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CLUSTERING BY A FUZZY METRIC : APPLICATIONS TO THE CLUSTER-MEDIAN PROBLEM

By

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Abstract

This paper is the second part of our study of the clustering problem with a fuzzy metric. The fuzzy metric between any two elements will be constructed from the multi-dimensional fuzzy data available for each element and our clustering criterion is to minimize the total sum of the fuzzy distances from all the elements in a cluster to its median, called the cluster-median problem. An optimal clustering is concretely sought by applying an interval method by α-cuts of monotone fuzzy numbers. Also, a numerical example is given.

1. Introduction

Given a set N, of n entries, and the metric d_{ij} of each $i, j \in N$, we wish to partition this set into a number of subsets (called clusters), in some sense optimality. In this mathematical clustering problem, the required metrics d_{ij} $(i, j \in N)$ are assumed to be exact while in practice these data are estimated and approximated. So, it naturally includes imprecision or ambiguity in the sense of fixing approximate values. In order to deal with these vagueness and to answer the flexible requirements, Kamimura and Kurano(1999) has used a fuzzy set representation for clustering problems, in which the metric d_{ij} 's are considered by fuzzy metric. The fuzzy metric has been discribed by a monotone fuzzy number, whose idea was first given by Kramosil and Michalek(1975) and developed by George and Veeramani(1994).

This paper considers the some problem as that discussed by Kamimura and Kurano(1999) and propose the new method for constructing the fuzzy metric and solve a fuzzy version of another clustering problem. In a fuzzy treatment of a clustering problem, we must suppose that for each element belonging to the set N there is a multi-dimensional fuzzy data giving the measurements on the characteristics for attributes that we wish to use as the input information for grouping. Here, the fuzzy

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metric between any two elements will be constructed from the multi-dimensional fuzzy data available for each elements, and our clustering criterion is to minimize the total sum of the fuzzy distances from all the elements in a cluster to its median, called the cluster-median (or homogeneous) problem (cf. Arthanari and Dodge(1981), Mulvey and Crowder(1979), Rao(1971)). An optimal clustering is concretely sought by applying an interval method (cf. Dubois and Prade(1979), Kurano et al(1992), Li and Yam(1995)) by α-cuts of monotone fuzzy numbers. Also, a numerical example is given to illustrate our approach. For various application of fuzzy theory to clustering problems, see, for example, Ruspini(1973), Sato et al(1995, 1997), Watada(1982), Zadeh(1971).

The mathematical facts described in the following lemma is used in the sequel, whose proof is easy and leaved to the reader.

LEMMA 1.1.

(i) *If the map* $h : [0,\infty) \to [0,1]$ *is right-continuous and increasing with* $h(t) = 1$ *for some* $t \geq 0$, $h^{-1} : [0,1] \to [0,\infty)$ *is left-continuous and increasing,where*

$$h^{-1}(\alpha) = \min\{t \in [0,\infty) \mid h(t) \geq \alpha\}, \tag{1.1}$$

(ii) *If the map* $g : [0,1] \to [0,\infty)$ *is left-continuous and increasing, then* $g^{+1} : [0,\infty) \to [0,1]$ *is right-continuous and increasing, where*

$$g^{+1}(t) = \max\{\alpha \in [0,1] \mid g(\alpha) \leq t\}. \tag{1.2}$$

(iii) *For* h *and* g *in* (i) *and* (ii), $(h^{-1})^{+1} = h$ *and* $(g^{+1})^{-1} = g$,

(iv) *For* g *in* (ii) *with* $g(\alpha) \geq a$ *for all* $\alpha \in [0,1]$,

$$(g-a)^{+1}(t) = g^{+1}(t+a) \quad (0 \leq t < \infty).$$

Note that min and max in (1.1) and (1.2) are attainable. In Section 2, we provide some relevant preliminaries on fuzzy metric and propose the method of constructing fuzzy metric from the multi-dimensional fuzzy data. In section 3, we specify a fuzzy version of the cluster-median problem which is solved by an interval method by α-cuts. In section 4, a numerical example is given.

2. Fuzzy Metric and Its Construction

In this section, we shall introduce the concept of a fuzzy metric refering to George and Veermani(1994), Kavba and Seikkala(1984), Kramosil and Michalek(1975), Syau(19 97), whose metric will be constructed from the multi-dimensional fuzzy data. Throughout this paper, we will denote a fuzzy set by the menbership function. For the theory of fuzzy sets, we refer to Zadeh(1971) and Novák(1989).

Let R_+ denote the set of all non-negative real numbers. A monotone fuzzy number on R_+ is a fuzzy set $\tilde{a} : R_+ \to [0,1]$ satisfying the following properties:

(i) $\tilde{a}(t_1) \leq \tilde{a}(t_2)$ for $t_1, t_2 \in R_+$ with $t_1 \leq t_2$,

(ii) $\tilde{a}(t)$ is right-continuous and has at most finite discontinuous points,

(iii) there exists $t \in R_+$ with $\tilde{a}(t) = 1$.

Let $\mathcal{F}_+$ denote the set of all monotone fuzzy numbers on R_+. We note that $\tilde{a}$ with $\tilde{a} \equiv 1$ is denoted by $\tilde{0}$ and $\tilde{0} \in \mathcal{F}_+$. If $\tilde{a}(t) = 0$ if $t < a$, $= 1$ if $t \geq a$ for some positive number a, $\tilde{a}$ means the real number a, which is denoted by $\tilde{1}_a$. The addition and the scalar multiplication in $\mathcal{F}_+$ are defined as follows (cf. Nguyeen(1978)) : For $\tilde{a}, \tilde{b} \in \mathcal{F}_+$ and $\lambda \in R_+$, define

$$(\tilde{a} + \tilde{b})(t) := \sup_{\substack{t_1, t_2 \in R_+ \\ t_1 + t_2 = t}} \tilde{a}(t_1) \wedge \tilde{b}(t_2)$$

and

$$(\lambda\tilde{a})(t) := \begin{cases} \tilde{a}(t/\lambda) & \text{if } \lambda > 0 \\ I_{R_+}(t) & \text{if } \lambda = 0 \end{cases} \qquad (t \in R_+),$$

where $a \wedge b = \min\{a, b\}$ and $I_A(\cdot)$ is the classical indicator function.
The α-cut ($\alpha \in [0,1]$) of the monotone fuzzy number $\tilde{a}$ is defined as

$$\tilde{a}_\alpha := \{t \in R_+ \mid \tilde{a}(t) \geq \alpha\} \quad (\alpha > 0) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{t \in R_+ \mid \tilde{a}(t) > 0\},$$

where cl denotes the closure of the set.
It is easily seen that , for $\alpha \in [0,1]$,

$$(\tilde{a} + \tilde{b})_\alpha = \tilde{a}_\alpha + \tilde{b}_\alpha \quad \text{and} \quad (\lambda\tilde{a})_\alpha = \lambda\tilde{a}_\alpha \quad (\lambda \in (0,1]).$$

Here, the operation on sets is defined as $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for $A, B \subset R_+$.
Then, it is easily shown that for any $\tilde{a}, \tilde{b} \in \mathcal{F}_+$, $\tilde{a} + \tilde{b} \in \mathcal{F}_+$ and $\lambda\tilde{a} \in \mathcal{F}_+$ ($\lambda \in R_+$). The following results can be easily proved from Lemma 1.1 (cf. Kamimura and Kurano(1999)).

LEMMA 2.1.

(i) *For any $\tilde{a} \in \mathcal{F}_+$, $\tilde{a}_\alpha = [\tilde{a}^{-1}(\alpha), \infty)$ for each $\alpha \in [0,1]$, where $\tilde{a}^{-1} : [0,1] \to [0,\infty)$ is defined in (1.1).*

(ii) *Conversely, for any map $h : [0,1] \to [0,\infty)$ such that h is left-continuous and increasing, $h^{+1} \in \mathcal{F}_+$ and $(h^{+1})_\alpha = [h(\alpha), \infty)$ for all $\alpha \in [0,1]$, where $h^{+1} : [0,\infty) \to [0,1]$ is defined in* (1.2).

(iii) *For any $\tilde{a} \in \mathcal{F}_+$, $(\tilde{a}^{-1})^{+1} = \tilde{a}$.*

A partial order on $\mathcal{F}_+$ is defined as

$$\tilde{a} \preceq \tilde{b} \quad (\tilde{a}, \tilde{b} \in \mathcal{F}_+) \quad \text{if } \tilde{a}(t) \geq \tilde{b}(t) \quad \text{for all } t \in R_+$$

(cf. Kramosil and Michalek(1975)). Then, it clearly holds that for $\tilde{a}, \tilde{b} \in \mathcal{F}_+$, $\tilde{a} \preceq \tilde{b}$ is equivalent to $\tilde{a}^{-1}(\alpha) \leq \tilde{b}^{-1}(\alpha)$ for all $\alpha \in [0,1]$.

Let $N = \{1, 2, \cdots, n\}$ be a finite set, of n-entries. The map $\tilde{d} : N \times N \to \mathcal{F}_+$ is called a fuzzy metric function if the following conditions (i)-(iii) are fulfilled (cf. George and Veermani(1994), Kramosil and Michalek(1975), Syau(1997)):

(i) $\tilde{d}(i,j) = 0 \quad \text{iff} \quad i = j$,

(ii) $\tilde{d}(i,j) = \tilde{d}(j,i)$ for $i, j \in N$,

(iii) for any $t_1, t_2 \in R_+$ and $i, j, k \in N$, $\tilde{d}(i,j)(t_1 + t_2) \geq \tilde{d}(i,k)(t_1) \wedge \tilde{d}(k,j)(t_2)$.

Note that $\tilde{d}(i,j)(t)$ can be thought of as the degree of nearness between i and j with respect to t. If $\tilde{d}(i,j)(t) = 0$ if $t < t_0$, $= 1$ if $t \geq t_0$ for some $t_0 \in R_+$, then $\tilde{d}(i,j)$ means non-fuzzy metric t_0 between i and j.

In the following lemma, it is shown that the last condition (iii) for a fuzzy metric function means the triangle inequality with respect to the partial order $\preceq$ and the addition on $\mathcal{F}_+$, whose proof is given in Kramosil and Michalek(1975).

LEMMA 2.2. (Kramosil and Michalek(1975)). *In a fuzzy metric function, the condition* (iii) *is equivalent to*

$$\tilde{d}(i,j) \preceq \tilde{d}(i,k) + \tilde{d}(k,j) \tag{2.1}$$

for all $i, j, k \in N$.

Let $\mathcal{F}_+^l$ be the set of all l-dimensional vectors whose elements are in $\mathcal{F}_+$, i.e.,

$$\mathcal{F}_+^l = \left\{ \tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_l) \mid \tilde{a}_i \in \mathcal{F}_+ \quad (1 \leq i \leq l) \right\}.$$

For any $\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_l)$, $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_l) \in \mathcal{F}_+^l$, let $p(\,\cdot \mid \tilde{\mathbf{a}}, \tilde{\mathbf{b}}) : [0,1] \to R_+$ be defined by

$$p(\alpha | \tilde{\mathbf{a}}, \tilde{\mathbf{b}}) = \sup_{0 \leq \alpha' \leq \alpha} \sqrt{\sum_{i=1}^{l} (\tilde{a}_i^{-1}(\alpha') - \tilde{b}_i^{-1}(\alpha'))^2} \qquad (\alpha \in [0,1]). \tag{2.2}$$

Using this function p, we define the fuzzy distance $\|\tilde{\mathbf{a}} - \tilde{\mathbf{b}}\|$ between $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ as follows:

$$\|\tilde{\mathbf{a}} - \tilde{\mathbf{b}}\|(t) = p^{+1}(\,\cdot \mid \tilde{\mathbf{a}}, \tilde{\mathbf{b}})(t) \qquad (0 \leq t < \infty). \tag{2.3}$$

We note that since $\tilde{a}_i^{-1}$ and $\tilde{b}_i^{-1}$ are left continuous and increasing for each i $(1 \leq i \leq l)$, $p(\,\cdot \mid \tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ is also so, such that from Lemma 2.1(ii) it holds $\|\tilde{\mathbf{a}} - \tilde{\mathbf{b}}\| \in \mathcal{F}_+$.

We will make use of the following easy lemmma.

LEMMA 2.3. *Let* $p, q : [0,1] \to R_+$ *be left continuous and increasing. Then, the following (i) and (ii) hold.*

(i) $(p+q)^{+1} = p^{+1} + q^{+1}$.

(ii) *if* $p \leq q$ *then* $p^{+1} \preceq q^{+1}$.

PROOF. By lemma 2.1(ii), p^{+1}, q^{+1}, $(p+q)^{+1} \in \mathcal{F}_+$. From Lemma 1.1(i), the left-side extreme point of $((p+q)^{+1})_\alpha$ is $((p+q)^{+1})^{-1}(\alpha)$ and by Lemma 1.1(ii),

$$((p+q)^{+1})^{-1}(\alpha) = (p+q)(\alpha) = p(\alpha) + q(\alpha).$$

Concerning the left-side extreme point of $(p^{+1}+q^{+1})_\alpha = p^{+1}_\alpha + q^{+1}_\alpha$,

$$(p^{+1})^{-1}(\alpha) + (q^{+1})^{-1}(\alpha) = p(\alpha) + q(\alpha).$$

Thus, $((p+q)^{+1})_\alpha = (p^{+1}+q^{+1})_\alpha$ for all $\alpha \in [0,1]$, which implies (i). Also, (ii) obviously follows. □

The metric properties of $\|\cdot\|$ are given in the following.

THEOREM 2.4. *For* $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}} \in \mathcal{F}^l_+$, *then*

(i) $\|\tilde{\mathbf{a}} - \tilde{\mathbf{a}}\| = \tilde{\mathbf{0}}$,

(ii) $\|\tilde{\mathbf{a}} - \tilde{\mathbf{b}}\| = \|\tilde{\mathbf{b}} - \tilde{\mathbf{a}}\|$

(iii) $\|\tilde{\mathbf{a}} - \tilde{\mathbf{b}}\| \preceq \|\tilde{\mathbf{a}} - \tilde{\mathbf{c}}\| + \|\tilde{\mathbf{c}} - \tilde{\mathbf{b}}\|$.

PROOF. By (2.2), $p(\,\cdot\mid \tilde{\mathbf{a}}, \tilde{\mathbf{a}}) \equiv 0$, which implies $\|\tilde{\mathbf{a}} - \tilde{\mathbf{a}}\| = p^{+1}(\,\cdot\mid \tilde{\mathbf{a}}, \tilde{\mathbf{a}}) \equiv \tilde{\mathbf{0}}$. From the definition, (ii) follows. For (iii), it holds that

$$\begin{aligned} p(\alpha|\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) &= \sup_{0\leq\alpha'\leq\alpha} \sqrt{\sum_{i=1}^{l} (\tilde{a}_i^{-1}(\alpha') - \tilde{b}_i^{-1}(\alpha'))^2} \\ &\leq \sup_{0\leq\alpha'\leq\alpha} \sqrt{\sum_{i=1}^{l} (\tilde{a}_i^{-1}(\alpha') - \tilde{c}_i^{-1}(\alpha'))^2} + \sup_{0\leq\alpha'\leq\alpha} \sqrt{\sum_{i=1}^{l} (\tilde{c}_i^{-1}(\alpha') - \tilde{b}_i^{-1}(\alpha'))^2} \\ &= p(\alpha\,|\tilde{\mathbf{a}}, \tilde{\mathbf{c}}) + p(\alpha\,|\tilde{\mathbf{c}}, \tilde{\mathbf{b}}). \end{aligned}$$

Applying Lemma 2.3, we get

$$\begin{aligned} p^{+1}(\,\cdot\,|\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) &\preceq (p(\,\cdot\,|\tilde{\mathbf{a}}, \tilde{\mathbf{c}}) + p(\,\cdot\,|\tilde{\mathbf{c}}, \tilde{\mathbf{b}}))^{+1} \\ &= p^{+1}(\,\cdot\,|\tilde{\mathbf{a}}, \tilde{\mathbf{c}}) + p^{+1}(\,\cdot\,|\tilde{\mathbf{c}}, \tilde{\mathbf{b}}) \\ &= \|\tilde{\mathbf{a}} - \tilde{\mathbf{c}}\| + \|\tilde{\mathbf{c}} - \tilde{\mathbf{b}}\|. \end{aligned}$$

This complete the proof. □

The following corollary shows that the fuzzy distance defined in the above is reasonable and thought of as an extension of Euclidian metric.

COROLLARY 2.5.

(i) *For* $\tilde{1}_{\mathbf{a}} = (\tilde{1}_{a_1}, \tilde{1}_{a_2}, \ldots, \tilde{1}_{a_l})$, $\tilde{1}_{\mathbf{b}} = (\tilde{1}_{b_1}, \tilde{1}_{b_2}, \ldots, \tilde{1}_{b_l})$ *with* $\mathbf{a} = (a_1, a_2, \ldots, a_l), \mathbf{b} = (b_1, b_2, \ldots, b_l) \in R^l_+$, *it holds that*

$$\|\tilde{1}_{\mathbf{a}} - \tilde{1}_{\mathbf{b}}\| = \tilde{1}_{\|\mathbf{a}-\mathbf{b}\|}, \tag{2.4}$$

where $\|\mathbf{a} - \mathbf{b}\| = \sqrt{\sum_{i=1}^{l}(a_i - b_i)^2}$.

(ii) *In the case of* $l = 1$, *we have the following :*

(a) *For any* $\tilde{a} \in \mathcal{F}_+$ *and* $b \in R_+$ *with* $\tilde{a} \succeq \tilde{1}_b$,

$$\|\tilde{a} - \tilde{1}_b\|(t) = \tilde{a}(t + b); \tag{2.5}$$

(b) *For* $\tilde{a}$, $\tilde{b} \in \mathcal{F}_+$ *and* c, $d \in R_+$ *with* $\tilde{1}_c \preceq \tilde{a}$, $\tilde{b} \preceq \tilde{1}_d$,

$$\|\tilde{a} - \tilde{b}\| \preceq \tilde{1}_{(d-c)}. \tag{2.6}$$

PROOF. For (i), we observe that $\tilde{1}_{a_i}^{-1}(\alpha) = 0$ if $\alpha = 0$, $= a_i$ if $\alpha > 0$ and $\tilde{1}_{b_i}^{-1}(\alpha) = 0$ if $\alpha = 0$, $= b_i$ if $\alpha > 0$, so that $p(\ \alpha|\ \tilde{1}_{\mathbf{a}}, \tilde{1}_{\mathbf{b}}) = 0$ if $\alpha = 0$, $= \|\mathbf{a} - \mathbf{b}\|$ if $\alpha > 0$, which obviously implies (2.4). Applying Lemma 1.1(iv), (ii)(a) is easily proved. Also, (ii)(b) clearly holds. □

REMARK. Recently, Voxman(1998) has introduced a fuzzy metric between fuzzy numbers which satisfies the triangle inequality with respect to the order defined by an idea of a reducing function. Here, we have described a metric with fuzziness by a monotone fuzzy number, so that a fuzzy distance defined by (2.3) is different from that of Voxman(1998).

3. The Cluster-median Problem

In this section, we shall formulate the clustering problem, called the cluster-median problem, and find the optimal clustering by an interval method by α-cuts. In the fuzzy treatment of a clustering problem, we can assumed that the l-dimentional fuzzy data $\tilde{\mathbf{X}}_i = (X_{1i}, X_{2i}, \ldots, X_{li}) \in \mathcal{F}^l_+$ is avairable for all $i \in N$. Next, we also require a fuzzy distance $\tilde{d}(i,j)$ between any elements $i, j \in N$, defined by

$$\tilde{d}(i,j) = \|\tilde{\mathbf{X}}_i - \tilde{\mathbf{X}}_j\|, \tag{3.1}$$

where $\|\tilde{\mathbf{X}}_i - \tilde{\mathbf{X}}_j\|$ is defined in Section 2 and $\|\tilde{\mathbf{X}}_i - \tilde{\mathbf{X}}_j\| \in \mathcal{F}_+$. Then, by Lemma 2.2 and Theorem 2.4 we observe that $\tilde{d} : N \times N \to \mathcal{F}_+$ is a fuzzy metric function. Given a subset $A \subset N$, the sum of the fuzzy metric $\tilde{d}(i,j)$ from all the elements in A to $j \in A$ is given by

$$\tilde{d}(A|j) = \sum_{i \in A} \tilde{d}(i,j) \tag{3.2}$$

For any fixed m with $1 \le m \le n$, $J = (J_1, J_2, \ldots, J_m)$ is called m-partion of N if each J_i is non-empty cluster of N, $J_i \cap J_j = \phi$ $(i \ne j)$ and $\cup_{i=1}^{m} J_i = N$. Let $\mathcal{J}_m$ be the set of all m-partition of N.

For each $J = (J_1, J_2, \ldots, J_m) \in \mathcal{J}_m$, the total sum of the fuzzy metric $\tilde{d}(i,j)$ from all the elements in a cluster to a given $j_k \in J_k$ $(1 \le k \le m)$ is given by

$$\tilde{D}(J|j_1, j_2, \ldots, j_m) := \sum_{k=1}^{m} \tilde{d}(J_k|j_k).$$

Note $\tilde{D}(J|j_1, j_2, \ldots, j_m) \in \mathcal{F}_+$.
Then, given a fuzzy metric function $\tilde{d}$ and the number of cluster m, the problem is to minimize $\tilde{D}(J|j_1, j_2, \ldots, j_m)$ over $J = (J_1, J_2, \ldots, J_m) \in \mathcal{J}_m$ and $j_1 \in J_1, j_2 \in J_2, \ldots, j_m \in J_m$ with respect to the partial order $\preceq$ on $\mathcal{F}_+$.

The following lemma is crucial for an approach by α-cuts, whose proof is easily done and omitted.

LEMMA 3.1.

(i) *For any cluster A of N and $j \in A$, $\tilde{d}(A|j)_\alpha = [d_A(\alpha|j), \infty)$ $(0 \le \alpha \le 1)$, where*

$$d_A(\alpha|j) = \sum_{i \in A} d_{ij}(\alpha),$$

$$d_{ij}(\alpha) = \tilde{d}(i,j)^{-1}(\alpha)$$

and

$$\tilde{d}(i,j)_\alpha = [d_{ij}(\alpha), \infty).$$

(ii) *For any clustering $J = (J_1, J_2, \ldots, J_m) \in \mathcal{J}_m$, and $j_k \in J_k$ $(1 \le k \le m)$,*

$$\tilde{D}(J|j_1, j_2, \ldots, j_m)_\alpha = [D_J(\alpha|j_1, j_2, \ldots, j_m), \infty),$$

where

$$D_J(\alpha|j_1, j_2, \ldots, j_m) = \sum_{k=1}^{m} d_{J_k}(\alpha|j_k).$$

In order to minimize $\tilde{D}(J|j_1, j_2, \ldots, j_m)$ as possible under the partial order $\preceq$ on $\mathcal{F}_+$, we conclude from Lemma 3.1 that for each $\alpha \in [0,1]$ we must minimize $D_J(\alpha|j_1, j_2, \ldots, j_m)$ over all $J \in \mathcal{J}_m$, and $j_k \in J_k$ $(1 \le k \le m)$, which motivates us to solve the non-fuzzy α-level cluster-median problem (cf.Arthanari and Dodge(1981), Mulvey and Crowder(1979)) specified by a metric $d_{ij}(\alpha)$ between i and j in N.
We note that $d_{ij}(\alpha)$ is a usual metric between i and j for each $\alpha \in [0,1]$.

The quantity $D(\alpha) := \min_{J\in\mathcal{J}_m} D_J(\alpha)$ is the best that can be achieved, where

$$D_J(\alpha) = \sum_{k=1}^{m} d_{J_k}(\alpha) \tag{3.3}$$

and

$$d_{J_k}(\alpha) = \min_{j_k\in J_k} d_{J_k}(\alpha|j_k). \tag{3.4}$$

And the set of

$$\mathcal{J}_{m,\alpha} := \{J \in \mathcal{J}_m | D(\alpha) = D_J(\alpha)\}.$$

For each $\alpha \in [0,1]$, we can find $D(\alpha)$ and $\mathcal{J}_{m,\alpha}$ with use of integer programming and Lagrangian multiplier (cf.Rao(1971), Mulvey and Crowder(1979)).
From finiteness of discontinuous points of $d_{ij}(\cdot)$, the following lemma is easily shown.

LEMMA 3.2. *There exists a sequense* $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_{l-1} < \alpha_l = 1$ *and* $\mathcal{J}_m^{(i)} \subset \mathcal{J}_m \quad (i = 1, 2, \ldots, l)$ *such that*

$$\mathcal{J}_{m,\alpha} = \mathcal{J}_m^{(i)} \qquad for \ \ \alpha_{i-1} < \alpha < \alpha_i \quad (i = 1, 2, \ldots, l).$$

By Lemma 1.1 and the definition, $D(\cdot) : [0,1] \to [0,\infty)$ is left-continuous and increasing. Thus, by Lemma 2.1, $D^{+1} \in \mathcal{F}_+$. Let $p : [0,1] \to \mathcal{J}_m$ be such that $p(\alpha) \in \mathcal{J}_m^{(i)}$ for $\alpha_{i-1} < \alpha < \alpha_i \ \ (i = 1, 2, \ldots, l)$ and $p(\alpha_i) \in \mathcal{J}_{m,\alpha_i} \ \ (i = 0, 1, \ldots, m)$, where α_i and $\mathcal{J}_m^i$ is as in Lemma 3.2.

For any $J = (J_1, J_2, \ldots, J_m) \in \mathcal{J}_m$ and $\alpha \in [0,1]$, we define a α-median vector, $med(J)(\alpha)$, by $med(J)(\alpha) = (j_1, j_2, \ldots, j_m)$ if $d_{J_k}(\alpha) = d_{J_K}(\alpha|j_k)$ for $k = 1, 2, \ldots, m$, where $d_{J_k}(\alpha)$ is given in (3.4). For simplisity, we put $med(\alpha) := med(p(\alpha))(\alpha)$.

Then, we have the following.

THEOREM 3.3.

(i) $\tilde{D}(J|\ j_1, j_2, \ldots, j_m) \succeq D^{+1}$ *for all* $J = (J_1, J_2, \ldots, J_m) \in \mathcal{J}_m$ *and* $j_k \in J_k$ *(*$1 \le k \le m$*).*

(ii) *If* $D^{+1}(t) = \alpha$*, then* $\tilde{D}(p(\alpha) \mid med(\alpha)(t) = D^{+1}(t)$.

PROOF. For (i), from the definition, $D_J(\alpha|j_1, j_2, \ldots, j_m) \ge D_J(\alpha) \ge D(\alpha)$, thus, $D_J^{+1}(\ \cdot\ |j_1, j_2, \ldots, j_m) \le D^{+1}(t)$. Since $D_J^{+1}(\ \cdot\ |j_1, j_2, \ldots, j_m)(t) = \tilde{D}(J|j_1, j_2, \ldots, j_m)$, (i) follows. For (ii), let (t, α) be such that $D^{+1}(t) = \alpha$. Then, by (i), $\tilde{D}(p(\alpha)|\ med(\alpha))(t) \le \alpha$. Suppose that $\tilde{D}(p(\alpha)|med(\alpha))(t) < \alpha$. Since $\tilde{D}(p(\alpha)|med(\alpha)(\cdot)$ is right-continuous and increasing, $D_{p(\alpha)}(\alpha|med(\alpha)\ ) > t$. On the otherhand, by the definition (1.2), $D^{+1}(t) = \alpha$ implies $D(\alpha) \le t$. Thus, $D_{p(\alpha)}(\alpha|med(\alpha)) > D(\alpha)$, which contradict the definition of $p(\alpha)$. Thus, it follows that $\tilde{D}(p(\alpha)|med(\alpha)) = \alpha$ follows, which proves (ii). □

REMARK. From Theorem 3.3, we see that the fuzzy metric D^{+1} gives a lower bound of $\tilde{D}(J|j_1, j_2, \ldots, j_m))$, $J = (J_1, J_2, \ldots, J_m) \in \mathcal{J}_m$ and $j_k \in J_k$ $(1 \le k \le m)$, which is obtained by the cluster $p(\alpha)$ and median vector $med(p(\alpha))(\alpha)$ depending on $\alpha \in [0,1]$. That is to say, if $D^{+1}(t) = \alpha$, the degree of nearness of the minimum total sum of the metrics from all the elements in a cluster to its median with respect to t is realized by the clustering $p(\alpha)$ and the median vector $med(\alpha)$.

4. A Numerical Example

We shall give a numerical example to illustrate the theoretical results in Section 3. Consider the problem of clustering the elements of $N = \{1,2,3,4,5,6,7\}$ into two clusters, that is, $m = 2$. Table 1 gives the fuzzy metric $\tilde{d}(i,j)$, where, for any $a, b \in R_+$ with $a < b$, let

$$[a,b](t) = \begin{cases} 0, & \text{if } 0 \le t < a \\ (t-a)/(b-a), & \text{if } a \le t \le b \\ 1, & \text{if } t > b, \end{cases}$$

and

$$[a,a](t) = \begin{cases} 0, & \text{if } 0 \le t < a \\ 1, & \text{if } t > a. \end{cases}$$

By solving the corresponding non-fuzzy α-level cluster median problem ($\alpha \in [0,1]$), we get

$$D(\alpha) = \begin{cases} 12\alpha + 11, & \text{if } 0 \le \alpha \le 1/2 \\ 10\alpha + 12, & \text{if } 1/2 < \alpha \le 2/3 \\ 7\alpha + 14, & \text{if } 2/3 < \alpha \le 1 \end{cases}$$

So,

$$D^{+1}(t) = \begin{cases} 0, & \text{if } 0 \le t \le 11 \\ (t-11)/12, & \text{if } 11 < t \le 17 \\ (t-12)/10, & \text{if } 17 < t \le 56/3 \\ (t-14)/7, & \text{if } 56/3 < t \le 21 \\ 1, & \text{if } t \ge 21, \end{cases}$$

which is given in Fig.1.

Also, we can find $\mathcal{J}_{2,\alpha}$ $(\alpha \in [0,1])$ given by

$$\mathcal{J}_{2,\alpha} = \begin{cases} (1236, 457), (12346, 57) & \text{if } 0 \le \alpha \le 1/2 \\ (12346, 57) & \text{if } 1/2 < \alpha < 2/3 \\ (1234, 567), (12346, 57) & \text{if } \alpha = 2/3 \\ (1234, 567) & \text{if } 2/3 < \alpha \le 1. \end{cases}$$

For example, let

$$p(\alpha) = \begin{cases} (1236, 457), & \text{if } 0 \le \alpha \le 1/2 \\ (12346, 57), & \text{if } 1/2 < \alpha \le 2/3 \\ (1234, 567), & \text{if } 2/3 < \alpha \le 1 \end{cases}$$

The α-median vector is given by $med(\alpha) \equiv (2, 7)$.

Observing Fig.1, we find $D^{+1}(19.25) = 0.75$, which means that the degree 0.75 of nearness of the minimum total distance with respect to 19.25 is realized by the clustering $p(0.75) = (1234, 567)$ and the median vector (2,7).

i	j						
	1	2	3	4	5	6	7
1		[1, 2]	[2, 4]	[3, 4]	[10, 11]	[3, 5]	[3, 4]
2			[1, 3]	[2, 3]	[11,12]	[2, 4]	[4, 6]
3				[3, 4]	[12, 13]	[1, 2]	[5, 7]
4					[9, 10]	[4, 6]	[1, 5]
5						[13, 15]	[7, 8]
6							[3, 3]
7							

Table 1. Fuzzy metric $\tilde{d}(i,j)$ $(\tilde{d}(i,j) = \tilde{d}(j,i),\ \tilde{d}(i,i) = 0)$

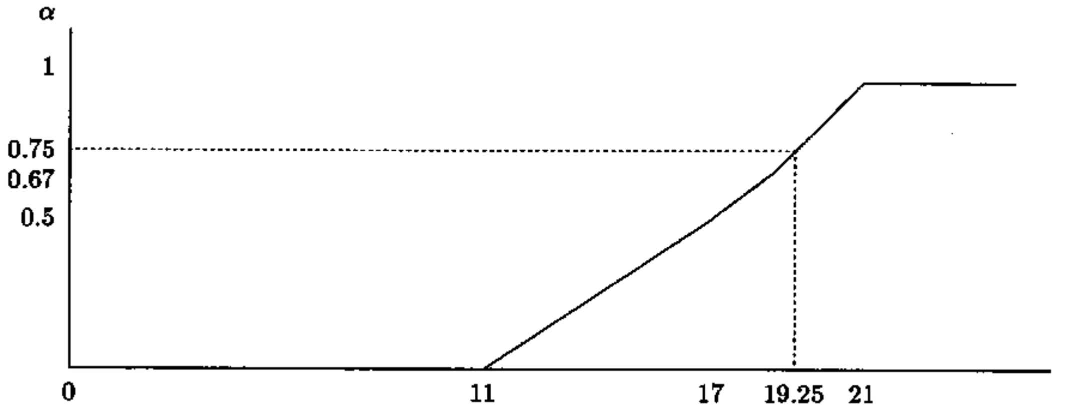

Fig. 1. The graph of D^{+1}

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