \longrightarrow [1,c] \ \exists J \subsetneq [1,l] \ \exists f : J \longrightarrow [1,m] \ \sharp F(L(f)) = 1$$

where

$$L(f) = \left\{ (i_1, \dots, i_l) \in [1,m]^l \mid \prod_{j \in J} \{i_j\} = \prod_{j \in J} \{f(j)\}, \prod_{j \notin J} \{i_j\} \in \bigcup_{i \in [1,m]} \{i\}^{l - \sharp J} \right\}.$$

Let $HJ(m,c)$ be the least such integer.

DEFINITION 4.1. For positive integers $p, N \geqslant 2$ we define $X_j^i = X_j^i(p,N)$ $(0 \leqslant j \leqslant i-1,\ 1 \leqslant i \leqslant p)$ as follows. Define m_j^p and X_j^p $(1 \leqslant j \leqslant p-1)$ by downward induction on j by $X_{p-1}^p = [1, N^2], m_j^p = HJ(\sharp X_j^p, N)$ and $X_{j-1}^p = (X_j^p)^{m_j^p}$. We set $X^p = X_0^p$. Define m_j^i and X_j^i $(1 \leqslant j \leqslant i-1,\ 1 \leqslant i \leqslant p-1)$ by downward induction on i and j by

$$X_{i-1}^i = \left[1, N \cdot \prod_{k=i+1}^{p} \prod_{j=1}^{i} \left\{ (\sharp X_j^k + 1)^{m_j^k} - (\sharp X_j^k)^{m_j^k} \right\} \right],$$
$$m_j^i = HJ\left(\sharp X_j^i, N^{\sharp X_{j-1}^{i+1} \cdots \sharp X_{j-1}^p}\right) \text{ and } X_{j-1}^i = (X_j^i)^{m_j^i}.$$

We set $X^i = X_0^i$ and $M(N,p) = \sharp X^1$. $\square$

If $1 \leqslant j \leqslant k \leqslant i-1 \leqslant p$ then we can write $w \in X_{j-1}^i$ as

$$w = \left(w_{(s_j, s_{j+1}, \dots, s_k)}\right)_{1 \leqslant s_j \leqslant m_j^i,\ 1 \leqslant s_{j+1} \leqslant m_{j+1}^i,\ \dots,\ 1 \leqslant s_k \leqslant m_k^i}$$

or

$$w = \left(w_{s_j}\right)_{1 \leqslant s_j \leqslant m_j^i} = \left(w_1, w_2, \dots, w_{m_j^i}\right)$$

where $w_{(s_j, s_{j+1}, \dots, s_k)} \in X_k^i$ $(1 \leqslant s_j \leqslant m_j^i,\ 1 \leqslant s_{j+1} \leqslant m_{j+1}^i,\ \dots,\ 1 \leqslant s_k \leqslant m_k^i)$ and $w_{s_j} \in X_j^i$ $(1 \leqslant s_j \leqslant m_j^i)$.

In the following definitions and lemmas let $p \in [2, [n/2]]$ be fixed. Set

$$\mathfrak{F}_j^i = \bigcup_{J \subsetneq [1, m_j^i]} {}^J X_j^i \quad (2 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant i-1),$$
$$\mathfrak{C}(f) = [1, m_j^i] \setminus \mathfrak{D}(f) \quad (f \in \mathfrak{F}_j^i).$$

DEFINITION 4.2. For $j \in [1, i-1], i \in [2, p]$ and $f_k \in \mathfrak{F}_k^i$ $(1 \leqslant k \leqslant j)$ we set

$$L(f_1, \dots, f_j) = \Bigg\{ x^i \in X^i \mid \prod_{(s_1, \dots, s_j) \in \mathfrak{C}(f_1) \times \cdots \times \mathfrak{C}(f_j)} \{x_{(s_1, \dots, s_j)}^i\} \in \bigcup_{w \in X_j^i} \{w\}^{\sharp \mathfrak{C}(f_1) \cdots \sharp \mathfrak{C}(f_j)},$$
$$x_{(s_1, \dots, s_k)}^i = f_k(s_k) \quad \left((s_1, \dots, s_k) \in \mathfrak{C}(f_1) \times \cdots \times \mathfrak{C}(f_{k-1}) \times \mathfrak{D}(f_k) \quad (1 \leqslant k \leqslant j)\right) \Bigg\},$$
$$I\big(L(f_1, \dots, f_j)\big) = \mathfrak{C}(f_1) \times \cdots \times \mathfrak{C}(f_j),$$
$$J\big(L(f_1, \dots, f_j)\big) = \left([1, m_1^i] \times \cdots \times [1, m_j^i]\right) \setminus I\big(L(f_1, \dots, f_j)\big) \quad \square$$

DEFINITION 4.3. We set

$$\mathfrak{L}^j(X^i) = \{L(f_1, \dots, f_j) \subset X^i \mid f_k \in \mathfrak{F}_k^i \quad (1 \leqslant k \leqslant j)\},$$

$$\mathfrak{L}^j(X^{j+1} \times \cdots \times X^p)$$

$$= \{L_j^{j+1} \times \cdots \times L_j^p \subset X^{j+1} \times \cdots \times X^p \mid L_j^{j+1} \in \mathfrak{L}^j(X^{j+1}), \dots, L_j^p \in \mathfrak{L}^j(X^p)\},$$

$$\mathfrak{L}(X_{j-1}^i) = \Bigg\{ \bigg\{ \Big(w_1, \dots, w_{m_j^i}\Big) \in X_{j-1}^i \mid w_k = f(k) \quad (k \in \mathfrak{D}(f)),$$

$$\prod_{k \in \mathfrak{C}(f)} \{w_k\} \in \bigcup_{w \in X_j^i} \{w\}^{\sharp \mathfrak{C}(f)} \bigg\} \subset X_{j-1}^i \mid f \in \mathfrak{F}_j^i \Bigg\} \quad (1 \leqslant j \leqslant i-1,\ 2 \leqslant i \leqslant p).$$

For $f_1 \in \mathfrak{F}_1^i, \dots, f_j \in \mathfrak{F}_j^i$ $(1 \leqslant j \leqslant i-2,\ 2 \leqslant i \leqslant p)$ we set

$$\mathfrak{L}\big(L(f_1, \dots, f_j)\big) = \{L(f_1, \dots, f_j, f_{j+1}) \subset L(f_1, \dots, f_j) \mid f_{j+1} \in \mathfrak{F}_{j+1}^i\}. \quad \square$$

For $L_k^i \in \mathfrak{L}^k(X^i)$ $(2 \leqslant i \leqslant p,\ 1 \leqslant k \leqslant i-1)$ let $\lambda_{L_k^i}$ be a bijection from X_k^i onto L_k^i such that for all $w \in X_k^i$

$$x^i = \lambda_{L_k^i}(w) \Longleftrightarrow x^i_{(s_1, \dots, s_k)} = w \qquad \big((s_1, \dots, s_k) \in I(L_k^i)\big).$$

For $L_k^j \in \mathfrak{L}^k(X^j), \dots, L_k^i \in \mathfrak{L}^k(X^i)$ $(1 \leqslant k \leqslant p-1,\ k+1 \leqslant j \leqslant p,\ j \leqslant i \leqslant p)$ let $\lambda_{L_k^j \times \cdots \times L_k^i}$ be a bijection from $X_k^j \times \cdots \times X_k^i$ onto $L_k^j \times \cdots \times L_k^i$ such that for all $(w^j, \dots, w^i) \in X_k^j \times \cdots \times X_k^i$

$$\lambda_{L_k^j \times \cdots \times L_k^i}(w^j, \dots, w^i) = \big(\lambda_{L_k^j}(w^j), \dots, \lambda_{L_k^i}(w^i)\big).$$

Note that

$$\sharp \mathfrak{F}_j^i = \sum_{r=0}^{m_j^i - 1} \binom{m_j^i}{r} \cdot (\sharp X_j^i)^r = (\sharp X_j^i + 1)^{m_j^i} - (\sharp X_j^i)^{m_j^i} \quad (2 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant i-1)$$

so

$$\sharp \mathfrak{L}^k(X^i) = \prod_{j=1}^{k} \left\{ (\sharp X_j^i + 1)^{m_j^i} - (\sharp X_j^i)^{m_j^i} \right\} \quad (1 \leqslant k \leqslant p-1,\ k+1 \leqslant i \leqslant p).$$

For each $k \in [1, p-1]$ let α^k be a fixed bijection from $\mathfrak{L}^k(X^{k+1} \times \cdots \times X^p) \times [1, N]$ onto X_{k-1}^k and let α^p be a fixed bijection from $[1, N]^2$ onto X_{p-1}^p. We make the following three definitions depending on these $\alpha^1, \alpha^2, \dots, \alpha^p$.

DEFINITION 4.4. (1) We define

$$Q^k : X^1 \times \cdots \times X^k \longrightarrow \mathfrak{L}^k(X^{k+1} \times \cdots \times X^p) \cup \{\phi\} \quad (1 \leqslant k \leqslant p-1)$$

by induction on k as follows. For $a^1 \in \alpha^1\big(\{L_1^2 \times \cdots \times L_1^p\} \times [1, N]\big)$ where $L_1^j \in \mathfrak{L}(X^j)$ $(2 \leqslant j \leqslant p)$, $Q^1(a^1) = L_1^2 \times \cdots \times L_1^p$. For $k \in [2, p-1]$ if

$$Q^{k-1}(a^1, \dots, a^{k-1}) = L_{k-1}^k \times \cdots \times L_{k-1}^p \in \mathfrak{L}^{k-1}(X^k \times \cdots \times X^p)$$

and $a^k \in \lambda_{L^k_{k-1}} \circ \alpha^k(\{L^{k+1}_k \times \cdots \times L^p_k\} \times [1,N])$ where $L^j_k \in \mathfrak{L}(L^j_{k-1})$ $(k+1 \leqslant j \leqslant p)$ then $Q^k(a^1,\dots,a^k) = L^{k+1}_k \times \cdots \times L^p_k$. Otherwise $Q^k(a^1,\dots,a^k) = \phi$.

(2) We define

$$Q^p : X^1 \times \cdots \times X^p \longrightarrow [1,N] \cup \{\phi\}$$

as follows. If $a^1 = \alpha^1\big(Q^1(a^1), b^1\big)$ for some $b^1 \in [1,N]$ and $a^p \in \lambda_{Q^{p-1}(a^1,\dots,a^{p-1})} \circ \alpha^p(\{b^1\} \times [1,N])$ then $Q^p(a^1,\dots,a^p) = b^1$. Otherwise $Q^p(a^1,\dots,a^p) = \phi$.

(3) For $k \in [1,p-1], (a^1,\dots,a^k) \in X^1 \times \cdots \times X^k$ if

$$Q^k(a^1,\dots,a^k) = L^{k+1}_k \times \cdots \times L^p_k \in \mathfrak{L}^k(X^{k+1} \times \cdots \times X^p),$$

let $Q^k_1(a^1,\dots,a^k) = L^{k+1}_k$. If $Q^k(a^1,\dots,a^k) = \phi$, let $Q^k_1(a^1,\dots,a^k) = \phi$. $\square$

DEFINITION 4.5. We define $\beta^1 : X^1 \longrightarrow [1,N]$ by $\beta^1(a^1) = b^1$ if $a^1 \in X^1$ and $b^1 \in [1,N]$ are such that $a^1 = \alpha^1\big(Q^1(a^1), b^1\big)$. For $k \in [2,p]$ we define $\beta^k : X^1 \times \cdots \times X^k \longrightarrow [0,N]$ by $\beta^k(a^1,\dots,a^k) = b^k$ if $a^1 \in X^1,\dots,a^k \in X^k$ and $b^k \in [1,N]$ are such that $a^k = \lambda_{Q^{k-1}_1(a^1,\dots,a^{k-1})} \circ \alpha^k\big(Q^k(a^1,\dots,a^k), b^k\big)$ and $\beta^k(a^1,\dots,a^k) = 0$ otherwise. $\square$

DEFINITION 4.6.

$$T = T(p,N) = \Big\{(x^1,\dots,x^p,y^1,\dots,y^p) \in X^1 \times \cdots \times X^p \times [1,N]^p \mid \\ y^k = \beta^k(x^1,\dots,x^k),\ 1 \leqslant k \leqslant p\Big\}. \quad \square$$

LEMMA 4.3. *For each $k \in [1,p]$ let $\varphi^k : [1,N]^{p-1} \longrightarrow X^k$ and let d^k_1, d^k_2 be distinct elements of $[1,N]$. Then*

$$\Big\{\big(\varphi^1(y^1,y^3,\dots,y^p),\dots,\varphi^{p-1}(y^1,\dots,y^{p-1}),\varphi^p(y^2,\dots,y^p),y^1,\dots,y^p\big) \\ \in X^1 \times \cdots \times X^p \times [1,N]^p \mid y^k \in \{d^k_1,d^k_2\},\ 1 \leqslant k \leqslant p\Big\} \not\subset T.$$

PROOF. Assume to obtain a contradiction that

$$\Big\{\big(\varphi^1(y^1,y^3,\dots,y^p),\dots,\varphi^{p-1}(y^1,\dots,y^{p-1}),\varphi^p(y^2,\dots,y^p),y^1,\dots,y^p\big) \\ \in X^1 \times \cdots \times X^p \times [1,N]^p \mid y^k \in \{d^k_1,d^k_2\},\ 1 \leqslant k \leqslant p\Big\} \subset T.$$

Then by induction we show that

$$
\begin{aligned}
&\forall y^1 \in \{d_1^1, d_2^1\} \cdots \forall y^k \in \{d_1^k, d_2^k\}\ \forall y^{k+2} \in \{d_1^{k+2}, d_2^{k+2}\} \cdots \forall y^p \in \{d_1^p, d_2^p\} \\
&Q^k\Big(\varphi^1(y^1, y^3, \dots, y^k, d_1^{k+1}, y^{k+2}, \dots, y^p), \dots, \\
&\qquad \varphi^{k-1}(y^1, \dots, y^{k-1}, d_1^{k+1}, y^{k+2}, \dots, y^p), \varphi^k(y^1, \dots, y^k, y^{k+2}, \dots, y^p)\Big) \\
&\quad = Q^k\Big(\varphi^1(y^1, y^3, \dots, y^k, d_2^{k+1}, y^{k+2}, \dots, y^p), \dots, \\
&\qquad\quad \varphi^{k-1}(y^1, \dots, y^{k-1}, d_2^{k+1}, y^{k+2}, \dots, y^p), \varphi^k(y^1, \dots, y^k, y^{k+2}, \dots, y^p)\Big) \\
&\quad \neq \phi \quad (1 \leqslant k \leqslant p-1), \\
&\forall y^2 \in \{d_1^2, d_2^2\} \cdots \forall y^p \in \{d_1^p, d_2^p\} \\
&Q^p\big(\varphi^1(d_1^1, y^3, \dots, y^p), \dots, \varphi^{p-1}(d_1^1, y^2, \dots, y^{p-1}), \varphi^p(y^2, \dots, y^p)\big) \\
&\quad = Q^p\big(\varphi^1(d_2^1, y^3, \dots, y^p), \dots, \varphi^{p-1}(d_2^1, y^2, \dots, y^{p-1}), \varphi^p(y^2, \dots, y^p)\big) \neq \phi.
\end{aligned}
$$

The case $k = 1$ is obvious. We write

$$
\begin{aligned}
&\boldsymbol{\varphi}^k(y^1, y^2, \dots, y^p) \\
&\quad = \big(\varphi^1(y^1, y^3, \dots, y^p), \varphi^2(y^1, y^2, y^4, \dots, y^p), \dots, \varphi^k(y^1, \dots, y^k, y^{k+2}, \dots, y^p)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqslant k \leqslant p-1).
\end{aligned}
$$

Suppose that $2 \leqslant k \leqslant p-1$ and

$$
\begin{aligned}
&Q^{k-1}\big(\boldsymbol{\varphi}^{k-1}(y^1, \dots, y^{k-1}, d_1^k, y^{k+1}, \dots, y^p)\big) \\
&\quad = Q^{k-1}\big(\boldsymbol{\varphi}^{k-1}(y^1, \dots, y^{k-1}, d_2^k, y^{k+1}, \dots, y^p)\big) \neq \phi
\end{aligned}
$$

for each $y^j \in \{d_1^j, d_2^j\}$ ($j \in [1,p] \setminus \{k\}$). Then, since the assumption implies that

$$
d_j^k = \beta^k\big(\boldsymbol{\varphi}^k(y^1, \dots, y^{k-1}, d_j^k, y^{k+1}, \dots, y^p)\big) \neq 0 \quad (j = 1, 2),
$$

we have

$$
\begin{aligned}
\varphi^k(y^1, \dots, y^{k-1}, d_j^k, y^{k+2}, \dots, y^p) &\in Q_1^{k-1}\big(\boldsymbol{\varphi}^{k-1}(y^1, \dots, y^{k-1}, d_1^k, y^{k+1}, \dots, y^p)\big) \\
&= Q_1^{k-1}\big(\boldsymbol{\varphi}^{k-1}(y^1, \dots, y^{k-1}, d_2^k, y^{k+1}, \dots, y^p)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi^k(y^1, \dots, y^{k-1}, d_j^k, y^{k+1}, \dots, y^p) \\
&= \lambda_{Q_1^{k-1}\left(\boldsymbol{\varphi}^{k-1}(y^1, \dots, y^{k-1}, d_1^k, y^{k+1}, \dots, y^p)\right)} \circ \alpha^k\big(Q^k(\boldsymbol{\varphi}^k(y^1, \dots, y^{k-1}, d_j^k, y^{k+1}, \dots, y^p)), d_j^k\big) \\
&= \lambda_{Q_1^{k-1}\left(\boldsymbol{\varphi}^{k-1}(y^1, \dots, y^{k-1}, d_2^k, y^{k+1}, \dots, y^p)\right)} \circ \alpha^k\big(Q^k(\boldsymbol{\varphi}^k(y^1, \dots, y^{k-1}, d_j^k, y^{k+1}, \dots, y^p)), d_j^k\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, 2).
\end{aligned}
$$

Therefore, since α^k is a bijection,

$$
\varphi^k(y^1, \dots, y^{k-1}, d_1^k, y^{k+2}, \dots, y^p) \neq \varphi^k(y^1, \dots, y^{k-1}, d_2^k, y^{k+2}, \dots, y^p)
$$

and since y^{k+1} was arbitrary,

$$\begin{aligned}&\varphi^k(y^1,\dots,y^{k-1},d_1^k,y^{k+2},\dots,y^p),\varphi^k(y^1,\dots,y^{k-1},d_2^k,y^{k+2},\dots,y^p)\\&\in Q_1^{k-1}\big(\varphi^{k-1}(y^1,\dots,y^{k-1},d_j^k,d_1^{k+1},y^{k+2},\dots,y^p)\big)\\&\cap Q_1^{k-1}\big(\varphi^{k-1}(y^1,\dots,y^{k-1},d_j^k,d_2^{k+1},y^{k+2},\dots,y^p)\big)\quad (j=1,2).\end{aligned}$$

Then for each $y^j\in\{d_1^j,d_2^j\}$ $(j\in[1,p]\setminus\{k+1\})$

$$\sharp\Big\{Q_1^{k-1}\big(\varphi^{k-1}(y^1,\dots,y^k,d_1^{k+1},y^{k+2},\dots,y^p)\big)\\ \cap Q_1^{k-1}\big(\varphi^{k-1}(y^1,\dots,y^k,d_2^{k+1},y^{k+2},\dots,y^p)\big)\Big\}\geqslant 2.$$

Note that if $L,L'\in\mathfrak{L}^{k-1}(X^k)$ then $\sharp(L\cap L')\geqslant 2$ implies $L=L'$. Hence

$$\begin{aligned}\varphi^k(y^1,\dots,y^k,y^{k+2},\dots,y^p)&\in Q_1^{k-1}\big(\varphi^{k-1}(y^1,\dots,y^k,d_1^{k+1},y^{k+2},\dots,y^p)\big)\\&=Q_1^{k-1}\big(\varphi^{k-1}(y^1,\dots,y^k,d_2^{k+1},y^{k+2},\dots,y^p)\big).\end{aligned}$$

Then, since the assumption implies that

$$\begin{aligned}&\varphi^k(y^1,\dots,y^k,y^{k+2},\dots,y^p)\\&=\lambda_{Q_1^{k-1}\left(\varphi^{k-1}(y^1,\dots,y^k,d_1^{k+1},y^{k+2},\dots,y^p)\right)}\circ\alpha^k\big(Q^k(\varphi^k(y^1,\dots,y^k,d_j^{k+1},y^{k+2},\dots,y^p)),y^k\big)\\&=\lambda_{Q_1^{k-1}\left(\varphi^{k-1}(y^1,\dots,y^k,d_2^{k+1},y^{k+2},\dots,y^p)\right)}\circ\alpha^k\big(Q^k(\varphi^k(y^1,\dots,y^k,d_j^{k+1},y^{k+2},\dots,y^p)),y^k\big)\\&\qquad (j=1,2).\end{aligned}$$

we have

$$Q^k\big(\varphi^k(y^1,\dots,y^k,d_1^{k+1},y^{k+2},\dots,y^p)\big)=Q^k\big(\varphi^k(y^1,\dots,y^k,d_2^{k+1},y^{k+2},\dots,y^p)\big)\neq\phi.$$

Similarly, from the $k=p-1$ case, it follows that

$$\begin{aligned}&Q^p\big(\varphi^1(d_1^1,d_1^3,\dots,d_1^p),\dots,\varphi^{p-1}(d_1^1,\dots,d_1^{p-1}),\varphi^p(d_1^2,\dots,d_1^p)\big)\\&=Q^p\big(\varphi^1(d_2^1,d_1^3,\dots,d_1^p),\dots,\varphi^{p-1}(d_2^1,\dots,d_1^{p-1}),\varphi^p(d_1^2,\dots,d_1^p)\big)\neq\phi.\end{aligned}$$

Hence $\beta^1\big(\varphi^1(d_1^1,d_1^3,\dots,d_1^p)\big)=\beta^1\big(\varphi^1(d_2^1,d_1^3,\dots,d_1^p)\big)$. Then, since the assumption implies that $d_1^1=\beta^1\big(\varphi^1(d_1^1,d_1^3,\dots,d_1^p)\big)$ and $d_2^1=\beta^1\big(\varphi^1(d_2^1,d_1^3,\dots,d_1^p)\big)$, we have $d_1^1=d_2^1$ which contradicts $d_1^1\neq d_2^1$. □

LEMMA 4.4. *For $k\in[1,p]$ let $\Phi^k={}^{(X^1\times\cdots\times X^{k-1}\times X^{k+1}\times\cdots\times X^p)}[1,N]$. Then*

$$\forall\varphi^1\in\Phi^1\cdots\forall\varphi^p\in\Phi^p\ \exists x^1\cdots\exists x^p\in X^p\\ \big(x^1,\dots,x^p,\varphi^1(x^2,\dots,x^p),\dots,\varphi^p(x^1,\dots,x^{p-1})\big)\in T.$$

PROOF. Let $\varphi^k\in\Phi^k$ $(1\leqslant k\leqslant p)$. Using the theorem of Hales-Jewett we will find $a^1,a^2,\dots,a^p$ inductively so that

$$\big(a^1,\dots,a^p,\varphi^1(a^2,\dots,a^p),\dots,\varphi^p(a^1,\dots,a^{p-1})\big)\in T.$$

Since $m_1^2 = HJ\left(\sharp X_1^2, N^{\sharp X^3 \cdot \dots \cdot \sharp X^p}\right)$ and $X^2 = (X_1^2)^{m_1^2}$ there exists $L_1^2 \in \mathfrak{L}(X^2)$ such that

$$\forall x^3 \in X^3 \cdots \forall x^p \in X^p \ \sharp\{\varphi^1(x^2, \dots, x^p) \in [1, N] \mid x^2 \in L_1^2\} = 1.$$

Let $3 \leqslant i \leqslant p$ and $L_1^j \in \mathfrak{L}(X^j)$ $(2 \leqslant j \leqslant i-1)$. Suppose that

$$\forall x^i \in X^i \cdots \forall x^p \in X^p \\ \sharp\{\varphi^1(x^2, \dots, x^p) \in [1, N] \mid (x^2, \dots, x^{i-1}) \in L_1^2 \times \cdots \times L_1^{i-1}\} = 1.$$

Then, since $m_1^i = HJ\left(\sharp X_1^i, N^{\sharp X^{i+1} \cdot \dots \cdot \sharp X^p}\right)$ and $X^i = (X_1^i)^{m_1^i}$, there exists $L_1^i \in \mathfrak{L}(X^i)$ such that

$$\forall x^{i+1} \in X^{i+1} \cdots \forall x^p \in X^p \\ \sharp\{\varphi^1(x^2, \dots, x^p) \in [1, N] \mid (x^2, \dots, x^i) \in L_1^2 \times \cdots \times L_1^i\} = 1.$$

By induction there exist $L_1^2 \in \mathfrak{L}(X^2), \dots, L_1^p \in \mathfrak{L}(X^p)$ such that $\sharp\varphi^1(L_1^2 \times \cdots \times L_1^p) = 1$. Let $b^1 \in \varphi^1(L_1^2 \times \cdots \times L_1^p)$ and $a^1 = \alpha^1(L_1^2 \times \cdots \times L_1^p, b^1)$. Then we have $\beta^1(a^1) = b^1$ and $Q^1(a^1) = L_1^2 \times \cdots \times L_1^p$.

If $2 \leqslant k \leqslant p-1$ and if $a^1 \in X^1, \dots, a^{k-1} \in X^{k-1}$ and $b^1, \dots, b^{k-1} \in [1, N]$ are defined so that for each $j \in [1, k-1]$ there exist $L_j^{j+1} \in \mathfrak{L}(L_{j-1}^{j+1}), \dots, L_j^p \in \mathfrak{L}(L_{j-1}^p)$ (where $L_0^2 = X^2, \dots, L_0^p = X^p$) such that

$$Q^j(a^1, \dots, a^j) = L_j^{j+1} \times \cdots \times L_j^p,$$
$$\beta^j(a^1, \dots, a^j) = b^j = \varphi^j(a^1, \dots, a^{j-1}, x^{j+1}, \dots, x^p) \quad \left((x^{j+1}, \dots, x^p) \in Q^j(a^1, \dots, a^j)\right)$$

then we define

$$\varphi^k_{(a^1, \dots, a^{k-1})} : L_{k-1}^{k+1} \times \cdots \times L_{k-1}^p \longrightarrow [1, N]$$

by

$$\varphi^k_{(a^1, \dots, a^{k-1})}(x^{k+1}, \dots, x^p) = \varphi^k(a^1, \dots, a^{k-1}, x^{k+1}, \dots, x^p).$$

Then, since $\varphi^k_{(a^1, \dots, a^{k-1})} \circ \lambda_{L_{k-1}^{k+1} \times \cdots \times L_{k-1}^p} : X_{k-1}^{k+1} \times \cdots \times X_{k-1}^p \longrightarrow [1, N]$, similarly as in the case of φ^1, there exist $U^{k+1} \in \mathfrak{L}(X_{k-1}^{k+1}), U^{k+2} \in \mathfrak{L}(X_{k-1}^{k+2}), \dots, U^p \in \mathfrak{L}(X_{k-1}^p)$ such that

$$\sharp\varphi^k_{(a^1, \dots, a^{k-1})} \circ \lambda_{L_{k-1}^{k+1} \times \cdots \times L_{k-1}^p}(U^{k+1} \times U^{k+2} \times \cdots \times U^p) = 1.$$

Let $L_k^i = \lambda_{L_{k-1}^i}(U^i)$ $(i \in [k+1, p])$, let $b^k \in \varphi^k_{(a^1, \dots, a^{k-1})}(L_k^{k+1} \times \cdots \times L_k^p)$ and let

$$a^k = \lambda_{L_{k-1}^k} \circ \alpha^k(L_k^{k+1} \times \cdots \times L_k^p, b^k).$$

Then $Q^k(a^1, \dots, a^k) = L_k^{k+1} \times \cdots \times L_k^p$ and

$$\beta^k(a^1, \dots, a^k) = b^k = \varphi^k(a^1, \dots, a^{k-1}, x^{k+1}, \dots, x^p) \\ \left((x^{k+1}, \dots, x^p) \in Q^k(a^1, \dots, a^k)\right).$$

Thus, inductively, we can find $a^1 \in X^1, \dots, a^{p-1} \in X^{p-1}$ such that for all $k \in [1, p-1]$

$$\beta^k(a^1,\dots,a^k) = \varphi^k(a^1,\dots,a^{k-1},x^{k+1},\dots,x^p) \quad \big((x^{k+1},\dots,x^p) \in Q^k(a^1,\dots,a^k)\big).$$

Let

$$a^p = \lambda_{Q^{p-1}(a^1,\dots,a^{p-1})} \circ \alpha^p\big(\beta^1(a^1), \varphi^p(a^1,\dots,a^{p-1})\big).$$

Then $\varphi^p(a^1,\dots,a^{p-1}) = \beta^p(a^1,\dots,a^p)$ is clear by the definition of β^p and for $k \leqslant p-1$ $\varphi^k(a^1,\dots,a^{k-1},a^{k+1},\dots,a^p) = \beta^k(a^1,\dots,a^k)$ follows from $(a^{k+1},\dots,a^p) \in Q^k(a^1,\dots,a^k)$ and the above result. Hence

$$\begin{aligned}&\big(a^1,\dots,a^p,\varphi^1(a^2,\dots,a^p),\dots,\varphi^p(a^1,\dots,a^{p-1})\big)\\ &\quad = \big(a^1,\dots,a^p,\beta^1(a^1),\dots,\beta^p(a^1,\dots,a^p)\big) \in T. \quad \square\end{aligned}$$

PROPOSITION 4.5. *If* $\sharp Z^k > 2^{\sharp\Sigma}$ $(1 \leqslant k \leqslant n)$, *then*

$$\forall \mu \in \Sigma_- \ \exists F \in^{(Z^1\times\cdots\times Z^n)} \boldsymbol{R} \quad \mu(F) < \min_{\tau \not\leqslant \mu} \tau(F).$$

PROOF. We may assume that $\{0,1\}^{\sharp\Sigma} \subset Z^k$ $(1 \leqslant k \leqslant n)$. Let

$$E_i^s = \big\{(e_1,\dots,e_{\sharp\Sigma}) \in \{0,1\}^{\sharp\Sigma} \mid e_s = i\big\} \quad (1 \leqslant s \leqslant \sharp\Sigma,\ i = 0,1).$$

Suppose that $\mu = \sigma_-(l,\pi,\gamma)$ and $\{\tau \in \Sigma \mid \tau \not\leqslant \mu\} = \{\tau_1,\dots,\tau_t\}$.

For $s \in [1,t]$ if $C(\mu,\tau_s,0) \neq \phi$ then we take $k \in C(\mu,\tau_s,0)$ and define a function F^s by

$$F^s(z^1,\dots,z^n) = \begin{cases} 0 & (z^k \in E_0^s),\\ 1 & (z^k \notin E_0^s).\end{cases}$$

If $C(\mu,\tau_s,0) = \phi$ then, since $\bigcup_{i=1}^{[n/2]} C(\mu,\tau_s,i) \neq \phi$, we take $p_s \in \big[1,[n/2]\big]$ and $\eta_s \in C(\mu,\tau_s,p_s)$ and define F^s by

$$F^s(z^1,\dots,z^n) = \begin{cases} 1 & \big(z^{\eta_s(k)} \in E_0^s \Longleftrightarrow z^{\eta_s(p_s+k)} \in E_0^s \quad (1 \leqslant k \leqslant p_s)\big),\\ 0 & (\text{otherwise}).\end{cases}$$

Let $F : Z^1\times\cdots\times Z^n \longrightarrow \boldsymbol{R}$ be defined by $F(z^1,\dots,z^n) = \max_{1\leqslant s\leqslant t} F^s(z^1,\dots,z^n)$ and let

$$\varphi^k : Z^{\gamma_k(1)} \times \cdots \times Z^{\gamma_k(h_k)} \longrightarrow Z^{\pi(k)} \quad (1 \leqslant k \leqslant l)$$

be such that

$$\varphi^k(z^{\gamma_k(1)},\dots,z^{\gamma_k(h_k)}) \in \left(\bigcap_{s\in U_k} E_0^s\right) \cap \left(\bigcap_{s\in[1,\sharp\Sigma]\setminus U_k} E_1^s\right)$$

where

$$\begin{aligned}U_k &= U_k(z^{\gamma_k(1)}, \dots, z^{\gamma_k(h_k)}) \\ &= \{s \in [1, \sharp\Sigma] \mid C(\mu, \tau_s, 0) \neq \phi\} \cup \{s \in [1, \sharp\Sigma] \mid \pi(k) = \eta_s(p_s + 1), z^{\eta_s(p_s)} \notin E_0^s\} \\ &\cup \left(\bigcup_{j=2}^{p_s} \{s \in [1, \sharp\Sigma] \mid \pi(k) = \eta_s(p_s + j), z^{\eta_s(j-1)} \in E_0^s\} \right)\end{aligned}$$

and $\mathfrak{D}(\gamma_k) = [1, h_k]$. Then

$$\forall z^{\pi(l+1)} \in Z^{\pi(l+1)} \dots \forall z^{\pi(n)} \in Z^{\pi(n)} \ (\gamma * \pi * F)(\varphi^1, \dots, \varphi^l, z^{\pi(l+1)}, \dots, z^{\pi(n)}) = 0$$

so $\mu(F) = 0$. Since $\tau_s(F) \geqslant \tau_s(F^s) = 1$ $(1 \leqslant s \leqslant t)$ is clear we have $\mu(F) < \min_{\tau \not\leqslant \mu} \tau(F)$. □

PROPOSITION 4.6. *If* $\sharp Z^k \geqslant M(N([n/2], 2, \sharp\Sigma), [n/2])$ $(1 \leqslant k \leqslant n)$, *then*

$$\forall \nu \in \Sigma_+ \ \exists F \in^{(Z^1 \times \dots \times Z^n)} \boldsymbol{R} \ \ \nu(F) < \min_{\tau \not\leqslant \nu} \tau(F).$$

PROOF. Let $N = N([n/2], 2, \sharp\Sigma)$. We may assume that $[1, N] \subset Z^k$ $(1 \leqslant k \leqslant n)$. For $p \in [2, [n/2]]$, $\eta \in \Pi_{2p}$, $k \in [1, p]$ let $\theta_k^\eta : X^k(p, N) \longrightarrow Z^{\eta(k)}$ be an injection. Define

$$\theta^\eta : X^1(p, N) \times \dots \times X^p(p, N) \times [1, N]^p \longrightarrow Z^{\eta(1)} \times \dots \times Z^{\eta(p)} \times [1, N]^p$$

by

$$\theta^\eta(x^1, \dots, x^p, y^1, \dots, y^p) = (\theta_1^\eta(x^1), \dots, \theta_p^\eta(x^p), y^1, \dots, y^p).$$

Suppose that $\nu = \sigma_+(l, \pi, \gamma)$ and $\tau \not\leqslant \nu$.

If $C(\nu, \tau, 0) \neq \phi$ then we take $k \in C(\nu, \tau, 0)$ and define F^τ by

$$F^\tau(z^1, \dots, z^n) = \begin{cases} 0 & (z^k \in [1, N]), \\ 1 & (z^k \notin [1, N]). \end{cases}$$

Then

$$\begin{aligned}&\forall \varphi^1 \in \Phi(\pi(1), \gamma_1) \cdots \forall \varphi^l \in \Phi(\pi(l), \gamma_l) \\ &\sharp\{(z^{\pi(l+1)}, \dots, z^{\pi(n)}) \in [1, N]^{n-l} \mid (\gamma * \pi * F^\tau)(\varphi^1, \dots, \varphi^l, z^{\pi(l+1)}, \dots, z^{\pi(n)}) = 1\} = 0.\end{aligned}$$

Clearly $\tau(F^\tau) = 1$.

If $C(\nu, \tau, 0) = \phi$ and $C(\nu, \tau, 1) \neq \phi$ then we take $\eta \in C(\nu, \tau, 1)$ and define F^τ by

$$F^\tau(z^1, \dots, z^n) = \begin{cases} 0 & \left((z^{\eta(1)}, z^{\eta(2)}) \in (Z^{\eta(1)} \times [1, N]) \setminus \bigcup_{s=1}^{N} \{(s, s)\} \right), \\ 1 & \text{(otherwise)}. \end{cases}$$

Then for $\varphi^1 \in \Phi(\pi(1),\gamma_1),\dots,\varphi^l \in \Phi(\pi(l),\gamma_l)$

$$\begin{aligned}&\sharp\{(z^{\pi(l+1)},\dots,z^{\pi(n)}) \in [1,N]^{n-l} \,|(\gamma * \pi * F^\tau)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}) = 1\}\\ &\leqslant N^{n-l-1} \leqslant \frac{N^{n-l}}{\sharp\Sigma}.\end{aligned}$$

Clearly $\tau(F^\tau) = 1$.

If $C(\nu,\tau,0) = \phi$ and $C(\nu,\tau,1) = \phi$ then, since $\bigcup_{i=2}^{[n/2]} C(\nu,\tau,i) \neq \phi$, we take $p \in [2,[n/2]]$ and $\eta \in C(\nu,\tau,p)$ and define F^τ by

$$\begin{aligned}&F^\tau(z^1,\dots,z^n)\\ &= \begin{cases} 0 & \big((z^{\eta(p+1)},\dots,z^{\eta(2p)}) \in [1,N]^p,\ (z^{\eta(1)},\dots,z^{\eta(2p)}) \notin \theta^\eta(T(p,N))\big),\\ 1 & \text{(otherwise)}.\end{cases}\end{aligned}$$

Then, since $N(p,2,\sharp\Sigma) \leqslant N = N([n/2],2,\sharp\Sigma)$, it follows from the theorem of Erdős and Lemma 4.3 that

$$\begin{aligned}&\forall\varphi^1 \in \Phi(\pi(1),\gamma_1)\cdots\forall\varphi^l \in \Phi(\pi(l),\gamma_l)\\ &\sharp\{(z^{\pi(l+1)},\dots,z^{\pi(n)}) \in [1,N]^{n-l} \,|(\gamma * \pi * F^\tau)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}) = 1\}\\ &\leqslant \frac{N^{n-l}}{\sharp\Sigma}.\end{aligned}$$

Lemma 4.4 implies that $\tau(F^\tau) = 1$.

Define F by $F(z^1,\dots,z^n) = \max\limits_{\tau\not\leqslant\nu} F^\tau(z^1,\dots,z^n)$ $\big((z^1,\dots,z^n) \in Z^1 \times\cdots\times Z^n\big)$.

Then for $\varphi^1 \in \Phi(\pi(1),\gamma_1),\dots,\varphi^l \in \Phi(\pi(l),\gamma_l)$

$$\begin{aligned}&\sharp\{(z^{\pi(l+1)},\dots,z^{\pi(n)}) \in [1,N]^{n-l} \mid (\gamma * \pi * F)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}) = 1\}\\ &\leqslant \sum_{\tau\not\leqslant\nu} \sharp\{(z^{\pi(l+1)},\dots,z^{\pi(n)}) \in [1,N]^{n-l}|(\gamma * \pi * F^\tau)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}) = 1\}\\ &\leqslant \sharp\{\tau \in \Sigma \mid \tau \not\leqslant \nu\} \cdot \frac{N^{n-l}}{\sharp\Sigma} < N^{n-l}.\end{aligned}$$

So

$$\begin{aligned}&\forall\varphi^1 \in \Phi(\pi(1),\gamma_1)\cdots\forall\varphi^l \in \Phi(\pi(l),\gamma_l)\ \exists z^{\pi(l+1)} \in Z^{\pi(l+1)}\cdots\exists z^{\pi(n)} \in Z^{\pi(n)}\\ &\qquad\qquad (\gamma * \pi * F)(\varphi^1,\dots,\varphi^l,z^{\pi(l+1)},\dots,z^{\pi(n)}) = 0.\end{aligned}$$

That is, $\nu(F) = 0$. Clearly $\tau(F) \geqslant \tau(F^\tau) = 1$ $(\tau \not\leqslant \nu)$. Hence $\nu(F) < \min_{\tau\not\leqslant\nu} \tau(F)$. □

By Proposition 4.5 and Proposition 4.6 with the dual results we have the following.

THEOREM 4.7. *Assume that $\sharp Z^1,\dots,\sharp Z^n$ are sufficiently large. Then for all $\tau \in \Sigma$ there exist $F,G : Z^1\times\cdots\times Z^n \longrightarrow \boldsymbol{R}$ so that $\tau(F) < \min_{\tau'\not\leqslant\tau} \tau'(F)$ and $\max_{\tau\not\leqslant\tau'} \tau'(G) < \tau(G)$.*

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