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ON LIMIT CYCLES AND TRANSIENT LENGTHS OF CELLULAR AUTOMATA WITH THRESHOD RULES

By

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Abstract

In this paper, we investigate behaviors of 1-D finite cellular automata with triplet local transitions rules of fixed boundary condition. We make the number of limit cycles and transient length clear, and we show 2 theorems, one is for period length of their limit cycles and the other is for their transient length.

1. Introduction

Cellular automata were introduced by J. von Neumann as theoretical model to demonstrate a system capable of self-reproduction and universal computation in 1950s. In 1980s important roles of cellular automata are rerealized by S. Wolfram and other physicists, and many researchers and engineers have been investigated and applied them. Cellular automata have very simple structure. While cellular automata obey simple transition rules, their behaviors are very complicated. The complication is caused by interaction between cells and is similar to behaviors of complex systems as fractal, chaotic phenomena. The cellular automata have been applied to biology, physics, mathematics, economics, computer science and so on.

Various cellular automata have been analised such as one dimensional and two dimensional cellular automata, cellular automata with cyclic cell array or liner cell array and so on. Wolfram(1986) classified 1-D cellular automata into four complexity classes according to pattern generated by the synchronous dynamics. With respect to linear cellular automata Kawahara(1991,1995) and Lee(1992) analyzed behaviors of cellular automata with rule 60 and 90. With respect to non-linear cellular automata the following are reported. Shingai(1978) stated that period lengths of limit cycles of 1-D cellular automata of null boundary are four or less. Sato(1996a,1996b,1997,1998) and Inokuchi(1996,1998,2000) investigated behaviors of 1-D cellular automata with triplet local transition rules and showed that the numbers of limit cycles and transient lengths obey simple rules like Fibonacci sequence. Furthermore, Lee and Kawahara(1996), and

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Inokuchi and Kawahara(1999) investigated transition diagrams of cellular automata and make an attempt to representing them by algebraic formulas called tree expressions.

However many non-linear cellular automata are not investigated. In order to see characters of non-linear cellular automata we investigate behaviors of simple cellular automata and see a character of a cellular automaton by behaviors of simple cellular automata. In this paper we investigate behaviors of 1-D cellular automata with threshold transition rules of fixed boundary conditions, and prove rules of limit cycles and transient length.

2. Preliminaries

In this section, we define 1-dimensional finite cellular automata and introduce notations used in this paper.

DEFINITION 2.1. The triplet local transition rule f is a function $\{0,1\}^3 \longrightarrow \{0,1\}$, and f is represented as follows;

$$\left| \begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ r_7 & r_6 & r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{array} \right|.$$

Rule number R of f is defined as follows;

$$R = 2^7 r_7 + 2^6 r_6 + \cdots + 2^0 r_0.$$

Usually the triplet local transition rule of rule number R is denoted by rule R for short.

DEFINITION 2.2. Let $I = \{1, 2, \cdots, m\}$ be a set of cells. We call m-dimensional vector space $\{0,1\}^m$ a configuration space. And a configuration is a vector
$x = (x_1, x_2, \cdots, x_m) \in \{0,1\}^m$.

The positive integer m is called cell-size. Usually a vector $(x_1, x_2, \cdots, x_m)$ denotes $x_1 x_2 \cdots x_m$ for short.

DEFINITION 2.3. The global transition function δ is defined as follows;

$$\delta(x_1, x_2, \cdots, x_m) = (f(\alpha, x_1, x_2), f(x_1, x_2, x_3), \cdots, f(x_{m-1}, x_m, \beta))$$

where f is rule R and $\alpha, \beta \in \{0,1\}$.

We call a pair (α, β) a boundary. We say that the boundary condition is cyclic if and only if $\alpha = x_m$ and $\beta = x_1$, and the boundary condition is fixed if and only if α and β are fixed. We say that the boundary condition is a-b if $\alpha = a$ and $\beta = b$.

CA-$R_{\alpha-\beta}(m)$ denotes an one-dimensional cellular automaton such that m cells exist and its transition rule is rule R and its boundary condition is α-β.

DEFINITION 2.4. The local transition rule f is threshold if there exist integers a, b, c, θ such that

$$f(u, v, w) = \begin{cases} 1 & if \;\; au + bv + cw + \theta \geq 0 \\ 0 & otherwise \end{cases}.$$

DEFINITION 2.5. A configuration x is a Garden-of-Eden (GOE, for short) configuration if there exists no configuration y such that $x = \delta(y)$. And a configuration x is a non-GOE configuration if there exists a configuration y such that $x = \delta(y)$.

DEFINITION 2.6. Let x be a configuration of $CA\text{-}R_{\alpha-\beta}(m)$. The configuration x is on a limit cycle of period length T if there exists a positive integer s such that $\delta^s(x) = x$, and $T = min\{s \geq 1 | \delta^s(x) = x\}$. And the configurations $x(1), x(2), \cdots, x(T-1)$ form a limit cycle of period length T if $x(i+1) = \delta(x(i))$ and $x(T) = x(1)$, and $x(i)$ is on a limit cycle of period length T where $1 \leq i \leq T-1$.

A limit cycle of period length T is denoted by a T-cycle, in particular, a limit cycle of period length 1 is denoted by a fixed point. And a number of a limit cycle of period length T is denoted by $\gamma_T(m)$.

DEFINITION 2.7. $h(x)$ is defined as follows;

$$h(x) = min\{s \geq 0 | \delta^s(x) \text{ is on a limit cycle}\}.$$

Then the transient length $H(m)$ of $CA\text{-}R(m)_{\alpha-\beta}$ is defined as follows;

$$H(m) = max\{h(x) | x \in \{0,1\}^m\}.$$

DEFINITION 2.8. The reverse transition rule $\overline{f}$ of a local transition rule f is defined as follows;

$$\overline{f}(x, y, z) = 1 - f(1-x, 1-y, 1-z).$$

DEFINITION 2.9. The symmetric transition rule f^T of a local transition rule f is defined as follows;

$$f^T(x, y, z) = f(z, y, x).$$

Let rule $\overline{R}$ and rule R^T be the reverse rule and symmetric rule of rule R respectively. Then we can identify $CA\text{-}\overline{R}_{\overline{\alpha}-\overline{\beta}(m)}$ and $CA\text{-}R^T_{\beta-\alpha}(m)$ with $CA\text{-}R_{\alpha-\beta}(m)$. So $CA\text{-}R_{\alpha-\beta}(m)$, $CA\text{-}\overline{R}_{\overline{\alpha}-\overline{\beta}}(m)$, $CA\text{-}R^T_{\beta-\alpha}(m)$ and $CA\text{-}\overline{R^T}_{\overline{\beta}-\overline{\alpha}}(m)$ can be identified with each other.

We use the following notations;

Let A be a subsequence.

- A^k : sequence composed of k A's.
- A^*_k : sequence composed of k bits taken from the left edge when some A's are arrayed.
- $*$: an irrelevant bit.
- $\alpha, \beta \in \{0,1\}$.
- $\delta^n(x)$ is denoted by $x(n)$ where $n \geq 0$. In particular $x(0) = x$
- The state of ith cell of a configuration x is denoted by x_i.
- x_0 and x_{m+1} mean the left and right boundary of x. That is, in CA-$R_{\alpha-\beta}(m)$ $x_0 + \alpha$ and $x_m = \beta$.

3. Behaviors of CA with Threshold Rules of Fixed Boundary Conditions

In this section we discuss behaviors of cellular automata (CA, for short) with threshold rules. There are 38 threshold rules except for symmetric, reverse and symmetric reverse rules. They are rule 0, 1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 19, 23, 32, 34, 35, 42, 43, 50, 51, 76, 77, 128, 136, 138, 140, 142, 160, 162, 168, 170, 178, 200, 204 and 232. With respect to behaviors of CA with threshold rules Shingai(1978) found that period lengths of limit cycles of CA with threshold rule of 0-0 boundary condition were bounded by 4. From the results by Inokuchi and Sato(1996) and others we can expect the following theorems.

THEOREM 3.1. *Period lengths of limit cycles of 1-D finite cellular automata with threshold rules of fixed boundary conditions are bounded by 4.*

THEOREM 3.2. *Transient lengths of 1-D finite cellular automata with threshold rules of fixed boundary conditions are* $3 \times$ *(cell-size)* $- 4$ *or less.*

In order to prove these theorems we investigate in the following subsections behaviors of cellular automata with various threshold rules and show rules of limit cycles and transient lengths.

3.1. CA with Rule 0

In this subsection we discuss behaviors of CA with rule 0. The symmetric rule of rule 0 is rule 0 and its reverse, symmetric reverse rule are both rule 255. Rule 0 is the function from $\{0,1\}^3$ to $\{0,1\}$ such that $f(u,v,w) = 0$ for any $u, v, w \in \{0,1\}$. That is, in CA-$0_{*-*}(m)$ for any configuration c

$$\delta(c) = 0^m.$$

So we get easily the following theorem.

THEOREM 3.3. *CA-*$0_{*-*}(m)$ *has a unique fixed point and its transient length is 1.*

3.2. CA with Rule 1

In this subsection we discuss behaviors of CA with rule 1. The symmetric, reverse and symmetric reverse rule of rule 1 are rule 1, rule 127 and rule 127 respectively. The following behaviors are obtained.

LEMMA 3.4. *Let c be an arbitrary configuration. Then the following hold;*

1. *The configuration* x *is on a 2-cycle of CA-*$1_{0-0}(m)$ *if and only if* x *satisfies the following conditions;*
    (a) *Neither* 101 *nor* 1001 *appears in* x.
    (b) $x_1x_2 \neq 01$.

(c) $x_{m-1}x_m \neq 10$.

2. *In* $CA\text{-}1_{0-0}(m)$ *neither* 101 *nor* 1001 *appears in* $c(1)$.

3. *In* $CA\text{-}1_{0-0}(m)$ $c(1)_1c(1)_2 \neq 01$ *and* $c(1)_{m-1}c(1)_m \neq 10$.

4. *In* $CA\text{-}1_{*-1}(m)$ $c(1)_m = 0$.

5. *In* $CA\text{-}1_{1-*}(m)$ $c(1)_1 = 0$.

PROPOSITION 3.5. *All 2-cycles of* $CA\text{-}1_{0-0}(m)$ *can be constructed without redundancy as follows;*

1. *If* $x_1x_2\cdots x_{m-i-1}0$ *and* $y_1y_2\cdots y_{m-i-1}1$ *form a 2-cycle of* $CA\text{-}1_{0-0}(m-i)$, *then* $x_1x_2\cdots x_{m-i-1}001^{i-1}$ *and* $y_1y_2\cdots y_{m-i-1}100^{i-1}$ *form a 2-cycle of* $CA\text{-}1_{0-0}(m)$ *where* $2 \leq i \leq m-1$.

By lemma 3.4 4 and lemma 3.4 5 behaviors of $\text{CA-}1_{0-1}(m)$ and $\text{CA-}1_{1-1}(m)$ after one step transition are regarded as behaviors of $\text{CA-}1_{0-0}(m-1)$ and $\text{CA-}1_{0-0}(m-2)$, respectively. And since rule 1 and its symmetric rule are the same $\text{CA-}1_{1-0}(m)$ and $\text{CA-}1_{0-1}(m)$ are isomorphic. So the following theorem results from lemma 3.4 and prop. 3.5.

THEOREM 3.6. $CA\text{-}1_{*-*}(m)$ *has 2-cycles only and for the number of those 2-cycles the following formula holds;*

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

And the transient lengths of $CA\text{-}1_{0-0}(m)$, $CA\text{-}1_{0-1}(m)$, $CA\text{-}1_{1-0}(m)$ *and* $CA\text{-}1_{1-1}(m)$ *are 1,2,2 and 2 respectively.*

3.3. CA with Rule 2

In this subsection we discuss behaviors of CA with rule 2. The symmetric, reverse and symmetric reverse rule of rule 2 are rule 16, rule 191 and rule 247 respectively. The following behaviors are found.

LEMMA 3.7. *Let* c *be an arbitrary configuration and* x *a configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}2_{*-0}(m)$.

2. *The configurations* $(100)^*_m$, $(001)^*_m$ *and* $(010)^*_m$ *form a 3-cycle of* $CA\text{-}2_{0-1}(m)$.

3. *The configurations* $0(100)^*_{m-1}$, $0(001)^*_{m-1}$ *and* $0(010)^*_{m-1}$ *form a 3-cycle of* $CA\text{-}2_{1-1}(m)$.

4. *In* $CA\text{-}2_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ *where* $1 \leq k \leq m$.

5. *In* $CA\text{-}2_{1-*}(m)$ $c(1)_1 = 0$.

6. *In* $CA\text{-}2_{0-1}(m)$ *if* $x_{m-i+1}x_{m-i+2}\cdots x_m = (001)^*_i$ *then* $x(1)_{i-i}x(1)_{m-i+1}\cdots x(1)_m = (001)^*_{i+1}$ *where* $2 \le i \le m-1$.

7. *In* $CA\text{-}2_{1-1}(m)$ *if* $x_{m-i+1}x_{m-i+2}\cdots x_m = (001)^*_i$ *then* $x(1)_{i-i}x(1)_{m-i+1}\cdots x(1)_m = (001)^*_{i+1}$ *where* $2 \le i \le m-2$.

8. *In* $CA\text{-}2_{*-1}(m)$ *if* $x_{m-1}x_m = 11$ *or* 10 *then* $x(1)_{m-1}x(1)_m = 00 = (001)^*_2$.

9. *In* $CA\text{-}2_{*-1}(m)$ *if* $x_{m-2}x_{m-1}x_m = 101$ *then* $x(1)_{m-1}x(1)_m = 00 = (001)^*_2$.

From lemma 3.7 5 we can regard behaviors of CA-$2_{1-\beta}(m)$ after one step transition as those of CA-$2_{0-\beta}(m-1)$. So the following theorem results from the above lemma.

THEOREM 3.8. *$CA\text{-}2_{*-0}(m)$ has a unique fixed point and $CA\text{-}2_{*-1}(m)$ has a unique 3-cycle. And transient length of $CA\text{-}2_{0-0}(m)$, $CA\text{-}2_{1-0}(m)$, $CA\text{-}2_{0-1}(m)$ and $CA\text{-}2_{1-1}(m)$ are m, $m-1$, $m-1$ and $m-2$, respectively.*

3.4. CA with Rule 3

In this subsection we discuss behaviors of CA with rule 3. The symmetric, reverse and symmetric reverse rule of rule 3 are rule 17, rule 63 and rule 119 respectively. The following behaviors are found.

LEMMA 3.9. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configurations 0^m and 1^m form a 2-cycle of $CA\text{-}3_{0-*}(m)$.*

2. *In* $CA\text{-}3_{0-*}(m)$ *if* $x_1x_2\cdots x_{k-1}c_k = 0^k$ *then* $x(1)_1x(1)_2\cdots x(1)_{k-1}x(1)_k = 1^k$ *where* $1 \le k \le m$.

3. *In* $CA\text{-}3_{0-*}(m)$ *if* $x_1x_2\cdots x_{k-1}x_k = 1^k$ *then* $x(1)_1x(1)_2\cdots x(1)_kx(1)_{k+1} = 0^{k+1}$ *where* $1 \le k \le m-1$.

4. *In* $CA\text{-}3_{1-*}(m)$ $c(1)_1 = 0$.

By lemma 3.9 4 we can regard behaviors of CA-$3_{1-\beta}(m)$ after one step transition as those of CA-$3_{0-\beta}(m-1)$. So the following theorem results from the above lemma.

THEOREM 3.10. *$CA\text{-}3_{*-*}(m)$ has a unique 2-cycle. Transient length of $CA\text{-}3_{0-*}(m)$ is $2m-2$ and that of $CA\text{-}3_{1-*}(m)$ is $2m-3$.*

3.5. CA with Rule 4

In this subsection we discuss behaviors of CA with rule 4. The symmetric, reverse and symmetric reverse rule of rule 4 are rule 4, rule 223 and rule 223 respectively. The following behaviors are obtained.

LEMMA 3.11. *Let c be an arbitrary configuration. Then the following hold;*

1. *The configuration x is a fixed point of CA-$4_{\alpha-\beta}(m)$ if and only if no* 11 *appears in $\alpha x\beta$.*

2. *In CA-$4_{\alpha-\beta}$ no* 11 *appears in $\alpha c(1)\beta$.*

3. *In CA-$4_{*-1}(m)$ $c(1)_m = 0$.*

4. *In CA-$4_{1-*}(m)$ $c(1)_1 = 0$.*

PROPOSITION 3.12. *All fixed points of CA-$4_{0-0}(m)$ can be constructed without redundancy as follows;*

1. *If $x_2x_3\cdots x_m$ is a fixed point of CA-$4_{0-0}(m-1)$, then $0x_2x_3\cdots x_m$ is a fixed point of CA-$4_{0-0}(m)$.*

2. *If $x_3x_4\cdots x_m$ is a fixed point of CA-$4_{0-0}(m-2)$, then $10x_3x_4\cdots x_m$ is a fixed point of CA-$4_{0-0}(m)$.*

Since the symmetric rule of rule 4 is rule 4 CA-$4_{0-1}(m)$ and CA-$4_{1-0}(m)$ are isomorphic. And by lemma 3.11 3 and lemma 3.11 4 behaviors of CA-$4_{0-1}(m)$ and CA-$4_{1-1}(m)$ after one step transition are regarded as those of CA-$4_{0-0}(m-1)$ and CA-$4_{0-0}(m-2)$, respectively. So the following theorem results from lemma 3.11 and prop. 3.12.

THEOREM 3.13. *CA-$4_{*-*}(m)$ has fixed points only and for the number of those fixed points the following formula holds;*

$$\gamma_1(m) = \gamma_1(m-1) + \gamma_1(m-2).$$

And its transient length is 1.

3.6. CA with Rule 5

In this subsection we discuss behaviors of CA with rule 5. The symmetric, reverse and symmetric reverse rule of rule 5 are rule 5, rule 95 and rule 95 respectively. The following behaviors are found.

LEMMA 3.14. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *In CA-$5_{0-0}(m)$ if $x_i = 1$ then $x(2)_i = 1$ where $1 \le i \le m$.*

2. *In CA-$5_{0-0}(m)$ if $x(1)_i = 0$ then $x(3)_i = 0$ where $1 \le i \le m$.*

3. *In CA-$5_{*-*}(c)$ if $x_{i-1}x_ix_{i+1} = 010$ then $x(1)_{i-1}x(1)_ix(1)_i = 010$ where $1 \le i \le m$.*

4. *In CA-$5_{1-*}(m)$ $c(1)_1 = 0$.*

5. *In CA-$5_{*-1}(m)$ $c(1)_m = 0$.*

PROPOSITION 3.15. *All fixed points of CA-$5_{0-0}(m)$ can be constructed without redundancy as follows;*

1. *If $x_3x_4\cdots x_m$ is a fixed point of CA-$5_{0-0}(m-2)$, then $10x_3x_4\cdots x_m$ is a fixed point of CA-$5_{0-0}(m)$.*

2. *If $x_4x_5\cdots x_m$ is a fixed point of CA-$5_{0-0}(m-3)$, then $010x_4x_5\cdots x_m$ is a fixed point of CA-$5_{0-0}(m)$.*

PROPOSITION 3.16. *All 2-cycles of CA-$5_{0-0}(m)$ can be constructed without redundancy as follows;*

1. *If $x_3x_4\cdots x_m$ and $y_3y_4\cdots y_m$ form a 2-cycle of CA-$5_{0-0}(m-2)$, then $10x_3x_4\cdots x_m$ and $10y_3y_4\cdots y_m$ form a 2-cycle of CA-$5_{0-0}(m)$.*

2. *If $x_4x_5\cdots x_m$ and $y_4y_5\cdots y_m$ form a 2-cycle of CA-$5_{0-0}(m-3)$, then $010x_4x_5\cdots x_m$ and $010y_4y_5\cdots y_m$ form a 2-cycle of CA-$5_{0-0}(m)$.*

3. *If $1x_{k+3}x_{k+4}\cdots x_m$ and $y_{k+2}y_{k+3}\cdots y_m$ form a 2-cycle of CA-$5_{0-0}(m-k-1)$, then $0^{k+1}1x_{k+3}x_{k+4}\cdots x_m$ and $1^k0y_{k+2}y_{k+3}\cdots y_m$ form a 2-cycle of CA-$5_{0-0}(m)$ where $2\le k\le m-2$.*

4. *$x_{k+2}x_{k+3}\cdots x_m$ is a fixed point of CA-$5_{0-0}(m-k-1)$, then $0^{k+1}x_{k+2}x_{k+3}\cdots x_m$ and $1_k0x_{k+2}x_{k+3}\cdots x_m$ form a 2-cycle of CA-$5_{0-0}(m)$ where $2\le k\le m-2$.*

5. *The configurations 0^m and 1^m form a 2-cycle of CA-$5_{0-0}(m)$.*

By lemma 3.14 4 and lemma 3.14 5 behaviors of CA-$5_{0-1}(m)$, CA-$5_{1-0}(m)$ and CA-$5_{1-1}(m)$ after one step transition are regarded as behaviors of CA-$5_{0-0}(m-1)$ or CA-$5_{0-0}(m-2)$. So the following theorem results from lemma 3.14, prop. 3.15 and prop. 3.16.

THEOREM 3.17. *CA-$5_{*-*}(m)$ has fixed points and 2-cycles, and for the numbers of those limit cycles the following formulas hold;*

$$\gamma_1(m)=\gamma_1(m-2)+\gamma_1(m-3),$$

$$\gamma_2(m)=\gamma_2(m-1)+\gamma_2(m-2)+\gamma_2(m-3)-\gamma_2(m-4)+\gamma_2(m-5)-\gamma_2(m-6)+\gamma_1(m-5).$$

And transient lengths of CA-$5_{0-0}(m)$, CA-$5_{0-1}(m)$, CA-$5_{1-0}(m)$ and CA-$5_{1-1}(m)$ are 1, 2, 2 and 2, respectively.

3.7. CA with Rule 7

In this subsection we discuss behaviors of CA with rule 7. The symmetric, reverse and symmetric reverse rule of rule 7 are rule 21, rule 31 and rule 87 respectively. The following behaviors are found.

LEMMA 3.18. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configuration $(10)^*_m$ is a fixed point of CA-$7_{0-0}(m)$.*

2. *If m is even then the configuration $(10)^{\frac{m}{2}}$ is a fixed point of CA-$7_{0-1}(m)$.*

3. *The configurations $(10)^*_{2k}0^{m-2k}$ and $(10)^*_{2k}1^{m-2k}$ form a 2-cycle of CA-$7_{0-0}(m)$ where $0 \le k \le [\frac{m-2}{2}]$.*

4. *The configurations $(10)^*_{2k}0^{m-2k}$ and $(10)^*_{2k}1^{m-2k}$ form a 2-cycle of CA-$7_{0-1}(m)$ where $0 \le k \le [\frac{m-1}{2}]$.*

5. *In CA-$7_{0-*}(m)$ if $x_1x_2\cdots x_{2k} = (10)^k$ then $x(1)_1x(1)_2\cdots x(1)_{2k} = (10)^k$ where $1 \le k \le [\frac{m+1}{2}]$.*

6. *In CA-$7_{0-*}(m)$ if $x(1)_1x(1)_2\cdots x(1)_{2k} = (10)^k$ then $x_1x_2\cdots x_{2k} = (10)^k$ or $x_1x_2\cdots x_{2k+1} = (10)^{k-1}011$ where $1 \le k \le [\frac{m}{2}]$.*

7. *In CA-$7_{0-*}(m)$ there exists no configuration x such that $x(1)_1x(1)_2\cdots x(1)_{2k+1} = (10)^{k-1}011$ where $1 \le k \le [\frac{m}{2}]$.*

8. *In CA-$7_{0-*}(m)$ if $x_1x_2\cdots x_{2k+i} = (10)^k0^i$ then $x(1)_1x(1)_2\cdots x(1)_{2k+i} = (10)^k1^i$ where $2 \le i \le m-2k$ and $0 \le k \le [\frac{m-2}{2}]$.*

9. *In CA-$7_{0-*}(m)$ if $x_1x_2\cdots x_{2k+i} = (10)^k1^i$ then $x(1)_1x(1)_2\cdots x(1)_{2k+i+1} = (10)^k0^{i+1}$ where $2 \le i \le m-2k-1$ and $0 \le k \le [\frac{m-3}{2}]$.*

10. *In CA-$7_{0-*}(m)$ if $x_1x_2\cdots x_{2k+3} = (10)^k010$ then $x(1)_1x(1)_2\cdots x(1)_{2k+2} = (10)^k11$ where $0 \le k \le [\frac{m-3}{2}]$.*

11. *In CA-$7_{0-0}(m)$ $\delta((10)^k0^{m-2k-1}1) = (10)^k1^{m-2k}$ where $0 \le k \le [\frac{m+1}{2}]$.*

12. *In CA-$7_{0-0}(2n+1)$ $\delta((10)^n0) = (10)^n1$.*

13. *In CA-$7_{1-*}(m)$ $c(1)_1 = 0$.*

By lemma 3.18 13 behaviors of CA-$7_{1-\beta}(m)$ after one step transition can be regarded as those of CA-$7_{0-\beta}(m-1)$. So the following theorem results from the above lemma.

THEOREM 3.19.

- *CA-$7_{0-0}(m)$ has a unique fixed point and 2-cycles, and for the number of those 2-cycles the following formulas hold;*

$$\gamma_2(m) = \left[\frac{m}{2}\right].$$

And its transient length is $2m-5$.

- *CA-$7_{0-1}(m)$ has fixed points and 2-cycles, and for the numbers of those limit cycles the following hold;*

$$\gamma_1(m) = \begin{cases} 0 & if\ m\ is\ odd \\ 1 & if\ m\ is\ even \end{cases},$$

$$\gamma_2(m) = \left[\frac{m+1}{2}\right].$$

And its transient length is $2m-4$.

- *CA-*$7_{1-0}(m)$ *has a unique fixed point and 2-cycles, and for the number of those 2-cycles the following hold;*

$$\gamma_2(m) = \left[\frac{m-1}{2}\right].$$

And its transient length is $2m - 6$.

- *CA-*$7_{1-1}(m)$ *has fixed points and 2-cycles, and for the numbers of those limit cycles the following hold;*

$$\gamma_1(m) = \begin{cases} 1 & if\ m\ is\ odd \\ 0 & if\ m\ is\ even \end{cases},$$

$$\gamma_2(m) = \left[\frac{m}{2}\right].$$

And its transient length is $2m - 5$.

3.8. CA with Rule 8

In this subsection we discuss behaviors of CA with rule 8. The symmetric, reverse and symmetric reverse rule of rule 8 are rule 64, rule 239 and rule 253 respectively. The following behaviors are found.

LEMMA 3.20. *Let* c *be an arbitrary configuration and* x *be a configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of CA-*$8_{*-*}(m)$.
2. *The configuration* $0^{m-1}1$ *is a fixed point of CA-*$8_{*-1}(m)$.
3. *In CA-*$8_{\alpha-*}(m)$ *no* 11 *appears in* $\alpha c(1)$.
4. *In CA-*$8_{\alpha-0}(m)$ *if no* 011 *appears in* αx *then* $x(1) = 0^m$, *that is,* $x(1)$ *is a fixed point.*
5. *In CA-*$8_{\alpha-1}(m)$ *if no* 011 *appears in* αx *then* $x(1) = 0^m$ *or* $0^{m-1}1$, *that is,* $x(1)$ *is a fixed point.*

So the following theorem results from the above lemma.

THEOREM 3.21. *CA-*$8_{*-0}(m)$ *has a unique fixed point and CA-*$8_{*-1}(m)$ *has two fixed points. And their transient lengths are both 2.*

3.9. CA with Rule 10

In this subsection we discuss behaviors of CA with rule 10. The symmetric, reverse and symmetric reverse rule of rule 10 are rule 80, rule 175 and rule 245 respectively. The following behaviors are found.

LEMMA 3.22. *Let* c *be an arbitrary configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}10_{*-0}(m)$.
2. *The configuration* x *is on a 4-cycle of* $CA\text{-}10_{0-1}(m)$ *if and only if no* 000, *no* 010, *no* 101 *and no* 111 *appear in* x.
3. *In* $CA\text{-}10_{1-*}(m)$ $c(1)_1 = 0$.
4. *In* $CA\text{-}10_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ *where* $1 \le k \le m$.
5. *In* $CA\text{-}10_{0-1}(m)$ *neither* 111 *nor* 101 *appears in* $c(1)$.
6. *In* $CA\text{-}10_{0-1}(m)$ *neither* 000 *nor* 010 *appears in* $c(k)_{m-k}c(k)_{m-k+1}\cdots c(k)_m$ *where* $2 \le k \le m-1$.
7. *In* $CA\text{-}10_{1-1}(m)$ *neither* 000 *nor* 010 *appears in* $c(k)_{m-k}c(k)_{m-k+1}\cdots c(k)_m$ *where* $2 \le k \le m-2$.

By lemma 3.22 3 behaviors of CA-$10_{1-\beta}(m)$ after one step transition can be regarded as those of CA-$10_{0-\beta}(m-1)$. So the following theorem results from the above lemma.

THEOREM 3.23.

- $CA\text{-}10_{0-0}(m)$ *has a unique fixed point. And its transient length is* m.
- $CA\text{-}10_{1-0}(m)$ *has a unique fixed point. And its transient length is* $m-1$.
- $CA\text{-}10_{0-1}(m)$ *has a unique 4-cycle. And its transient length is* $m-1$.
- $CA\text{-}10_{1-1}(m)$ *has a unique 4-cycle. And its transient length is* $m-2$.

3.10. CA with Rule 11

In this subsection we introduce behaviors of CA with rule 11. The symmetric, reverse and symmetric reverse rule of rule 11 are rule 81, rule 47 and rule 117 respectively. Sato(1996b) showed the following theorem.

THEOREM 3.24.

- $CA\text{-}11_{0-1}(m)$ *has a unique 3-cycle. And its transient length is* $2m-4$.
- $CA\text{-}11_{0-0}(m)$ *has only 3-cycles and for the number of those 3-cycles the following formula holds;*
$$\gamma_3(m) = \left[\frac{m}{2}\right].$$
And its transient length is as follows;
$$H(m) = 2\left[\frac{m+1}{2}\right] - 1.$$
- $CA\text{-}11_{1-1}(m)$ *has a unique 3-cycle. And its transient length is* $2m-6$.

- *CA*-$11_{1-0}(m)$ *has only 3-cycles and for the number of those 3-cycles the following formula holds;*
$$\gamma_3(m) = \left[\frac{m-1}{2}\right].$$
And its transient length is as follows;
$$H(m) = 2\left[\frac{m}{2}\right] - 1.$$

3.11. CA with Rule 12

In this subsection we discuss behaviors of CA with rule 12. The symmetric, reverse and symmetric reverse rule of rule 12 are rule 68, rule 207 and rule 221 respectively. The following behaviors are found.

LEMMA 3.25. *Let c be an arbitrary configuration. The following hold;*

1. *The configuration x is fixed point of CA*-$12_{*-*}(m)$ *if and only if no* 11 *appears in* x.
2. *In CA*-$12_{*-*}(m)$ *no* 11 *appears in* $c(1)$.
3. *In CA*-$12_{1-*}(m)$ $c(1)_1 = 0$.

PROPOSITION 3.26. *All fixed points of CA*-$12_{0-\beta}(m)$ *can be constructed without redundancy as follows;*

1. *If* $x_2x_3\cdots x_m$ *is a fixed point of CA*-$12_{0-\beta}(m-1)$, *then* $0x_2x_3\cdots x_m$ *is a fixed point of CA*-$12_{0-\beta}(m)$.
2. *If* $x_3x_4\cdots x_m$ *is a fixed point of CA*-$12_{0-\beta}(m-2)$, *then* $10x_3x_4\cdots x_m$ *is a fixed point of CA*-$12_{0-\beta}(m)$.

By lemma 3.25 3 we can regard behaviors of CA-$12_{1-\beta}(m)$ after one step transition as those of CA-$12_{0-\beta}(m-1)$. So the following theorem results from lemma 3.25 and prop. 3.26.

THEOREM 3.27. *CA*-$12_{*-*}(m)$ *has fixed points only and for the number of those fixed points the following formula holds;*
$$\gamma_1(m) = \gamma_1(m-1) + \gamma_1(m-2).$$
And its transient length is 1.

3.12. CA with Rule 13

In this subsection we discuss behaviors of CA with rule 13. The symmetric, reverse and symmetric reverse rule of rule 13 are rule 69, rule 79 and rule 93 respectively. The following behaviors are found.

LEMMA 3.28. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configuration x is a fixed point of CA-$13_{0-\beta}(m)$ if and only if no 000 appears in $0x\beta$ and no 11 appears in x.*
2. *In CA-$13_{*-*}(m)$ if $x_i x_{i+1} = 01$ then $x(1)_i x(1)_{i+1} = 01$ where $0 \le i \le m$.*
3. *In CA-$13_{0-*}(m)$ if $x(1)_i x(1)_{i+1} x(1)_{i+2} = 000$ then $x_{i-1}x_i = 11$ where $2 \le i \le m-1$.*
4. *In CA-$13_{0-*}(m)$ If $x(1)_i x(1)_{i+1} = 11$ then $x_{i-1}x_i x_{i+1} = 000$ where $1 \le i \le m-1$.*
5. *In CA-$13_{0-*}(m)$ no 000 appears in $0c(k)_1 c(k)_2 \cdots c(k)_k c(k)_{k+1}$ where $1 \le k \le m$.*
6. *In CA-$13_{0-*}(m)$ no 11 appears in $c(k)_1 c(k)_2 \cdots c(k)_{k-1} c(k)_k$ where $2 \le k \le m$.*
7. *In CA-$13_{*-1}(m)$ $c(1)_m = 0$.*
8. *In CA-$13_{1-*}(m)$ $c(1)_1 = 0$.*

PROPOSITION 3.29. *All fixed points of CA-$13_{0-\beta}(m)$ can be constructed without redundancy as follows;*

1. *If $x_3 x_4 \cdots x_m$ is a fixed point of CA-$13_{0-\beta}(m-2)$, then $10x_3x_4 \cdots x_m$ is a fixed point of CA-$13_{0-\beta}(m)$.*
2. *If $x_4 x_5 \cdots x_m$ is a fixed point of CA-$13_{0-\beta}(m-3)$, then $010x_4x_5 \cdots x_m$ is a fixed point of CA-$13_{0-\beta}(m)$.*

By lemma 3.28 8 we can regard behaviors of CA-$13_{1-\beta}(m)$ after one step transition as those of CA-$13_{0-\beta}(m-1)$. So the following theorem results from lemma 3.28 and prop. 3.29.

THEOREM 3.30. *CA-$13_{*-*}(m)$ has fixed points only and for the number of those fixed points the following formula holds;*

$$\gamma_1(m) = \gamma_2(m-2) + \gamma_2(m-3).$$

And the transient length of CA-$13_{*-0}(m)$ is m and that of CA-$13_{*-1}(m)$ is $m-1$.

3.13. CA with Rule 14

In this subsection we introduce behaviors of CA with rule 14. The symmetric, reverse and symmetric reverse rule of rule 14 are rule 84, rule 143 and rule 213 respectively. Inokuchi(1999) showed the following theorem.

THEOREM 3.31.

- *The cellular automaton* $CA\text{-}14_{0-0}(m)$ *has only fixed points and for the number of those fixed points the following formula holds;*

$$\gamma_1(m) = \left[\frac{m+3}{2}\right].$$

And its transient length is m.

- *The cellular automaton* $CA\text{-}14_{0-1}(m)$ *has a unique fixed point. And its transient length is* $2m-1$.

- *The cellular automaton* $CA\text{-}14_{1-0}(m)$ *has only fixed points and for the number of those fixed points the following formula holds;*

$$\gamma_1(m) = \left[\frac{m+2}{2}\right].$$

And its transient length is m.

- *The cellular automaton* $CA\text{-}14_{1-1}(m)$ *has a unique fixed point. And its transient length is* $2m-2$.

3.14. CA with Rule 15

In this subsection we discuss behaviors of CA with rule 15. The symmetric, reverse and symmetric reverse rule of rule 15 are rule 85, rule 15 and rule 85 respectively. The following behaviors are found.

LEMMA 3.32. *The following hold;*

1. *The configuration* $(10)^*_m$ *is a fixed point of* $CA\text{-}15_{0-*}(m)$.
2. *Let* c *be an arbitrary configuration.*
   Then in $CA\text{-}15_{0-*}(m)$ $c(k)_1 c(k)_2 \cdots c(k)_{k-1} c(k)_k = (10)^*_k$ *where* $1 \le k \le m$.

Since the reverse rule of rule 15 is rule 15 behaviors of CA-$15_{0-0}(m)$ and CA-$15_{0-1}(m)$ is isomorphic to those of CA-$15_{1-1}(m)$ and CA-$15_{1-0}(m)$, respectively. So the following theorem results from the above lemma.

THEOREM 3.33. $CA\text{-}15_{*-*}(m)$ *has a unique fixed point. And its transient length is* m.

3.15. CA with Rule 19

In this subsection we discuss behaviors of CA with rule 19. The symmetric, reverse and symmetric reverse rule of rule 19 are rule 19, rule 55 and rule 55 respectively. The following behaviors are found.

LEMMA 3.34. *Let* c *be an arbitrary configuration. The following hold;*

1. *The configuration x is on a 2-cycle of CA-$19_{\alpha-\beta}(m)$ if and only if x satisfies the following condition;*
   - *No* 101 *appears in* $\alpha x \beta$.
   - *No* 010 *appears in* x.
   - $x_1 x_2 \neq 10$ *if* $\alpha = 1$.
   - $x_{m-1} x_m \neq 01$ *if* $\beta = 1$.
2. *In CA-$19_{*-*}(m)$ no* 010 *appears in* $c(1)$.
3. *In CA-$19_{\alpha-\beta}(m)$ no* 101 *appears in* $\alpha c(2) \beta$.
4. *In CA-$19_{1-*}(m)$* $c(1)_1 c(1)_2 \neq 10$.
5. *In CA-$19_{*-1}(m)$* $c(1)_{m-1} c(1)_m \neq 01$.

PROPOSITION 3.35. *All 2-cycles of CA-$19_{\alpha-0}(m)$ can be constructed without redundancy as follows;*

1. *If $x_1 x_2 \cdots x_{m-2}0$ and $y_1 y_2 \cdots y_{m-2}1$ form a 2-cycle of CA-$19_{\alpha-0}(m-1)$, then $x_1 x_2 \cdots x_{m-2}00$ and $y_1 y_2 \cdots y_{m-2}11$ form a 2-cycle of CA-$19_{\alpha-0}(m)$.*
2. *If $x_1 x_2 \cdots x_{m-3}0$ and $y_1 y_2 \cdots y_{m-3}1$ form a 2-cycle of CA-$19_{\alpha-0}(m-2)$, then $x_1 x_2 \cdots x_{m-3}001$ and $y_1 y_2 \cdots y_{m-2}110$ form a 2-cycle of CA-$19_{\alpha-0}(m)$.*

PROPOSITION 3.36. *All 2-cycles of CA-$19_{\alpha-1}(m)$ can be constructed without redundancy as follows;*

1. *If $x_1 x_2 \cdots x_{m-2}0$ and $y_1 y_2 \cdots y_{m-2}1$ form a 2-cycle of CA-$19_{1-1}(m-1)$, then $x_1 x_2 \cdots x_{m-2}00$ and $y_1 y_2 \cdots y_{m-2}11$ form a 2-cycle of CA-$19_{1-1}(m)$.*
2. *If $x_1 x_2 \cdots x_{m-3}0$ and $y_1 y_2 \cdots y_{m-3}1$ form a 2-cycle of CA-$19_{1-1}(m-2)$, then $x_1 x_2 \cdots x_{m-3}011$ and $y_1 y_2 \cdots y_{m-2}100$ form a 2-cycle of CA-$19_{1-1}(m)$.*

By lemma 3.34, prop. 3.35 and prop. 3.36 we get the following theorem.

THEOREM 3.37. *CA-$19_{*-*}(m)$ has 2-cycles only, and for the number of those 2-cycles the following formula holds;*

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

And its transient length is 2.

3.16. CA with Rule 23

In this subsection we discuss behaviors of CA with rule 23. The symmetric, reverse and symmetric reverse rule of rule 2 are rule 23 each. The following behaviors are found.

LEMMA 3.38. *Let x be a configuration. The following hold;*

1. *The configuration x is a fixed point of CA-$23_{0-\beta}(m)$ if and only if neither* 00 *nor* 11 *appears in $0x\beta$.*
2. *The configuration x is on a 2-cycle of CA-$23_{0-*}(m)$ if and only if x satisfies the following conditions;*
   - *Neither* 010 *nor* 101 *appears in x.*
   - $x_1 = x_2$.
   - $x_{m-1} = x_m$.
3. *In CA-$23_{*-*}(m)$ if $x_i = x_{i+1}$ then $x(1)_i = x(1)_{i+1} \neq x_i$ where $1 \leq i \leq m-1$.*
4. *In CA-$23_{*-*}(m)$ if $x_{i+1}x_{i+2}\cdots x_{i+2k+3} = 00(10)^k0$ then $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+2k+3} = 11(10)^{k-1}111$ where $0 \leq i \leq m-5$ and $1 \leq k \leq \lceil \frac{m-i-3}{2} \rceil$.*
5. *In CA-$23_{*-*}(m)$ if $x_{i+1}x_{i+2}\cdots x_{i+2k+2} = 1(10)^k0$ then $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+2k+2} = 000(10)^{k-2}111$ where $0 \leq i \leq m-6$ and $2 \leq k \leq \lceil \frac{m-i-2}{2} \rceil$.*
6. *In CA-$23_{*-*}(m)$ if $x_{i+1}x_{i+2}\cdots x_{i+2k+4} = 00(10)^k11$ then $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+2k+4} = 11(10)^k00$ where $0 \leq i \leq m-6$ and $1 \leq k \leq \lceil \frac{m-i-4}{2} \rceil$.*
7. *In CA-$23_{*-*}(m)$ if $x_{i+1}x_{i+2}\cdots x_{i+2k+3} = 1(10)^k11$ then $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+2k+3} = 000(10)^{k-1}00$ where $0 \leq i \leq m-5$ and $1 \leq k \leq \lceil \frac{m-i-3}{2} \rceil$.*
8. *Let $x_{m-k-1}x_{m-k}\cdots x_m = 00(10)^*_k$. Then in CA-$23_{*-\beta}(m)$ the following hold;*
   - *If $\beta = 0$ and k is even then $x(1)_{m-k-1}x(1)_{m-k}\cdots x(1)_m = 11(10)^*_{k-2}11$.*
   - *If $\beta = 1$ and k is even then $x(1)_{m-k-1}x(1)_{m-k}\cdots x(1)_m = 11(10)^*_k$.*
   - *If $\beta = 0$ and k is odd then $x(1)_{m-k-1}x(1)_{m-k}\cdots x(1)_m = 11(10)^*_k$.*
   - *If $\beta = 1$ and k is odd then $x(1)_{m-k-1}x(1)_{m-k}\cdots x(1)_m = 11(10)^*_{k-1}$.*

   Where $1 \leq k \leq m-2$.
9. *Let $x_{m-k}x_{m-k+1}\cdots x_m = 1(10)^*_k$. Then in CA-$23_{*-\beta}(m)$ the following hold;*
   - *If $\beta = 0$ and k is even then $x(1)_{m-k}x(1)_{m-k+1}\cdots x(1)_m = 000(10)^*_{k-4}11$ where $4 \leq k \leq m-1$.*

- *If* $\beta = 1$ *and* k *is even then* $x(1)_{m-k}x(1)_{m-k+1}\cdots x(1)_m = 000(10)^*_{k-2}$ *where* $2 \le k \le m-1$.
- *If* $\beta = 0$ *and* k *is odd then* $x(1)_{m-k}x(1)_{m-k+1}\cdots x(1)_m = 000(10)^*_{k-2}$ *where* $2 \le k \le m-1$.
- *If* $\beta = 1$ *and* k *is odd then* $x(1)_{m-k}x(1)_{m-k+1}\cdots x(1)_m = 000(10)^*_{k-2}$ *where* $2 \le k \le m-1$.
- *If* $\beta = 0$ *and* $k = 2$ *then* $x(1)_{m-2}x(1)_{m-1}x(1)_m = 001$.

10. *In CA-*$23_{0-*}(m)$ *if* $x_1x_2\cdots x_{2k+1} = (10)^k0$
then $x(1)_1x(1)_2\cdots x(1)_{2k+1} = (10)^{k-1}111$ *where* $1 \le k \le [\frac{m-1}{2}]$.

11. *In CA-*$23_{0-*}(m)$ *if* $x_1x_2\cdots x_{2k+2} = 0(10)^k0$
then $x(1)_1x(1)_2\cdots x(1)_{2k+2} = 1(10)^{k-1}111$ *where* $1 \le k \le [\frac{m-2}{2}]$.

12. *In CA-*$23_{0-*}(m)$ *if* $x_1x_2\cdots x_{2k+2} = (10)^k11$
then $x(1)_1x(1)_2\cdots x(1)_{2k+2} = (10)^k00$ *where* $1 \le k \le [\frac{m-2}{2}]$.

13. *In CA-*$23_{0-*}(m)$ *if* $x_1x_2\cdots x_{2k+3} = 0(10)^k11$
then $x(1)_1x(1)_2\cdots x(1)_{2k+3} = 1(10)^k00$ *where* $1 \le k \le [\frac{m-3}{2}]$.

14. *Let* $x = (10)^*_m$. *Then in CA-*$23_{0-\beta}(m)$ *the following hold;*
    - *If* $\beta = 0$ *and* m *is even then* $x(1) = (10)^*_{m-2}11$.
    - *If* $\beta = 1$ *and* m *is even then* x *is a fixed point.*
    - *If* $\beta = 0$ *and* m *is odd then* x *is a fixed point.*
    - *If* $\beta = 1$ *and* m *is odd then* $x(1) = (10)^*_{m-1}0$.

15. *Let* $x = 0(10)^*_m$. *Then in CA-*$23_{0-\beta}(m)$ *the following hold;*
    - *If* $\beta = 0$ *and* m *is even then* $x(1) = 1(10)^*_{m-1}$.
    - *If* $\beta = 1$ *and* m *is even then* $x(1) = 1(10)^*_{m-1}$.
    - *If* $\beta = 0$ *and* m *is odd then* $x(1) = 1(10)^*_{m-3}11$.
    - *If* $\beta = 1$ *and* m *is odd then* $x(1) = 1(10)^*_{m-1}$.

PROPOSITION 3.39. *All 2-cycles of CA-*$23_{0-\beta}(m)$ *can be constructed without redundancy as follows;*

1. *If* $1x_3x_4\cdots x_m$ *and* $0y_3y_4\cdots y_m$ *form a 2-cycle of CA-*$23_{0-\beta}(m-1)$
then $11x_3x_4\cdots x_m$ *and* $00y_3y_4\cdots y_m$ *form a 2-cycle of CA-*$23_{0-\beta}(m)$

2. *If* $1x_4x_5\cdots x_m$ *and* $0y_4y_5\cdots y_m$ *form a 2-cycle of CA-*$23_{0-\beta}(m-2)$
then $001x_4x_5\cdots x_m$ *and* $110y_4y_5\cdots y_m$ *form a 2-cycle of CA-*$23_{0-\beta}(m)$.

Since the reverse rule of rule 23 is rule 23 CA-$23_{0-0}(m)$ and CA-$23_{0-1}(m)$ are isomorphic to CA-$23_{1-1}(m)$ and CA-$23_{1-0}(m)$, respectively. So the following theorem results from lemma 3.38 and prop. 3.39.

THEOREM 3.40.

- *CA-*$23_{\alpha-\alpha}(m)$ *has fixed points and 2-cycles, and for the numbers of those limit cycles the following formula hold;*

$$\gamma_1(m) = \begin{cases} 1 & if\ m\ is\ odd \\ 0 & otherwise \end{cases},$$

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

And its transient length is as follows;

$$H(m) = 2\left[\frac{m}{2}\right] - 1.$$

- *CA-*$23_{\alpha-\overline{\alpha}}(m)$ *has fixed points and 2-cycles and for the numbers of those limit cycles the following formula hold;*

$$\gamma_1(m) = \begin{cases} 0 & if\ m\ is\ odd \\ 1 & otherwise \end{cases},$$

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

And its transient length is as follows;

$$H(m) = 2\left[\frac{m-1}{2}\right].$$

3.17. CA with Rule 32

In this subsection we discuss behaviors of CA with rule 32. The symmetric, reverse and symmetric reverse rule of rule 32 are rule 32, rule 251 and rule 251 respectively. The following behaviors are found.

LEMMA 3.41. *Let c be an arbitrary configuration and x a configuration. The following hold;*

1. *The configuration 0^m is a fixed point of CA-*$32_{*-*}(m)$.
2. *The configuration x is on a 2-cycle of CA-*$32_{1-1}(m)$ *if and only if neither* 00 *nor* 11 *appears in x.*
3. *In CA-*$32_{*-*}(m)$ *the following hold;*
   - *If* $x_1x_2 = 00$ *then* $x(1)_1x(1)_2x(1)_3 = 000$.
   - *If* $x_ix_{i+1} = 00$ *then* $x(1)_{i-1}x(1)_ix(1)_{i+1}x(1)_{i+2} = 0000$ *where* $2 \le i \le m-2$.
   - *If* $x_{m-1}x_m = 00$ *then* $x(1)_{m-2}x(1)_{m-1}x(1)_m = 000$.
4. *In CA-*$32_{*-*}(m)$ *If* $x_ix_{i+1} = 11$ *then* $x(1)_ix(1)_{i+1} = 00$ *where* $1 \le i \le m-1$.
5. *In CA-*$32_{0-*}(m)$ $c(k)_1c(k)_2\cdots c(k)_k = 0^k$.

6. *In* $CA\text{-}32_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$.

So the following theorem results from the above lemma.

THEOREM 3.42.

- $CA\text{-}32_{0-0}(m)$ *has a unique fixed point. And its transient length is as follows;*

$$H(m) = \left\lceil \frac{m+1}{2} \right\rceil .$$

- $CA\text{-}32_{\alpha-\overline{\alpha}}(m)$ *has a unique fixed point. And its transient length is* m.
- $CA\text{-}32_{1-1}(m)$ *has a unique fixed point and a unique 2-cycle. And its transient length is* $m-1$.

3.18. CA with Rule 34

In this subsection we discuss behaviors of CA with rule 34. The symmetric, reverse and symmetric reverse rule of rule 34 are rule 48, rule 187 and rule 243 respectively. The following behaviors are found.

LEMMA 3.43. *The following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}34_{*-0}(m)$.
2. *The configuration* x *is on a 2-cycle of* $CA\text{-}34_{*-1}(m)$ *if and only if neither* 00 *nor* 11 *appears in* x.
3. *Let* c *be an arbitrary configuration.*
   Then in $CA\text{-}34_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ *where* $1 \le k \le m$.
4. *Let* x *be a configuration such that* $x_m = 0$.
   Then in $CA\text{-}34_{*-1}(m)$ $x(k)_{m-k}x(k)_{m-k+1}\cdots x(k)_m = (01)^*_{k+1}$ *where* $1 \le k \le m-1$.
5. *Let* x *be a configuration. Then in* $CA\text{-}34_{*-1}(m)$ *if* $x_m = 1$ *then* $x(1)_m = 0$.

The following theorem results from the above lemma.

THEOREM 3.44. $CA\text{-}34_{*-0}(m)$ *has a unique fixed point and* $CA\text{-}34_{*-1}(m)$ *has a unique 2-cycle. And their transient lengths are both* m.

3.19. CA with Rule 35

In this subsection we discuss behaviors of CA with rule 35. The symmetric, reverse and symmetric reverse rule of rule 35 are rule 49, rule 59 and rule 115 respectively. The following behaviors are found.

LEMMA 3.45. *Let* c *be an arbitrary configuration and* x *a configuration. The following hold;*

1. *The configurations* 0^m *and* 1^m *form a 2-cycle of* $CA\text{-}35_{0-0}(m)$.

2. *The configurations* $1^k(01)^*_{m-k}$ *and* $0^k(10)^*_{m-k}$ *form a 2-cycle of* $CA\text{-}35_{0-1}(m)$ *where* $1 \leq k \leq m$.

3. *The configurations* $(010)^*_m, (100)^*_m$ *and* $(001)^*_m$ *form a 3-cycles of* $CA\text{-}35_{1-0}(m)$.

4. *The configurations* $(01)^*_m$ *and* $(10)^*_m$ *form a 2-cycle of* $CA\text{-}35_{1-1}(m)$.

5. *In* $CA\text{-}35_{*-*}(m)$ *if* $x(1)_i x(1)_{i+1} x(1)_{i+2} = 101$ *then* $x_{i+1}x_{i+2}x_{i+3} = 101$ *where* $1 \leq i \leq m-2$.

6. *In* $CA\text{-}35_{*-0}(m)$ *no* 101 *appears in* $c(k)_{m-k-1}c(k)_{m-k}\cdots c(k)_m$ *where* $1 \leq k \leq m-2$.

7. *Let* x *such that no* 101 *appears in* x. *In* $CA\text{-}35_{0-0}(m)$ *if* $x_1x_2\cdots x_{k+1} = 0^k1$ *then* $x(1)_1x(1)_2\cdots x(1)_{k+1} = 1^k0$ *where* $1 \leq k \leq m-1$.

8. *Let* x *such that no* 101 *appears in* x. *In* $CA\text{-}35_{0-0}(m)$ *if* $x_1x_2\cdots x_{k+1} = 1^k0$ *then* $x(1)_1x(1)_2\cdots x(1)_{k+2} = 0^{k+1}1$ *where* $1 \leq k \leq m-2$.

9. *In* $CA\text{-}35_{0-0}(m)$ $\delta(1^{m-1}0) = 0^m$.

10. *In* $CA\text{-}35_{1-*}(m)$ *if* $x(1)_i x(1)_{i+1} x(1)_{i+2} = 000$ *then* $x_ix_{i+1} = 11$ *where* $1 \leq i \leq m-2$.

11. *In* $CA\text{-}35_{1-*}(m)$ *if* $x(1)_i x(1)_{i+1} = 11$ *then* $x_{i-1}x_ix_{i+1} = 000$ *where* $2 \leq i \leq m-1$.

12. *In* $CA\text{-}35_{1-*}(m)$ $c(1)_1c(1)_2 \neq 11$.

13. *In* $CA\text{-}35_{1-*}(m)$ *no* 000 *appears in* $c(2m-4)$.

14. *In* $CA\text{-}35_{1-*}(m)$ *no* 11 *appears in* $c(2m-3)$.

15. *Let* x *such that neither* 000 *nor* 11 *appears in* x, *and* $x_{m-1}x_m = 01$. *Then in* $CA\text{-}35_{1-1}(m)$
$$x(2k)_{m-2k-1}x(2k)_{m-2k}\cdots x(2k)_m = (01)^{k+1}$$
where $1 \leq k \leq [\frac{m-2}{2}]$.

16. *In* $CA\text{-}35_{0-1}(m)$ *if* $x_1 = x_2 = \cdots = x_k$ *then* $x(1)_1 = x(1)_2 = \cdots = x(1)_k \neq x_1$ *where* $1 \leq k \leq m$.

17. *In* $CA\text{-}35_{0-1}(m)$ $c(1)_1c(1)_2c(1)_3 \neq 011$.

18. *In* $CA\text{-}35_{0-1}(m)$ *no* 1011 *appears in* $c(1)$.

19. *In* $CA\text{-}35_{0-1}(m)$ *if* $x(1)_i x(1)_{i+1} x(1)_{i+2} x(1)_{i+3} = 1000$ *then* $x_ix_{i+1}x_{i+2} = 011$ *where* $1 \leq i \leq m-3$.

20. *In* $CA\text{-}35_{0-1}(m)$ *if* $x(1)_i x(1)_{i+1} x(1)_{i+2} = 011$ *then* $x_{i-1}x_ix_{i+1}x_{i+2} = 1000$ *where* $2 \leq i \leq m-2$.

21. In CA-$35_{0-1}(m)$ no 011 appears in $c(2m-5)$.

22. In CA-$35_{0-1}(m)$ no 100 appears in $c(2m-4)$.

The following theorem results from the above lemma.

THEOREM 3.46.

- *CA-$35_{0-0}(m)$ has a unique 2-cycle. And its transient length is $3m-4$.*
- *CA-$35_{1-1}(m)$ has a unique 2-cycle. And its transient length is $3m-4$.*
- *CA-$35_{1-0}(m)$ has a unique 3-cycle. And its transient length is $2m-3$.*
- *CA-$35_{0-1}(m)$ has m 2-cycles. And its transient length is $2m-4$.*

3.20. CA with Rule 42

In this subsection we discuss behaviors of CA with rule 42. The symmetric, reverse and symmetric reverse rule of rule 42 are rule 112, rule 171 and rule 241 respectively. The following behaviors are found.

LEMMA 3.47. *Let c be an arbitrary configuration. Then the following hold;*

1. *The configuration 0^m is a fixed point of CA-$42_{*-0}(m)$.*
2. *The configurations $(101)^*_m, (011)^*_m$ and $(110)^*_m$ form a 3-cycle of CA-$42_{0-1}(m)$.*
3. *The configurations $1(101)^*_{m-1}$, $0(011)^*_{m-1}$ and $0(110)^*_{m-1}$ form a 3-cycle of CA-$42_{1-1}(m)$.*
4. *In CA-$42_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ where $1 \le k \le m$.*
5. *In CA-$42_{*-1}(m)$ no 111 appears in $c(1)$.*
6. *In CA-$42_{*-1}(m)$ no 00 appears in $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m$ where $2 \le k \le m-1$.*
7. *In CA-$42_{*-1}(m)$ no 010 appears in $c(k)_{m-k-1}c(k)_{m-k}\cdots c(k)_m$ where $1 \le k \le m-2$.*
8. *In CA-$42_{1-1}(m)$ $c(m-1)_1c(m-1)_2 \neq 10$.*

The following theorem results from the above lemma.

THEOREM 3.48.

- *CA-$42_{*-0}(m)$ has a unique fixed point. And its transient length is m.*
- *CA-$42_{0-1}(m)$ has a unique 3-cycle. And its transient length is m*
- *CA-$42_{1-1}(m)$ has a unique 3-cycle. And its transient length is $m-1$*

3.21. CA with Rule 43

In this subsection we introduce behaviors of CA with rule 43. The symmetric, reverse and symmetric reverse rule of rule 43 are rule 113, rule 43 and rule 113 respectively. Sato(1997) showed the following theorem.

THEOREM 3.49.

- *CA-$43_{\alpha-\alpha}(m)$ has 3-cycles only and for the number of those 3-cycles the following formula holds;*

$$\gamma_3(m) = \left[\frac{m}{2}\right].$$

And its transient length is $m-1$.

- *CA-$43_{\alpha-\overline{\alpha}}(m)$ has 3-cycles only and for the number of those 3-cycles the following formula holds;*

$$\gamma_3(m) = \left[\frac{m+1}{2}\right].$$

And its transient length is as follows;

$$H(m) = \left[\frac{m}{2}\right].$$

3.22. CA with Rule 50

In this subsection we discuss behaviors of CA with rule 50. The symmetric, reverse and symmetric reverse rule of rule 50 are rule 50, rule 179 and rule 179 respectively. The following behaviors are found.

LEMMA 3.50. *Let c be an arbitrary configuration and x a configuration. The following hold;*

1. *The configuration x is on a 2-cycle of CA-$50_{\alpha-\beta}(m)$ if and only if x satisfies the following conditions;*
   - *No 000 appears in $\alpha x\beta$.*
   - *No 111 appears in x.*
   - *$x_1x_2 \neq 11$ if $\alpha = 0$.*
   - *$x_{m-1}x_m \neq 11$ if $\beta = 0$.*
2. *In CA-$50_{0-*}(m)$ if $x(1)_1x(1)_2 = 00$ then $x_1 = x_2$.*
3. *In CA-$50_{*-*}(m)$ if $x(1)_ix(1)_{i+1}x(1)_{i+2} = 000$ then $x_i = x_{i+1} = x_{i+2}$ where $1 \leq i \leq m-2$.*
4. *In CA-$50_{*-0}(m)$ if $x(1)_{m-1}x(1)_m = 00$ then $x_{m-1} = x_m$.*
5. *In CA-$50_{*-*}(m)$ no 111 appears in $c(1)$.*

6. *Let x be non-GOE. In CA-$50_{0-*}(m)$ if $x_1x_2\cdots x_ix_{i+1} = 0^i1$ then $x(1)_1x(1)_2\cdots x(1)_i = 0^{i-1}1$ where $i \geq 2$.*

7. *Let x be non-GOE. Then in CA-$50_{*-*}(m)$ if $x_kx_{k+1}\cdots x_{k+i}x_{k+i+1} = 10^i1$ then $x(1)_{k+1}x(1)_{k+2}\cdots x(1)_{k+i} = 10^{i-2}1$ where $k \geq 0$ and $3 \leq i \leq m-k$.*

8. *Let x be non-GOE. Then in CA-$50_{*-0}(m)$ if $x_{m-i}x_{m-i+1}\cdots x_m = 10^i$ then $x(1)_{m-i+1}x(1)_{m-i+2}\cdots x(1)_m = 10^{i-1}$ where $i \geq 2$.*

9. *In CA-$50_{0-*}(m)$ $c(1)_1c(1)_2 \neq 11$.*

10. *In CA-$50_{*-0}(m)$ $c(1)_{m-1}c(1)_m \neq 11$.*

PROPOSITION 3.51. *All 2-cycles of CA-$50_{0-\beta}(m)$ can be constructed without redundancy as follows;*

1. *If $1x_3x_4\cdots x_m$ and $0y_3y_4\cdots y_m$ form a 2-cycle of CA-$50_{0-\beta}(m-1)$ then $01x_3x_4\cdots x_m$ and $10y_3y_4\cdots y_m$ form a 2-cycle of CA-$50_{0-\beta}(m)$.*

2. *If $1x_4x_5\cdots x_m$ and $0y_4y_5\cdots y_m$ form a 2-cycle of CA-$50_{0-\beta}(m-2)$ then $011x_4x_5\cdots x_m$ and $100y_4y_5\cdots y_m$ form a 2-cycle of CA-$50_{0-\beta}(m)$.*

PROPOSITION 3.52. *All 2-cycles of CA-$50_{1-1}(m)$ can be constructed without redundancy as follows;*

1. *If $1x_3x_4\cdots x_m$ and $0y_3y_4\cdots y_m$ form a 2-cycle of CA-$50_{1-1}(m-1)$ then $01x_3x_4\cdots x_m$ and $10y_3y_4\cdots y_m$ form a 2-cycle of CA-$50_{1-1}(m)$*

2. *If $1x_4x_5\cdots x_m$ and $0y_4y_5\cdots y_m$ form a 2-cycle of CA-$50_{1-1}(m-2)$ then $001x_4x_5\cdots x_m$ and $110y_4y_5\cdots y_m$ form a 2-cycle of CA-$50_{1-1}(m)$.*

Since the symmetric rule of rule 50 is rule 50 CA-$50_{0-1}(m)$ and CA-$50_{1-0}(m)$ are isomorphic. So the following theorem results from lemma 3.50, prop. 3.51 and prop. 3.52.

THEOREM 3.53.

- *The cellular automaton CA-$50_{0-0}(m)$ has a unique fixed point and 2-cycles, and for the number of those 2-cycles the following formula hold;*

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

  And its transient length is $m-2$.

- *The cellular automata CA-$50_{0-1}(m)$, CA-$50_{1-0}(m)$ and CA-$50_{1-1}(m)$ have 2-cycles only and for the number of those 2-cycles the following formula holds;*

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

  And their transient lengths are m each.

3.23. CA with Rule 51

In this subsection we discuss behaviors of CA with rule 51. Rule 51 is the reversible rule, that is, $f(u,v,w) = \overline{v}$ for any $u,v,w \in \{0,1\}$. The symmetric, reverse and symmetric reverse rule of rule 51 are rule 51 each. So we get the following theorem easily.

THEOREM 3.54. *CA-$51_{*-*}(m)$ has 2^{m-1} 2-cycles. And its transient length is* 0.

3.24. CA with Rule 76

In this subsection we discuss behaviors of CA with rule 76 The symmetric, reverse and symmetric reverse rule of rule 76 are rule 76, rule 205 and rule 205 respectively. The following behaviors are found.

LEMMA 3.55. *The following hold;*

1. *The configuration x is a fixed point of CA-$76_{\alpha-\beta}(m)$ if and only if no* 111 *appears in $\alpha x \beta$.*

2. *Let c be an arbitrary configuration. Then in CA-$76_{\alpha-\beta}(m)$ no* 111 *appears in $\alpha c(1)\beta$.*

PROPOSITION 3.56. *All fixed point of CA-$76_{0-0}(m)$ can be constructed without redundancy by the followings;*

1. *If $x_1x_2\cdots x_{m-1}$ is a fixed point of CA-$76_{0-0}(m-1)$, then $x_1x_2\cdots x_{m-1}0$ is a fixed point of CA-$76_{0-0}(m)$.*

2. *If $x_1x_2\cdots x_{m-2}$ is a fixed point of CA-$76_{0-0}(m-2)$, then $x_1x_2\cdots x_{m-2}01$ is a fixed point of CA-$76_{0-0}(m)$.*

3. *If $x_1x_2\cdots x_{m-3}$ is a fixed point of CA-$76_{0-0}(m-3)$, then $x_1x_2\cdots x_{m-3}011$ is a fixed point of CA-$76_{0-0}(m)$.*

PROPOSITION 3.57. *All fixed point of CA-$76_{\alpha-1}(m)$ can be constructed without redundancy by the followings;*

1. *If $x_1x_2\cdots x_{m-2}0$ is a fixed point of CA-$76_{\alpha-1}(m-1)$, then $x_1x_2\cdots x_{m-2}01$ is a fixed point of CA-$76_{\alpha-1}(m)$.*

2. *If $x_1x_2\cdots x_{m-i-1}$ is a fixed point of CA-$76_{\alpha-1}(m-i-1)$, then $x_1x_2\cdots x_{m-i-1}10^i$ is a fixed point of CA-$76_{\alpha-1}(m)$ where $1 \le i \le m-2$.*

3. *The configurations 0^m and 10^{m-1} are fixed points of CA-$76_{\alpha-1}(m)$.*

Since the symmetric rule of rule 76 is rule 76 CA-$76_{0-1}(m)$ and CA-$76_{1-0}(m)$ are isomorphic. So the following theorem results from lemma 3.55, prop. 3.56 and prop. 3.57.

THEOREM 3.58. *CA-76$_{*-*}(m)$ has fixed points only, and the number of those fixed points conforms the following formula;*

$$\gamma_1(m) = \gamma_1(m-1) + \gamma_1(m-2) + \gamma_1(m-3).$$

And its transient length is 1.

3.25. CA with Rule 77

In this subsection we discuss behaviors of CA with rule 77. The symmetric, reverse and symmetric reverse rule of rule 77 are rule 77 each. The following behaviors are found.

LEMMA 3.59. *Let x be a configuration. The following hold;*

1. *The configuration x is a fixed point of CA-77$_{\alpha-0}(m)$ if and only if neither* 000 *nor* 111 *appears in $\alpha x0$.*
2. *In CA-77$_{*-0}(m)$ if $x_i x_{i+1} \cdots x_{i+k+1} = 01^k0$ then $x(1)_i x(1)_{i+1} \cdots x(1)_{i+k+1} = 010^{k-2}10$ where $3 \le k \le m$ and $0 \le i \le m-k$.*
3. *In CA-77$_{*-0}(m)$ if $x_i x_{i+1} \cdots x_{i+k+1} = 10^k1$ then $x(1)_i x(1)_{i+1} \cdots x(1)_{i+k+1} = 101^{k-2}01$ where $3 \le k \le m-1$ and $0 \le i \le m-k-1$.*
4. *In CA-77$_{0-0}(m)$ if $x_1 x_2 \cdots x_{k+1} = 0^k1$ then $x(1)_1 x(1)_2 \cdots x(1)_{k+1} = 1^{k-1}01$ where $2 \le k \le m-1$.*
5. *In CA-77$_{*-0}(m)$ if $x_{m-k} x_{m-k+1} \cdots x_m = 10^k$ then $x(1)_{m-k} x(1)_{m-k+1} \cdots x(1)_m = 101^{k-1}$ where $2 \le k \le m$.*
6. *In CA-77$_{1-0}(m)$ if $x_1 x_2 \cdots x_{k+1} = 1^k0$ then $x(1)_1 x(1)_2 \cdots x(1)_{k+1} = 0^{k-1}10$ where $2 \le k \le m$.*
7. *In CA-77$_{*-0}(m)$ if $x(1)_i x(1)_{i+1} x(1)_{i+2} = 000$ then $x_{i-1} x_i x_{i+1} x_{i+2} x_{i+3} = 11111$ where $1 \le i \le m-3$.*
8. *In CA-77$_{*-0}(m)$ if $x(1)_i x(1)_{i+1} x(1)_{i+2} = 111$ then $x_{i-1} x_i x_{i+1} x_{i+2} x_{i+3} = 00000$ where $1 \le i \le m-2$.*
9. *In CA-77$_{0-0}(m)$ $\delta(0^m) = 1^m$.*

PROPOSITION 3.60. *All fixed points of CA-77$_{\alpha-0}(m)$ can be constructed without redundancy as follows;*

1. *If $x_1 x_2 \cdots x_{m-2}$ is a fixed point of CA-77$_{\alpha-0}(m-2)$ then $x_1 x_2 \cdots x_{m-2}01$ is a fixed point of CA-77$_{\alpha-0}(m)$.*
2. *If $x_1 x_2 \cdots x_{m-3}$ is a fixed point of CA-77$_{\alpha-0}(m-3)$ then $x_1 x_2 \cdots x_{m-2}011$is a fixed point of CA-77$_{\alpha-0}(m)$.*

3. *If* $x_1x_2\cdots x_{m-3}$ *is a fixed point of* CA-$77_{\alpha-0}(m-3)$ *then* $x_1x_2\cdots x_{m-2}010$ *is a fixed point of* CA-$77_{\alpha-0}(m)$.

4. *If* $x_1x_2\cdots x_{m-4}$ *is a fixed point of* CA-$77_{\alpha-0}(m-4)$ *then* $x_1x_2\cdots x_{m-2}0110$ *is a fixed point of* CA-$77_{\alpha-0}(m)$.

Since rule 77 and its reverse rule are the same CA-$77_{1-1}(m)$ and CA-$77_{0-1}(m)$ are isomorphic to CA-$77_{0-0}(m)$ and CA-$77_{1-0}(m)$ respectively. So the following theorem results from lemma 3.59 and prop. 3.60.

THEOREM 3.61. *CA-$77_{*-*}(m)$ has fixed points only and for the number of those fixed points the following formula holds;*

$$\gamma_1(m)=\gamma_1(m-2)+2\gamma_1(m-3)+\gamma_1(m-4).$$

The transient length of CA-$77_{\alpha-\alpha}(m)$ is as follows;

$$H(m)=\left[\frac{m+1}{2}\right].$$

The transient length of CA-$77_{\alpha-\overline{\alpha}}(m)$ is as follows;

$$H(m)=\left[\frac{m}{2}\right].$$

3.26. CA with Rule 128

In this subsection we discuss behaviors of CA with rule 128. The symmetric, reverse and symmetric reverse rule of rule 128 are rule 128, rule 254 and rule 254 respectively. The following behaviors are found.

LEMMA 3.62. *Let x be a configuration. Then the following hold;*

1. *The configuration x is a fixed point of CA-$128_{0-*}(m)$ if and only if no 1 appears in x. That is, the configuration 0^m is a unique fixed point of CA-$128_{0-*}(m)$.*

2. *The configurations 1^m and 0^m are two fixed points of CA-$128_{1-1}(m)$.*

3. *In CA-$128_{*-*}(m)$ if $x(1)_i=1$ then $x_{i-1}x_ix_{i+1}=111$ where $1\leq i\leq m$.*

4. *In CA-$128_{*-*}(m)$ if $x_ix_{i+1}=00$ then $x(1)_ix(1)_{i+1}=00$ where $0\leq i\leq m$.*

5. *In CA-$128_{1-1}(m)$ if $x(1)=1^m$ then $x=1^m$.*

6. *In CA-$128_{*-*}(m)$ if $x_ix_{i+1}\cdots x_{i+k+1}=01^k0$ then $x(1)_ix(1)_{i+1}\cdots x(1)_{i+k+1}=001^{k-2}00$. where $1\leq k\leq m$ and $0\leq i\leq m-k$.*

7. *In CA-$128_{*-1}(m)$ if $x_{m-k}x_{m-k+1}\cdots x_m=01^k$ then $x(1)_{m-k}x(1)_{m-k+1}\cdots x(1)_m=001^{k-1}$. where $1\leq k\leq m$.*

8. *In CA-$128_{1-*}(m)$ if $x_1x_2\cdots x_{k+1}=1^k0$ then $x(1)_1x(1)_2\cdots x(1)_{k+1}=1^{k-1}00$.*

9. *Let* 1^j *appear in* x *and* $i = \max\{\ j \mid 1^j$ *appears in* $x\ \}$. *Then in* $CA\text{-}128_{0-0}(m)$
$$x([\frac{i+1}{2}]) = 0^m.$$

10. *Let* 1^j *appear in* x *and* $i = \max\{\ j \mid 1^j$ *appears in* $x\ \}$. *Then in* $CA\text{-}128_{0-1}(m)$
$$x(i) = 0^m.$$

11. *Let* 1^j *appear in* x *and* $i = \max\{\ j \mid 1^j$ *appears in* $x\ \}$. *In* $CA\text{-}128_{1-1}(m)$ *if* $c \neq 1^m$ *then*
$$x(i) = 0^m.$$

Since the symmetric rule of rule 128 is rule 128 CA-$128_{0-1}(m)$ and CA-$128_{1-0}(m)$ are isomorphic. So the following theorem results from the above lemma.

THEOREM 3.63.

- $CA\text{-}128_{0-0}(m)$ *has a unique fixed point. And its transient length is as follows;*
$$H(m) = \left[\frac{m+1}{2}\right].$$
- $CA\text{-}128_{\alpha-\overline{\alpha}}(m)$ *has a unique fixed point. And its transient length is* m.
- $CA\text{-}128_{1-1}(m)$ *has only 2 fixed point. And its transient length is* $m-1$

3.27. CA with Rule 136

In this subsection we discuss behaviors of CA with rule 136. The symmetric, reverse and symmetric reverse rule of rule 136 are rule 192, rule 238 and rule 252 respectively. The following behaviors are found.

LEMMA 3.64. *The following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}136_{*-0}(m)$.
2. *The configuration* 1^m *is a fixed point of* $CA\text{-}136_{*-1}(m)$.
3. *Let* c *be an arbitrary configuration. Then in* $CA\text{-}136_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ *where* $1 \leq k \leq m$.
4. *Let* x *be a configuration. Then in* $CA\text{-}136_{*-1}(m)$ *if* $x_{m-i}x_{m-i+1}\cdots x_m = 01^i$ *then* $x(1)_{m-i}x(1)_{m-i+1}\cdots x(1)_m = 01^i$ *where* $0 \leq i \leq m-1$.

By lemma 3.64 4 we can regard behaviors of the configuration x such that $x_{m-i}x_{m-i+1}\cdots x_m = 01^i$ in CA-$136_{\alpha-1}(m)$ as behaviors of CA-$136_{\alpha-0}(m-i-1)$. So the following theorem results from the above lemma.

THEOREM 3.65.

- $CA\text{-}136_{*-0}(m)$ *has a unique fixed point. And it transient length is* m.
- $CA\text{-}136_{*-1}(m)$ *has* $m+1$ *fixed points. And its transient length is* $m-1$.

3.28. CA with Rule 138

In this subsection we discuss behaviors of CA with rule 138. The symmetric, reverse and symmetric reverse rule of rule 138 are rule 208, rule 174 and rule 244 respectively. The following behaviors are found.

LEMMA 3.66. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configuration 0^m is a fixed point of CA-$138_{*-0}(m)$.*
2. *In CA-$138_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ where $1 \le k \le m$.*
3. *The configuration 1^m is a fixed point of CA-$138_{0-1}(m)$.*
4. *In CA-$138_{*-1}(m)$ no 101 appears in $c(1)1$.*
5. *Let no 101 appear in $x1$. Then in CA-$138_{0-1}(m)$ $x(k)_{m-k+1}x(k)_{m-k+2}\cdots x(k)_m = 1^k$ where $1 \le k \le m$.*
6. *The configurations 1^m and 01^{m-1} are two fixed point of CA-$138_{1-1}(m)$.*
7. *In CA-$138_{1-1}(m)$ if $x(1) = 1^m$ then $x = 1^m$.*
8. *In CA-$138_{1-1}(m)$ if $x_1x_2\cdots x_{k+1} = 1^k0$ then $x(1)_1x(1)_2\cdots x(1)_k = 1^{k-1}0$ where $1 \le k \le m-1$.*
9. *In CA-$138_{1-*}(m)$ if $x_1 = 0$ then $x(1)_1 = 0$.*
10. *In CA-$138_{1-1}(m)$ $c(k)_{m-k+2}c(k)_{m-k+3}\cdots c(k)_m = 1^{k-1}$ where $2 \le k \le m$.*

So the following theorem results from the above lemma.

THEOREM 3.67.

- *CA-$138_{*-0}(m)$ has a unique fixed point. And its transient length is m.*
- *CA-$138_{0-1}(m)$ has a unique fixed point. And its transient length is $m+1$.*
- *CA-$138_{1-1}(m)$ has only 2 fixed point. And its transient length is m.*

3.29. CA with Rule 140

In this subsection we discuss behaviors of CA with rule 140. The symmetric, reverse and symmetric reverse rule of rule 140 are rule 196, rule 206 and rule 220 respectively. The following behaviors are found.

LEMMA 3.68. *Let x be a configuration. Then the following hold;*

1. *The configuration x is a fixed point of CA-$140_{*-0}(m)$ if and only if no 11 appears in x.*

2. *In* $CA\text{-}140_{*-0}(m)$ *if* $x_i x_{i+1} \cdots x_{i+k} x_{i+k+1} = 01^k 0$ *then* $x(1)_i x(1)_{i+1} \cdots x(1)_{i+k} x(1)_{i+k+1} = 01^{k-1}00$ *where* $2 \le k \le m$ *and* $0 \le i \le m-k$.

3. *In* $CA\text{-}140_{1-0}(m)$ *if* $x_1 x_2 \cdots x_k x_{k+1} = 1^k 0$ *then* $x(1)_1 x(1)_2 \cdots x(1)_{k-1} x(1)_k x(1)_{k+1} = 1^{k-1}00$ *where* $1 \le k \le m$.

4. *In* $CA\text{-}140_{*-0}(m)$ *if* $x(1)_i x(1)_{i+1} = 11$ *then* $x_i x_{i+1} x_{i+2} = 111$ *where* $0 \le i \le m-2$.

5. *In* $CA\text{-}140_{*-1}(m)$ *if* $x_{m-i} x_{m-i+1} \cdots x_m = 01^i$ *then* $x(1)_{m-i} x(1)_{m-i+1} \cdots x(1)_m = 01^i$ *where* $0 \le i \le m-1$.

PROPOSITION 3.69. *All fixed points of* $CA\text{-}140_{\alpha-0}(m)$ *can be constructed without redundancy as follows;*

1. *If* $x_1 x_2 \cdots x_{m-1}$ *is a fixed point of* $CA\text{-}140_{\alpha-0}(m-1)$ *then* $x_1 x_2 \cdots x_{m-1}0$ *is a fixed point of* $CA\text{-}140_{\alpha-0}(m)$.

2. *If* $x_1 x_2 \cdots x_{m-2}$ *is a fixed point of* $CA\text{-}140_{\alpha-0}(m-2)$ *then* $x_1 x_2 \cdots x_{m-2}01$ *is a fixed point of* $CA\text{-}140_{\alpha-0}(m)$.

By lemma 3.68 5 we can regard transition behaviors of the configurations x such that $x_{m-i} x_{m-i+1} \cdots x_m = 01^i$ in CA-$140_{\alpha-1}(m)$ as those of CA-$140_{\alpha-0}(m-i-1)$. So the following theorem results from lemma 3.68 and prop. 3.69.

THEOREM 3.70.

- $CA\text{-}140_{0-0}(m)$ *has only fixed points and for the number of those fixed point the following formula holds;*

$$\gamma_1(m) = \gamma_1(m-1) + \gamma_1(m-2)$$

  And its transient length is $m-1$.

- $CA\text{-}140_{1-0}(m)$ *has only fixed point and for the number of those fixed point the following holds;*

$$\gamma_1(m) = \gamma_1(m-1) + \gamma_1(m-2)$$

  And its transient length is m.

- $CA\text{-}140_{0-1}(m)$ *has only fixed point and for the number of those fixed points the following holds;*

$$\gamma_1(m) = 2\gamma_1(m-1) - \gamma_1(m-3).$$

  And its transient length is $m-2$.

- $CA\text{-}140_{1-1}(m)$ *has only fixed point and for the number of those fixed points the following holds;*

$$\gamma_1(m) = 2\gamma_1(m-1) - \gamma_1(m-3).$$

  And its transient length is $m-1$.

3.30. CA with Rule 142

In this subsection we introduce behaviors of CA with rule 142. The symmetric, reverse and symmetric reverse rule of rule 142 are rule 212, rule 142 and rule 212 respectively. Inokuchi(1999) showed the following theorem.

THEOREM 3.71.

- *CA*-$142_{\alpha-\alpha}(m)$ *has fixed points only and for the number of those fixed points the following holds;*
$$\gamma_1(m) = \left\lceil \frac{m+3}{2} \right\rceil.$$
And its transient length is m.

- *CA*-$142_{\alpha-\overline{\alpha}}(m)$ *has fixed points only and for the number of those fixed points the following holds;*
$$\gamma_1(m) = \left\lceil \frac{m+2}{2} \right\rceil.$$
And its transient length is m.

3.31. CA with Rule 160

In this subsection we discuss behaviors of CA with rule 160. The symmetric, reverse and symmetric reverse rule of rule 160 are rule 160, rule 250 and rule 250 respectively. The following behaviors are found.

LEMMA 3.72. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of CA*-$160_{*-0}(m)$ *and CA*-$160_{0-*}(m)$.
2. *The configurations* 0^m *and* 1^m *are two fixed points of CA*-$160_{1-1}(m)$.
3. *The configuration c is on a 2-cycle of CA*-$160_{1-1}(m)$ *if and only if neither* 11 *nor* 00 *appears in c.*
4. *In CA*-$160_{0-*}(m)$ $c(k)_1c(k)_2\cdots c(k)_k = 0^k$.
5. *In CA*-$160_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$.
6. *In CA*-$160_{1-1}(m)$ *if* $x(1) = 1^m$ *then* $x = 1^m$.
7. *Let* $x \neq 0^m$. *In CA*-$160_{1-1}(m)$ *the following hold;*
   - *If* $x_1x_2\cdots x_k = 0^k$ *then* $x(1)_1x(1)_2\cdots x(1)_{k+1} = 0^{k+1}$.
   - *If* $x_{i+1}x_{i+2}\cdots x_{i+k} = 0^k$ *then* $x(1)_ix(1)_{i+1}\cdots x(1)_{i+k+1} = 0^{k+2}$ *where* $1 \leq i \leq m-k-1$.
   - *If* $x_{m-k+1}x_{m-k+2}\cdots x_m = 0^k$ *then* $x(1)_{m-k}x(1)_{m-k+1}\cdots x(1)_m = 0^{k+1}$.

Where $2 \le k \le m-1$.

8. *Let no* 00 *appear in* x. *In* $CA\text{-}160_{1-1}(m)$ *if* $x(1)_i x(1)_{i+1} = 00$ *then* $x_{i-1}x_i x_{i+1} x_{i+2} = 0110$ *where* $2 \le i \le m-2$.
9. *In* $CA\text{-}160_{1-1}(m)$ *if* $x(1)_i x(1)_{i+1} = 11$ *then* $x_i x_{i+1} = 11$ *where* $1 \le i \le m-1$.
10. *In* $CA\text{-}160_{1-1}(m)$ *if* $x_i x_{i+1} \cdots x_{i+k+1} = 01^k 0$ *then* $x(1)_{i+1} x(1)_{i+2} \cdots x(1)_{i+k} = 01^{k-2}0$ *where* $1 \le i \le m-k-1$ *and* $2 \le k \le m-2$.
11. *In* $CA\text{-}160_{1-1}(m)$ *if* $x_1 x_2 \cdots x_{k+1} = 1^k 0$ *then* $x(1)_1 x(1)_2 \cdots x(1)_k = 1^{k-1}0$ *where* $2 \le k \le m-1$
12. *In* $CA\text{-}160_{1-1}(m)$ *if* $x_{m-k} x_{m-k+1} \cdots x_m = 01^k$ *then* $x(1)_{m-k+1} x(1)_{m-k+2} \cdots x(1)_m = 01^{k-1}$ *where* $2 \le k \le m-1$.

So the following theorem results from the above lemma.

THEOREM 3.73.

- $CA\text{-}160_{1-1}(m)$ *has 2 fixed points and a 2-cycle. And its transient length is* $m-2$.
- $CA\text{-}160_{0-0}(m)$ *has a unique fixed point. And its transient length is as follows;*

$$H(m) = \left\lceil \frac{m+1}{2} \right\rceil.$$

- $CA\text{-}160_{\alpha-\overline{\alpha}}(m)$ *has a unique fixed point. And its transient length is* m.

3.32. CA with Rule 162

In this subsection we discuss behaviors of CA with rule 162. The symmetric, reverse and symmetric reverse rule of rule 162 are rule 176, rule 186 and rule 242 respectively. The following behaviors are found.

LEMMA 3.74. *Let* c *be an arbitrary configuration and* x *a configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}162_{*-0}(m)$
2. *The configuration* 1^m *is a fixed point of* $CA\text{-}162_{*-1}(m)$
3. *The configuration* x *is on a 2-cycle of* $CA\text{-}162_{*-1}(m)$. *if and only if neither* 00 *nor* 11 *appears in* x.
4. *In* $CA\text{-}162_{*-0}(m)$ $c(k)_{m-k+1} c(k)_{m-k+2} \cdots c(k)_m = 0^k$ *where* $1 \le k \le m$.
5. *In* $CA\text{-}162_{*-1}(m)$ *if* $x(1) = 1^m$ *then* $x = 1^m$.
6. *In* $CA\text{-}162_{*-1}(m)$ *if* $x_{m-i-j+1} x_{m-i-j+2} \cdots x_m = (01)_i^* 1^j$ *then* $x(1)_{m-i-j} x(1)_{m-i-j+1} \cdots x(1)_m = (01)_{i+2}^* 1^{j-1}$ *where* $1 \le i \le m-2$ *and* $1 \le j \le m-i-1$.

7. *In* $CA\text{-}162_{*-1}(m)$ *if* $x_{m-i+1}x_{m-i+2}\cdots x_m = (01)^*_i$ *then* $x(1)_{m-i}x(1)_{m-i+1}\cdots x(1)_m = (01)^*_{i+1}$ *where* $1 \le i \le m-1$.
8. *In* $CA\text{-}162_{*-1}(m)$ *if* $x = (01)^*_i 1^{m-i}$ *then* $x(1) = (10)^*_{i+1}1^{m-i-1}$.
9. *In* $CA\text{-}162_{*-1}(m)$ *if* $x = (10)^*_i 1^{m-i}$ *then* $x(1) = (01)^*_{i+1}1^{m-i-1}$.

The following theorem results from the above lemma.

THEOREM 3.75.

- $CA\text{-}162_{1-1}(m)$ *has a unique fixed point and a unique 2-cycle. And its transient length is* $m-1$.
- $CA\text{-}162_{0-1}(m)$ *has a unique 2-cycle. And its transient length is* $m-1$.
- $CA\text{-}162_{*-0}(m)$ *has a unique fixed point. And its transient length is* m.

3.33. CA with Rule 168

In this subsection we discuss behaviors of CA with rule 168. The symmetric, reverse and symmetric reverse rule of rule 168 are rule 224, rule 234 and rule 248 respectively. The following behaviors are found.

LEMMA 3.76. *Let c be an arbitrary configuration. The following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}168_{*-0}(m)$.
2. *The configuration c is a fixed point of* $CA\text{-}168_{0-1}(m)$ *if and only if no* 10 *appears in c.*
3. *The configuration c is a fixed point of* $CA\text{-}168_{1-1}(m)$ *if and only if c satisfies the following conditions;*
   - *No* 10 *appears in c.*
   - $c_1c_2 \neq 01$.
4. *In* $CA\text{-}168_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ *where* $1 \le k \le m$.
5. *In* $CA\text{-}168_{*-1}(m)$ *if* $x(1)_i x(1)_{i+1} = 10$ *then* $x_{i+1}x_{i+2} = 10$ *where* $1 \le i \le m-2$.
6. *In* $CA\text{-}168_{*-1}(m)$ *no* 10 *appears in* $c(k)_{m-k}c(k)_{m-k+1}\cdots c(k)_m$ *where* $1 \le k \le m-1$.
7. *In* $CA\text{-}168_{1-1}(m)$ $c(m)_1c(m)_2 \neq 01$.

The following theorem results from the above lemma.

THEOREM 3.77.

- $CA\text{-}168_{*-0}(m)$ *has a unique fixed point. And its transient length is* m.
- $CA\text{-}168_{0-1}(m)$ *has* $m+1$ *fixed points. And its transient length is* $m-1$.
- $CA\text{-}168_{1-1}(m)$ *has* m *fixed points. And its transient length is* m.

3.34. CA with Rule 170

In this subsection we discuss behaviors of CA with rule 170. The symmetric, reverse and symmetric reverse rule of rule 170 are rule 240, rule 170 and rule 240 respectively. The following behaviors are found.

LEMMA 3.78. *Let c be an arbitrary configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}170_{*-0}(m)$.
2. *The configuration* 1^m *is a fixed point of* $CA\text{-}170_{*-1}(m)$.
3. *In* $CA\text{-}170_{*-0}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 0^k$ *where* $1 \le k \le m$.
4. *In* $CA\text{-}170_{*-1}(m)$ $c(k)_{m-k+1}c(k)_{m-k+2}\cdots c(k)_m = 1^k$ *where* $1 \le k \le m$.

So the following theorem results from the above lemma.

THEOREM 3.79. $CA\text{-}170_{*-*}(m)$ *has a unique fixed point. And its transient length is* m.

3.35. CA with Rule 178

In this subsection we discuss behaviors of CA with rule 178. The symmetric, reverse and symmetric reverse rule of rule 178 are rule 178 each. The following behaviors are found.

LEMMA 3.80. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configuration* 0^m *is a fixed point of* $CA\text{-}178_{0-0}(m)$.
2. *The configuration x is on a 2-cycle of* $CA\text{-}178_{0-\beta}(m)$ *if and only if x satisfies the following conditions;*
   - *Neither* 111 *nor* 000 *appears in* $0c\beta$.
   - $x_1 \ne x_2$.
   - $x_{m-1} \ne x_m$.
3. *In* $CA\text{-}178_{0-0}(m)$ *if* $\delta(c) = 0^m$ *then* $c = 0^m$.
4. *In* $CA\text{-}178_{0-*}(m)$ *if* $x(1)_{i-1} = x(1)_i = x(1)_{i+1}$ *then* $x_{i-1}x_ix_{i+1} = x(1)_{i-1}x(1)_ix(1)_{i+1}$ *where* $1 \le i \le m$.
5. *In* $CA\text{-}178_{0-0}(m)$ *if* $x_{m-k}x_{m-k+1}\cdots x_m = 10^k$ *then* $x(1)_{m-k+1}x(1)_{m-k+2}\cdots c_m = 10^{k-1}$ *where* $1 \le k \le m-1$.
6. *In* $CA\text{-}178_{0-*}(m)$ *if* $x_ix_{i+1}\cdots x_{i+k+1} = 01^k0$ *then* $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+k} = 01^{k-2}0$ *where* $2 \le k \le m$ *and* $0 \le i \le m-k$.

7. *In* $CA\text{-}178_{0-*}(m)$ *if* $x_i x_{i+1} \cdots x_{i+k+1} = 10^k 1$ *then* $x(1)_{i+1} x(1)_{i+2} \cdots x(1)_{i+k} = 10^{k-2}1$ *where* $2 \leq k \leq m-1$ *and* $1 \leq i \leq m-k$.

8. *In* $CA\text{-}178_{0-*}(m)$ *if* $x_1 x_2 \cdots x_{k+1} = 0^k 1$ *then* $x(1)_1 x(1)_2 \cdots x(1)_k = 0^{k-1}1$ *where* $1 \leq k \leq m$.

9. *In* $CA\text{-}178_{0-1}(m)$ *if* $x_{m-k} x_{m-k+1} \cdots x_m = 01^k$ *then* $x(1)_{m-k+1} x(1)_{m-k+2} \cdots x(1)_m = 01^{k-1}$ *where* $1 \leq k \leq m$.

10. *In* $CA\text{-}178_{0-*}(m)$ $c(1)_1 c(1)_2 \neq 11$.

11. *In* $CA\text{-}178_{*-0}(m)$ $c(1)_{m-1} c(1)_m \neq 11$.

PROPOSITION 3.81. *All 2-cycles of* $CA\text{-}178_{0-\beta}(m)$ *can be constructed without redundancy as follows;*

1. *If* $1x_3x_4 \cdots x_m$ *and* $0y_3y_4 \cdots y_m$ *from a 2-cycle of* $CA\text{-}178_{0-\beta}(m-1)$ *then* $01x_3x_4 \cdots x_m$ *and* $10y_3y_4 \cdots y_m$ *form a 2-cycle of* $CA\text{-}178_{0-\beta}(m)$

2. *If* $1x_4x_5 \cdots x_m$ *and* $0y_4y_5 \cdots y_m$ *form a 2-cycle of* $CA\text{-}178_{0-\beta}(m-2)$ *then* $011x_4x_5 \cdots x_m$ *and* $100y_4y_5 \cdots y_m$ *form a 2-cycle of* $CA\text{-}178_{0-\beta}(m)$

Since rule 178 and its reverse rule are the same CA-$178_{1-1}(m)$ and CA-$178_{1-0}(m)$ are isomorphic to CA-$178_{0-0}(m)$ and CA-$178_{0-1}(m)$ respectively. So the following theorem results from lemma 3.80 and prop. 3.81.

THEOREM 3.82.

- $CA\text{-}178_{\alpha-\alpha}(m)$ *has a unique fixed point and 2-cycles, and for the number of those 2-cycles the following hold;*

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

  And its transient length is $m-2$.

- $CA\text{-}178_{\alpha-\bar{\alpha}}(m)$ *has only 2-cycles and for the number of those 2-cycles the following holds;*

$$\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2).$$

  And its transient length is $m-2$.

3.36. CA with Rule 200

In this subsection we discuss behaviors of CA with rule 200. The symmetric, reverse and symmetric reverse rule of rule 200 are rule 200, rule 236 and rule 236 respectively. The following behaviors are found.

LEMMA 3.83. *The following hold;*

1. *The configuration x is a fixed point of CA-$200_{\alpha-\beta}(m)$ if and only if no* 010 *appears in* $\alpha x\beta$.

2. *For any configuration c in CA-$200_{\alpha-\beta}(m)$ no* 010 *appears in* $\alpha c(1)\beta$.

PROPOSITION 3.84. *All fixed points of CA-$200_{0-\beta}(m)$ can be constructed without redundancy by followings;*

1. *If $x_2x_3\cdots x_m$ is a fixed point of CA-$200_{0-\beta}(m-1)$, then $0x_2x_3\cdots x_m$ is a fixed point of CA-$200_{0-\beta}(m)$.*

2. *If $x_3x_4\cdots x_m$ is a fixed point of CA-$200_{0-\beta}(m-2)$, then $11x_3x_4\cdots x_m$ is a fixed point of CA-$200_{0-\beta}(m)$.*

3. *If $x_5x_6\cdots x_m$ is a fixed point of CA-$200_{0-\beta}(m-4)$, then $1110x_5x_6\cdots x_m$ is a fixed point of CA-$200_{0-\beta}(m)$.*

PROPOSITION 3.85. *All fixed points of CA-$200_{1-1}(m)$ can be constructed without redundancy by followings;*

1. *If $x_2x_3\cdots x_m$ is a fixed point of CA-$200_{1-1}(m-1)$, then $1x_2x_3\cdots x_m$ is a fixed point of CA-$200_{1-1}(m)$.*

2. *If $x_{i+3}x_{i+4}\cdots x_m$ is a fixed point of CA-$200_{1-1}(m-i-2)$, then $0^i11x_{i+3}x_{i+4}\cdots x_m$ is a fixed point of CA-$200_{1-1}(m)$.*

3. *The configurations $0^{m-2}11$, $0^{m-1}1$ and 1^m are fixed points of CA-$200_{1-1}(m)$.*

Since rule 200 and its symmetric rule are the same, CA-$200_{0-1}(m)$ is isomorphic to CA-$200_{1-0}(m)$. So the following theorem results from lemma 3.83, prop. 3.84 and prop. 3.85.

THEOREM 3.86. *CA-$200_{*-*}(m)$ has fixed points only and for the number of those fixed points the following formula holds;*

$$\gamma_1(m) = \gamma_1(m-1) + \gamma_1(m-2) + \gamma_1(m-4).$$

And its transient length is 1.

3.37. CA with Rule 204

In this subsection we discuss behaviors of CA with rule 204. Rule 204 is the identity transition function, that is, $f(u,v,w) = v$ for any $u,v,w \in \{0,1\}$. The symmetric, reverse and symmetric reverse rule of rule 204 are rule 204 each. So we get the following theorem easily.

THEOREM 3.87. *CA-$204_{*-*}(m)$ has 2^m fixed points And its transient length is 0.*

3.38. CA with Rule 232

In this subsection we discuss behaviors of CA with rule 232. The symmetric, reverse and symmetric reverse rule of rule 232 are rule 232 each. The following behaviors are found.

LEMMA 3.88. *Let c be an arbitrary configuration and x a configuration. Then the following hold;*

1. *The configuration x is a fixed point of CA-$232_{\alpha-\beta}(m)$ if and only if neither* 010 *nor* 101 *appears in $\alpha x\beta$.*
2. *In CA-$232_{*-*}(m)$ if $x_i = x_{i+1}$ then $x(1)_i = x(1)_{i+1} = x_i$ where $0 \le i \le m$.*
3. *In CA-$232_{0-*}(m)$ $c(1)_1c(1)_2 \ne 10$.*
4. *In CA-$232_{*-0}(m)$ $c(1)_{m-1}c(1)_m \ne 01$.*
5. *In CA-$232_{1-*}(m)$ $c(1)_1c(1)_2 \ne 01$.*
6. *In CA-$232_{*-1}(m)$ $c(1)_{m-1}c(1)_m \ne 10$.*
7. *In CA-$232_{*-*}(m)$ if $x(1)_{i-1}x(1)_ix(1)_{i+1} = 010$ then $x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2} = 01010$ where $2 \le i \le m-1$.*
8. *In CA-$232_{*-*}(m)$ if $x(1)_{i-1}x(1)_ix(1)_{i+1} = 101$ then $x_{i-2}x_{i-1}x_ix_{i+1}x_{i+2} = 10101$ where $2 \le i \le m-1$.*
9. *In CA-$232_{*-*}(m)$ if $x_ix_{i+1}\cdots x_{i+j} = (10)^*_{j+1}$ then $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+j-1} = (10)^*_{j-1}$ where $i \ge 0$, $j \ge 2$ and $i+j \le m+1$.*
10. *In CA-$232_{*-*}(m)$ if $x_ix_{i+1}\cdots x_{i+j} = (01)^*_{j+1}$ then $x(1)_{i+1}x(1)_{i+2}\cdots x(1)_{i+j-1} = (01)^*_{j-1}$ where $i \ge 0$, $j \ge 2$ and $i+j \le m+1$.*
11. *In CA-$232_{0-\beta}(m)$ neither* 010 *nor* 101 *appears in $0c([\frac{m+1}{2}])\beta$.*

PROPOSITION 3.89. *All fixed points of CA-$232_{0-\beta}(m)$ can be constructed without redundancy as follows;*

1. *If the configuration $x_2x_3\cdots x_m$ is a fixed point of CA-$232_{0-\beta}(m-1)$, then the configuration $0x_2x_3\cdots x_m$ is a fixed point of CA-$232_{0-\beta}(m)$*
2. *If the configuration $x_{i+1}x_{i+2}\cdots x_m$ is a fixed point of CA-$232_{0-\beta}(m-i)$, then the configuration $1^{i-2}00x_{i+1}x_{i+2}\cdots x_m$ is a fixed point of CA-$232_{0-\beta}(m)$ where $4 \le i \le m-1$.*
3. *The configurations 1^m and $1^{m-2}00$ are fixed points of CA-$232_{0-\beta}(m)$.*
4. *The configuration $1^{m-1}0$ is a fixed point of CA-$232_{0-0}(m)$.*

Since rule 232 and its reverse rule are the same CA-$232_{0-0}(m)$ and CA-$232_{0-1}(m)$ are isomorphic to CA-$232_{1-1}(m)$ and CA-$232_{1-0}(m)$ respectively. So the following theorem results from lemma 3.88 and prop. 3.89.

THEOREM 3.90. *CA-$232_{*-*}(m)$ has only fixed points and for the number of those fixed points the following formula holds;*

$$\gamma_1(m) = 2\gamma_1(m-1) - \gamma_1(m-2) + \gamma_1(m-4).$$

The transient length of CA-$232_{\alpha-\alpha}(m)$ is as follows;

$$H(m) = \left[\frac{m+1}{2}\right],$$

and that of CA-$232_{\alpha-\overline{\alpha}}(m)$ is as follows;

$$H(m) = \left[\frac{m}{2}\right].$$

4. Conclusion

In this paper we investigated behaviors of each 1-D finite cellular automata with threshold triplet local transition rules of fixed boundary conditions. We got two main results; One is that period lengths of limit cycles are bounded by 4 (theorem 3.1), and the other is that transient lengths are bounded by $3 \times (\textit{cell-size}) - 4$ (theorem 3.2).

There are some future works. The first is to investigate another 1-D cellular automata with triplet local transition rules and make a database of behaviors of cellular automata. The second is to apply the results to analysing cellular automata with another structure. The last is to analyze cellular automata closely, for example, to decide transition diagrams.

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