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ON EXPLICIT BOUNDS IN THE ERROR FOR THE H_0^1-PROJECTION INTO PIECEWISE POLYNOMIAL SPACES

By

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Abstract

The values of constants appearing in error estimates of approximations by finite element methods play an important role in numerical verification methods for elliptic equations (Nakao and Yamamoto(1998),Yamamoto and Nakao(1995) , etc.).For efficient implementation of the verification algorithms on computers, it is necessary that these constants can be estimated as close as possible to their optimal values. In Nakao,Yamamoto and Kimura(1998), the optimal constant was derived for quadratic elements as well as a nearly optimal value for cubic elements. In this paper, we establish a method to calculate the values of constants for approximation by piecewise polynomials of arbitrary degree and to give bounds on the difference between the constants so calculated and optimal values.

1. Introduction

We consider an interval $\mathrm{I} \subset \mathbf{R}$ and Sobolev spaces on it, denoted by $L^2(\mathrm{I})$, $H^1(\mathrm{I})$, $H_0^1(\mathrm{I})$, and so on, as usual. The inner product in $L^2(\mathrm{I})$ is represented by $(\cdot, \cdot)$, and the norm in the same space by $\|\cdot\|$.

We divide the interval I into subintervals and consider a piecewise polynomial space $S_h(\mathrm{I}) \subset H_0^1(\mathrm{I})$ of arbitrary degree. Let u be a function in $H_0^1(\mathrm{I}) \cap H^2(\mathrm{I})$ and $u_h \in S_h$ its approximation which satisfies

$$(u' - u_h', \psi') = 0 \qquad \text{for } {}^\forall\psi \in S_h(\mathrm{I}).$$

Since u_h is the orthogonal projection of u into S_h with respect to the inner-product in H_0^1,

$$\|u' - u_h'\| \le \|u' - \psi'\| \qquad \text{for } {}^\forall\psi \in S_h(\mathrm{I}),$$

holds, and we refer to u_h as the H_0^1-projection of u into $S_h(\mathrm{I})$.

For u and u_h, the following error estimate holds :

$$\|u' - u_h'\| \le C_0 h \|u''\|, \tag{1.1}$$

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where h is the maximal size of subintervals, and C_0 is a constant independent of u and h. The purpose of the present paper is to calculate the value of C_0.

As is well known, an H_0^1-projection into a piecewise polynomial space coincides with the interpolation of the function at each node (e.g., Schultz,1973). Therefore, we shall reduce our problem to the error estimation for local approximations of functions on an interval $e \equiv [0, h]$, vanishing at the terminals.

Let $Q_N(e)$ with $N \geq 0$ be a polynomial space defined by

$$Q_N(e) = \left\{ p(x) \middle| p \text{ is a polynomial of degree } N+1,\ p(0) = p(h) = 0 \right\},$$

and, for an arbitrary function $u \in H_0^1(e) \cap H^2(e)$, let $P_N u$ be the local H_0^1-projection of u into $Q_N(e)$ defined by

$$(u' - (P_N u)', v')_e = 0 \qquad \text{for } {}^\forall v \in Q_N(e),$$

where $(\cdot, \cdot)_e$ denotes the inner-product of $L^2(e)$. We look for the minimal constant C_0 which satisfies an error estimate of the form:

$$\|u' - (P_N u)'\|_e \leq C_0 h \|u''\|_e, \tag{1.2}$$

where $\|\cdot\|_e$ is the norm of $L^2(e)$. Note that the constant in (1.2) gives an upper bound in (1.1) when $S_h(I)$ is a piecewise polynomial space of degree $N+1$, and that it can also be extended into multi-dimensional problems (cf. Nakao,Yamamoto and Kimura,1998).

The following lemma provides a basis for error estimates of the approximation by piecewise linear functions.

LEMMA 1.1. *Wirtinger's inequality (Theorem 1.2 in Schultz,1973)*
If $u \in H_0^1(e)$, then

$$\|u\|_e \leq \frac{h}{\pi}\|u'\|_e.$$

Moreover, the equality holds if $u(x) = \sin(\pi x/h)$.

From Lemma 1.1, it can be seen that the optimal value for the constant C_0 in (1.2) for the error estimate of linear interpolant ($N = 0$) is $1/\pi$. Therefore, it is natural to expect that the corresponding constant is smaller than $1/\pi$ when we use higher degree polynomials. Actually, this is proved in Nakao,Yamamoto and Kimura(1998) in the quadratic and cubic cases. In this paper, we are concerned with an arbitrary degree.

2. Explicit Estimate of Constants

Now we introduce a set of polynomials $\{g_i\}_{i=1}^{\infty}$ of the form

$$g_i(x) = \frac{i + 1/2}{i!\, h^{i+1/2}} \frac{d^{i-1}}{dx^{i-1}} \left[x^i (h-x)^i \right].$$

The followings are well known properties of these polynomials:

$$
(g_i, g_j)_e = \begin{cases} \dfrac{h^2}{2(2i-1)(2i+3)} & (i = j) \\ -\dfrac{h^2}{4(i+j+1)(2i+1)^{1/2}(2j+1)^{1/2}} & (i = j \pm 2) \\ 0 & (\text{otherwise}) \end{cases} \tag{2.1}
$$

and

$$
(g_i', g_j')_e = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases} \tag{2.2}
$$

The fact that $\{g_i'\}_{i=1}^{\infty}$ gives a complete orthonormal system in $L^2(e)$ enables us to expand an arbitrary function $u \in H_0^1(e)$ as

$$
u(x) \sim \sum_{j=1}^{\infty} u_j\, g_j(x),
$$

where

$$
u_j \equiv (u', g_j')_e\,.
$$

Moreover, we can represent the H_0^1-projection of $u \in H_0^1(e)$ into $Q_N(e)$ as

$$
P_N u(x) = \sum_{j=1}^{N} u_j\, g_j(x),
$$

owing to the uniqueness of the H_0^1-projection for fixed $Q_N(e)$.

The following theorem gives a rough estimation for the case of an arbitrary integer $N \geq 0$.

THEOREM 2.1. *If $u \in H_0^1(e) \cap H^2(e)$, it follows that*

$$
\|u' - (P_N u)'\|_e^2 \leq \tilde{C}_N\, h^2 \|u''\|_e^2,
$$

where

$$
\tilde{C}_N \equiv \frac{1}{8}\left(\frac{1}{2N+1} + \frac{1}{2N+3}\right).
$$

PROOF.

$$
\begin{aligned}
\|u' - (P_N u)'\|_e^2 &= \sum_{i=N+1}^{\infty} (u', g_i')_e^2 \\
&= \sum_{i=N+1}^{\infty} (u'', g_i)_e^2 \qquad \text{(integrating by parts)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|u''\|_e^2 \sum_{i=N+1}^{\infty} \|g_i\|_e^2 \qquad \text{(by Schwarz' inequality)} \\
&= h^2 \|u''\|_e^2 \sum_{i=N+1}^{\infty} \frac{1}{2(2i-1)(2i+3)} \qquad \text{(from (2.1))} \\
&= \frac{h^2}{8} \|u''\|_e^2 \sum_{i=N+1}^{\infty} \left(\frac{1}{2i-1} - \frac{1}{2i+3} \right) \\
&= \frac{h^2}{8} \left(\frac{1}{2N+1} + \frac{1}{2N+3} \right) \|u''\|_e^2.
\end{aligned}
$$

□

In the following lemma, a smaller constant is derived by considering a finite dimensional eigenvalue problem in the case that u is a polynomial.

LEMMA 2.2. *Let $M \ge 1$ be an arbitrary integer. If $q \in Q_{N+M}(e)$, then the following holds:*

$$\|q - P_N q\|_e^2 \le \Lambda_N \, h^2 \, \|q' - (P_N q)'\|_e^2,$$

where

$$
\begin{aligned}
\Lambda_N \equiv \max\Bigg\{ & \frac{1}{2(2N+1)(2N+5)} + \frac{1}{4(2N+5)\sqrt{2N+3}\sqrt{2N+7}}, \\
& \frac{1}{4(2N+5)\sqrt{2N+3}\sqrt{2N+7}} + \frac{1}{2(2N+5)(2N+9)} + \\
& \frac{1}{4(2N+9)\sqrt{2N+7}\sqrt{2N+11}} \Bigg\}.
\end{aligned} \tag{2.3}
$$

PROOF.

Each polynomial $q \in Q_{N+M}(e)$ can be represented as

$$q = \sum_{i=1}^{N+M} q_i \, g_i,$$

and we have

$$
\begin{aligned}
\|q - P_N q\|_e^2 &= \sum_{i=N+1}^{N+M} \sum_{j=N+1}^{N+M} q_i \, (g_i, g_j)_e \, q_j \,, \\
\|q' - (P_N q)'\|_e^2 &= \sum_{i=N+1}^{N+M} {q_i}^2.
\end{aligned}
$$

Now let us consider an M-by-M matrix:

$$A_N^M \equiv \left(a_{i,j}^N \right)_{1 \le i,j \le M},$$

where

$$a_{i\,j}^N \quad \equiv \quad \frac{1}{h^2}(g_{N+i}, g_{N+j})_e\,,$$

and let q_N^M be an M-dimentional vector:

$$q_N^M \equiv \Big(q_i \Big)_{1 \le i \le M}$$

Then we have

$$\sup_{q \neq 0} \frac{\|q - P_N q\|_e^2}{h^2\, \|q' - (P_N q)'\|_e^2} = \sup_{q_N^M \neq 0} \frac{{q_N^M}^T\, A_N^M\, q_N^M}{{q_N^M}^T\, q_N^M}\,.$$

The right-hand side equals the largest eigenvalue of A_N^M, denoted by λ_N^M. From Gerschgorin's Theorem, λ_N^M can be estimated by

$$\lambda_N^M \le \max_{1 \le i \le M} \sum_{j=1}^{M} |a_{i,j}^N|.$$

Using (2.1), the right-hand side of the above inequality is calculated as

$$\begin{aligned} \lambda_N^M \quad &\le \quad \max\left\{ |a_{1\,1}^N| + |a_{1\,3}^N|\,,\, |a_{2\,2}^N| + |a_{2\,4}^N|\,,\, \max_{3 \le i \le M}\{|a_{i\,i-2}^N| + |a_{i\,i}^N| + |a_{i\,i+2}^N|\} \right\} \\ &= \quad \max\{|a_{1\,1}^N| + |a_{1\,3}^N|\,,\, |a_{3\,1}^N| + |a_{3\,3}^N| + |a_{3\,5}^N|\} \\ &= \quad \Lambda_N. \end{aligned}$$

This completes the proof.

□

The following theorem, the main result of this article, is an extension of Lemma 2.2.

THEOREM 2.3. *For any function $u \in H_0^1(e) \cap H^2(e)$, we have*

$$\|u' - (P_N u)'\|_e^2 \le \Lambda_N\, h^2 \|u''\|_e^2, \tag{2.4}$$

where the constant Λ_N is defined by (2.3).

PROOF.

Using Lemma 1.1, Lemma 2.2, Theorem 2.1, and the relation $\|(P_{N+M}u)' - (P_N u)'\|_e \le \|u' - (P_N u)'\|_e$, we have the following for an arbitrary integer $M \ge 1$:

$$\begin{aligned} \|u - P_N u\|_e^2 \quad &\le \quad (\|u - P_{N+M} u\|_e + \|P_{N+M} u - P_N u\|_e)^2 \\ &\le \quad \left\{ \left(\frac{h}{\pi}\right) \|u' - (P_{N+M} u)'\|_e + {\Lambda_N}^{1/2}\, h\, \|(P_{N+M} u)' - (P_N u)'\|_e \right\}^2 \\ &\le \quad \left\{ \left(\frac{h}{\pi}\right) \frac{h}{2\sqrt{2}} \left(\frac{1}{2N + 2M + 1} + \frac{1}{2N + 2M + 3}\right)^{1/2} \|u''\|_e \right. \\ &\qquad \left. + {\Lambda_N}^{1/2}\, h\, \|u' - (P_N u)'\|_e \right\}^2 . \end{aligned}$$

Letting M tend to infinity, we obtain

$$\|u - P_N u\|_e^2 \leq \Lambda_N h^2 \|u' - (P_N u)'\|_e^2. \tag{2.5}$$

Moreover, from (2.5),

$$\begin{aligned} \|u' - (P_N u)'\|_e^2 &= (u' - (P_N u)', u' - (P_N u)')_e \\ &= (u' - (P_N u)', u')_e \\ &= -(u - P_N u, u'')_e \\ &\leq \|u - P_N u\|_e \|u''\|_e \\ &\leq {\Lambda_N}^{1/2} h \|u' - (P_N u)'\|_e \|u''\|_e \end{aligned} \tag{2.6}$$

holds. Dividing both sides of (2.6) by $\|u' - (P_N u)'\|_e$, we have (2.4).

□

In Bernardi(1996), similar arguments (but not numerical) are presented. The above theorem assures (1.5) of Bernardi(1996) in the case of $s = 2$:

$$\|u - P_N u\|_{H^1(e)} \leq \frac{c}{N+1} \|u\|_{H^2(e)}$$

with numerical estimate of the constant c. For example, the theorem tells us that we can choose $c = 1/2$.

Moreover, we can estimate an upper bound of the difference between Λ_N and the optimal constant ${C_0}^2$. Since $\lambda_N^M \leq {C_0}^2 \leq \Lambda_N$ holds for an arbitrary integer $M \geq 1$, we have

$$|\Lambda_N - {C_0}^2| \quad \leq \quad |\Lambda_N - \lambda_N^M|.$$

In the following table, we estimate relative errors of Λ_N on ${C_0}^2$ by $\dfrac{|\Lambda_N - \lambda_N^M|}{\Lambda_N}$ taking $M = 4$ for $N = 0, \cdots, 10$. The exact values of ${C_0}^2$ obtained in Nakao,Yamamoto and Kimura(1998) are shown for $N = 0, 1$. For $N \geq 2$, exact values of ${C_0}^2$ are not yet known, which are represented by '?' in the table.

From this table, it is seen that the relative errors of Λ_N are less than 15% for $N = 0, \cdots, 10$.

N	λ_N^4	${C_0}^2$	Λ_N	$\frac{\lvert\Lambda_N - \lambda_N^4\rvert}{\Lambda_N}$
0	0.1013197	$(\frac{1}{\pi})^2 \cong 0.10132$	0.110911	0.0864766
1	0.0253155	$(\frac{1}{2\pi})^2 \cong 0.02533$	0.0291335	0.131053
2	0.0123516	?	0.0142767	0.134841
3	0.00748463	?	0.00859464	0.129152
4	0.00507143	?	0.00577062	0.121164
5	0.00368227	?	0.00415142	0.113011
6	0.00280337	?	0.00313354	0.105368
7	0.00220957	?	0.00245078	0.0984193
8	0.00178846	?	0.00197004	0.0921739
9	0.00147842	?	0.00162342	0.0893178
10	0.00124326	?	0.00138892	0.104871

3. Conclusion

We have obtained a method to get numerical estimates of the constants appearing in (1.1) for the approximation by polynomials of arbitrary degree. Since the estimates derived here are sharp enough for actual application, we believe that they will play an important role in the error estimates for the finite element method and in the numerical verification of solutions to nonlinear elliptic equations.

From the mathematical point of view, the relation between the degree of polynomials and the constant in (1.2) is very interesting. We leave it as an open problem to decide the optimal constants.

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