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INVERSE PARTITION PROBLEMS

By

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Abstract

We analyze a partition problem and its inverse problem both in discrete variables and in two continuous ones through dynamic programming. We show that an inverse relation and an envelopping relation hold in each case. It is shown that one optimal solution in discrete partition is expressed by the other through either upper-semi inverse function or lower-semi inverse function and that optimal solutions in continuous partition through the (regular) inverse function. As a result, we show that the optimal partition is to partition equally in essence any quantity into the quantities of the same size of e. We call this optimal policy *Euler partition rule*.

1. Introduction

The partition problem is one of traditional mathematical optimization problems. It is also called *division problem* in Beckmann and Lademan(1956), Golmb(1968,80) and others or *allocation problem* in Beckmann(1968), Ibaraki and Katoh(1988) and others. In this paper it is to partition any given nonnegative quantity into any number of nonnegative ones to maximize the product (multiplicated quantity). In the paper we consider three partitions; a discrete partition and two continuous partitions. We propose an inversion of each (main) partition problem from the view point of dynamic programming (Bellman(1975), Iwamoto(1987), Sniedovich(1992)). We show that an inverse relation holds between main and inverse partition problems. Further, by relating the partition problem with the family of fixed partition problems, we derive an envelopping relation between the partition problem and the family.

In section 2, we consider the discrete partition. We analyze both main and inverse partition problems through dynamic programming (Iwamoto(1977a,1977b,1977c)). We propose upper-semi inverse function and lower-semi inverse function. It is shown that one optimal solution is characterized by the other through either inverse function (inverse theorem).

In section 3, we consider the partition of any nonnegative real into finitely many ones. We state the inverse theorem through the (regular) inverse function.

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In section4, we are concerned with partition into continuously many ones. We derive the inverse relation by Bellman equation in continuous dynamic programming (Iwamoto and Wang(1983,1986)).

Throughout the analysis of three main partition problems, we show that the *Euler partition rule* is an optimal partition. That is, the optimal partition is to partition equally in essence any quantity into the quantities of the same size of the Euler number e.

2. Discrete Partition

In this section, we consider a pair of partition problems of any natural number n into natural numbers.

2.1. Regular Partition Problems

In this subsection, we are concerned with partition into *any number of* natural numbers. We consider the *main partition problem* (Golmb(1968,1980)):

$$\mathrm{MPP}(n) \qquad \begin{array}{ll} \text{Maximize} & n_1 n_2 \cdots n_t \\ \text{subject to} & \text{(i)} \quad n_1 + n_2 + \cdots + n_t = n \\ & \text{(ii)} \quad n_1, n_2, \ldots, n_t, t \in N \end{array} \tag{1}$$

and its *inverse partition problem*:

$$\mathrm{IPP}(n) \qquad \begin{array}{ll} \text{minimize} & n_1 + n_2 + \cdots + n_t \\ \text{subject to} & \text{(i)}' \quad n_1 n_2 \cdots n_t \geq n \\ & \text{(ii)} \quad n_1, n_2, \ldots, n_t, t \in N \end{array} \tag{2}$$

where $n \in N$.

Let $f(n)$, $g(n)$ be the maximum value and the minimum value, respectively. First we have the monotonicity of optimum value functions $f(\cdot)$, $g(\cdot)$ as follows:

LEMMA 2.1. *The maximum value function $f : N \to N$ is strictly increasing but the minimum value function $g : N \to N$ is nondecreasing. Both go to ∞ as so does n.*

Second we have the recursive equations:

THEOREM 2.2.

$$f(n) = \operatorname*{Max}_{1 \leq k < n} [kf(n-k)] \qquad n \geq 5 \tag{3}$$

$$f(n) = n \qquad n = 1, 2, 3, 4 \tag{4}$$

$$g(n) = \min_{\substack{1 < k, l < n \\ kl \geq n}} [k + g(l)] \qquad n \geq 6 \tag{5}$$

$$g(n) = n \qquad n = 1, 2, 3, 4, 5. \tag{6}$$

We note that the minimization in (5) is taken over the set of all pairs (k, l) satisfying the condition $1 < k, l < n$, $kl \geq n$.

Let $\pi^*(n)$ be the the maximizer in (3) and $\hat{\sigma}(n)$ be the value of k for the minimizer in (5), respectively.

2.2. Inverse Theorems

Furthermore, we have the inverse relationship between Main and Inverse Partition Problems:

THEOREM 2.3. *(Weak Inverse Theorem)*

$$\text{(i)} \qquad g(f(n)) = n, \quad f(g(n)) \geq n \qquad n \in N \tag{7}$$

$$\text{(ii)} \qquad f(g(n)) = n \quad n = 3^m,\ 4 \cdot 3^{m-1},\ 2 \cdot 3^m \ \textit{for}\ m \in N. \tag{8}$$

Proof (i) Let $n \in N$. First we have a maximum solution:

$$f(n) = n_1 n_2 \cdots n_t$$

for some $n_1, n_2, \ldots, n_t, t \in N$ satisfying

$$n_1 + n_2 + \cdots + n_t = n.$$

Thus we have

$$g(f(n)) \leq n. \tag{9}$$

On the other hand, set $g(f(n)) < n$. Then there exist $m_1, m_2, \ldots, m_s, s \in N$ such that

$$g(f(n)) = m_1 + m_2 + \cdots + m_s < n, \quad m_1 m_2 \cdots m_s \geq f(n).$$

Therefore we can choose $m_1^* > m_1$ satisfying

$$m_1^* + m_2 + \cdots + m_s = n, \quad m_1^* m_2 \cdots m_s > m_1 m_2 \cdots m_s \geq f(n).$$

This contradicts the maximality of $f(n)$. Thus we have

$$g(f(n)) \geq n. \tag{10}$$

From (9),(10), we have the desired equality.

Second, there exists a minimum solution:

$$g(n) = n_1 + n_2 + \cdots + n_t$$

for some $n_1, n_2, \ldots, n_t, t \in N$ satisfying

$$n_1 n_2 \cdots n_t \geq n.$$

This implies $f(g(n)) \geq n$.

(ii) Since

$$f(3m) = 3^m, \quad f(3m+1) = 4 \cdot 3^{m-1}, \quad f(3m+2) = 2 \cdot 3^m$$

implies

$$g(3^m) = 3m, \quad g(4 \cdot 3^{m-1}) = 3m+1, \quad g(2 \cdot 3^m) = 3m+2$$

we have the desired composite equalities, respectively. □

Let $h : N \to N$ be a nondecreasing function with $h(1) = 1$ and $h(n)$ go to ∞ as so does n. *Only in this section* we use two kinds of its inverse function as follows: One is the *upper-semi inverse function* $h^{-1} : N \to N$

$$h^{-1}(n) := \min\{ m \in N \mid h(m) \geq n \}. \tag{11}$$

The other is the *lower-semi inverse function* $h_{-1} : N \to N$

$$h_{-1}(n) := \mathrm{Max}\{ m \in N \mid h(m) \leq n \}. \tag{12}$$

We say that a value $n \in N$ is *attainable* if there exists some $m \in N$ satisfying $h(m) = n$. Then we have the following properties. The proof is straightforward and therefore omitted. For the same reason, some proofs are omitted in the remainder of this paper.

LEMMA 2.4.

(i) $h^{-1}, h_{-1} : N \to N$ *are nondecreasing*

(ii) $h_{-1}(n) \geq h^{-1}(n)$ *for attainable* $n \in N$ (13)

(iii) $h_{-1}(n) < h^{-1}(n)$ *for nonattainable* $n \in N$. (14)

Furthermore, for any nonattainable $n \in N$, *both* $h_{-1}(n)$ *and* $h^{-1}(n)$ *take two adjacent (neighbouring) values in* N.

LEMMA 2.5.

(i) $h(h^{-1}(n)) \geq n, \quad h(h_{-1}(n)) \leq n \quad n \in N$ (15)

(ii) $h^{-1}(h(m)) \leq m, \quad h_{-1}(h(m)) \geq m \quad m \in N$. (16)

Moreover, we have a rather strict inverse relations as follows:

THEOREM 2.6. *(Strong Inverse Theorem I)*

(i) $f^{-1}(n) = g(n), \quad g_{-1}(n) = f(n) \quad n \in N$ (17)

(ii) $\hat{\sigma}(n) \supseteq \pi^*(f^{-1}(n)) \quad n \in [6, 7, \ldots), \qquad \pi^*(n) = \hat{\sigma}(g_{-1}(n)) \quad n \in [5, 6, \ldots)$ (18)

Proof (i) Let $n \in N$. First, from Theorem 2.3 (Weak Inverse Theorem), we have $f(g(n)) \geq n$. This implies

$$f^{-1}(n) \leq g(n). \tag{19}$$

On the other hand, we see that $f(m) \geq n$ implies $m \geq g(n)$. Thus we get

$$f^{-1}(n) \geq g(n). \tag{20}$$

Then we have the desired equality.

Now we show the above-mentioned implication. Let $f(m) \geq n$. Then we have a maximum solution:

$$f(m) = n_1 n_2 \cdots n_t \geq n$$

for some $n_1, n_2, \ldots, n_t, t \in N$ satisfying

$$n_1 + n_2 + \cdots + n_t = m.$$

Thus we get $g(n) \leq m$..

Second, from $g(f(n)) = n$, we have

$$g_{-1}(n) \geq f(n). \tag{21}$$

Further we see that $g(m) \leq n$ implies $m \leq f(n)$. Thus we get

$$g_{-1}(n) \leq f(n). \tag{22}$$

Then we have the desired equality.

(ii) First take $n \geq 6$. Let $m := f^{-1}(n)$. Then we show

$$\pi^*(m) \subseteq \hat{\sigma}(n).$$

Let $k \in \pi^*(m)$. Then we have

$$f(m) = k \times f(m-k).$$

Letting $p := f(m)/k$, we get

$$p = f(m-k) = g_{-1}(m-k).$$

This in turn implies

$$k + g(p) \leq m.$$

Since $m = f^{-1}(n) = g(n)$, we have

$$k + g(p) \leq g(n). \tag{23}$$

On the other hand, we have $kp = f(m) \geq n$. This together with the recursiveness yields

$$k + g(p) \geq g(n). \tag{24}$$

From (23),(24), we have $k \in \hat{\sigma}(n)$.

Second, take $n \geq 5$. Let $m := g_{-1}(n)$. Then we show

$$\hat{\sigma}(m) = \pi^*(n).$$

Let $k \in \hat{\sigma}(m)$. Then we have

$$g(m) = k + g(l)$$

for some l with $kl = m$. Letting $q := g(m) - k$, we get

$$q = g(l) = f^{-1}(l).$$

This in turn implies

$$k \times f(q) \geq m.$$

Since $m = g_{-1}(n) = f(n)$, we have

$$k \times f(q) \geq f(n). \tag{25}$$

On the other hand, from $g(g_{-1}(n)) = n$, we have $k + q = g(m) = n$. This together with the recursiveness yields

$$k \times f(q) \leq f(n), \qquad k + q = n. \tag{26}$$

From (25),(26), we have $k \in \pi^*(n)$.

Conversely, letting $k \in \pi^*(n)$, we have

$$f(n) = k \times f(n-k), \quad 1 < k < n.$$

Let $p := f(n)/k$. Then we get

$$p = f(n-k) = g_{-1}(n-k).$$

This implies

$$k + g(p) \leq n.$$

Since $n = g(m)$, we have

$$k + g(p) \leq g(m). \tag{27}$$

On the other hand, we have $kp = f(n) = g_{-1}(n) = m$. This together with the recursiveness yields

$$k + g(p) \geq g(m). \tag{28}$$

From (27),(28), we have $k \in \hat{\sigma}(m)$. This completes the proof. □

Henceforth we write symbolically

$$f^{-1} = g, \quad g_{-1} = f \quad \text{on } N \tag{29}$$

$$\hat{\sigma} \supseteq \pi^* \circ f^{-1} \text{ on } [6, 7, \ldots), \qquad \pi^* = \hat{\sigma} \circ g_{-1} \text{ on } [5, 6, \ldots), \tag{30}$$

where $\circ$ is the composition operator between functions.

In fact, we have the following optimal solutions:
For $n \geq 5$ the MPP has the maximum solution

$$f(n) = \begin{cases} 3^m & \text{for } n = 3m \\ 4 \cdot 3^{m-1} & \text{for } n = 3m+1 \\ 2 \cdot 3^m & \text{for } n = 3m+2 \end{cases} \qquad m \in N \tag{31}$$

$$\pi^*(n) = \begin{cases} 3 & \text{for } n = 3m \\ 2,3,4 & \text{for } n = 3m+1 \\ 2,3 & \text{for } n = 3m+2 \end{cases} \qquad m \in N \tag{32}$$

$$t^*(n) = \begin{cases} m & \text{for } n = 3m \\ m, m+1 & \text{for } n = 3m+1 \\ m+1 & \text{for } n = 3m+2 \end{cases} \qquad m \in N. \tag{33}$$

For $n \geq 6$ the IPP has the minimum solution as follows :

$$g(n) = \begin{cases} 3m & \text{for } 2 \cdot 3^{m-1} < n \leq 3^m \\ 3m+1 & \text{for } 3^m < n \leq 4 \cdot 3^{m-1} \\ 3m+2 & \text{for } 4 \cdot 3^{m-1} < n \leq 2 \cdot 3^m \end{cases} \qquad m \in N \tag{34}$$

$$\hat{\sigma}(n) = \begin{cases} 3^\star & \text{for } 2 \cdot 3^{m-1} < n \leq 3^m \\ 2,3,4^\star & \text{for } 3^m < n \leq 4 \cdot 3^{m-1} \\ 2,3^\star & \text{for } 4 \cdot 3^{m-1} < n \leq 2 \cdot 3^m \end{cases} \qquad m \in N \tag{35}$$

$$\hat{t}(n) = \begin{cases} m^\star & \text{for } 2 \cdot 3^{m-1} < n \leq 3^m \\ m, m+1^\star & \text{for } 3^m < n \leq 4 \cdot 3^{m-1} \\ m+1^\star & \text{for } 4 \cdot 3^{m-1} < n \leq 2 \cdot 3^m \end{cases} \qquad m \in N, \tag{36}$$

where, $\star$ means that the specification is a minimal expression (subset) of $\hat{\sigma}(n)$ and a possibility of $\hat{t}(n)$. However, for the cases $n = 3^m,\ 4 \cdot 3^{m-1},\ 2 \cdot 3^m$, it specifies the complete expression. The full description of $\hat{\sigma}(n)$ is given in Figures 2.1, 2.2, ..., 2.6.

We are interested in the ultimate distribution of *average numbers* of optimal (maximum and minimum) partitions

$$\frac{n}{t^*(n)} \quad \text{and} \quad n^{1/\hat{t}(n)} \tag{37}$$

for the limitting case $n \to \infty$, respectively. Both will converge to a common *Euler number e*.

2.3. Fixed Partition Problems

In this subsection we consider, for any $k \in N$, the *main partition problem:*

$$\text{MPP}(n:k) \qquad \begin{array}{ll} \text{Maximize} & n_1 n_2 \cdots n_k \\ \text{subject to} & \text{(i)}\ \ n_1 + n_2 + \cdots + n_k = n \\ & \text{(ii)}\ \ n_1, n_2, \ldots, n_k \in N \end{array} \tag{38}$$

where $k \le n \in N$ and, its *inverse partition problem:*

$$\text{IPP}(n;k) \qquad \begin{array}{ll} \text{minimize} & n_1 + n_2 + \cdots + n_k \\ \text{subject to} & \text{(i)}' \quad n_1 n_2 \cdots n_k \ge n \\ & \text{(ii)} \quad n_1, n_2, \ldots, n_k \in N \end{array} \tag{39}$$

where $n \in N$.

Let $f_k(n)$, $g_k(n)$ be the maximum value and the minimum value, respectively. Then

LEMMA 2.7. *The maximum value functions* $f_k : [k, k+1, \ldots) \to N$ *are strictly increasing but* $g_k : N \to N$ *are nondecreasing. They go to* ∞ *as so does* n.

THEOREM 2.8.

$$f_k(n) = \operatorname*{Max}_{1 \le m < n} [m f_{k-1}(n-m)] \qquad k \ge 2 \tag{40}$$

$$f_1(n) = n \qquad n \in N \tag{41}$$

$$g_k(n) = \min_{\substack{1 < m, l < n \\ ml \ge n}} [m + g_{k-1}(l)] \qquad k \ge 2 \tag{42}$$

$$g_1(n) = n \qquad n \in N. \tag{43}$$

The minimization is restricted to the set of all pairs (m, l) *satisfying the condition.*

Let $\pi_k^*(n)$ be the minimizer in (40) and $\hat{\sigma}_k(n)$ the value of m for the maximizer in (42), respectively.

We have an envelopping relation between the regular problem and the fixed problems:

THEOREM 2.9. *(Envelopping Theorem)*

$$f(n) = \operatorname*{Max}_{k \ge 1} f_k(n) \qquad n \in N \tag{44}$$

$$g(n) = \min_{k \ge 1} g_k(n) \qquad n \in N. \tag{45}$$

Furthermore, we have the inverse relationship between main and inverse fixed partition problems:

THEOREM 2.10. *(Inverse Theorem)*

$$\text{(i)} \ g_k = f_k^{-1} \ \textit{ on } N, \quad f_k = (g_k)_{-1} \quad \textit{on } [k, k+1, \ldots), \quad k \ge 1 \tag{46}$$

$$\text{(ii)} \ \hat{\sigma}_k \supseteq \pi_k^* \circ f_k^{-1} \ \textit{ on } N, \quad \pi_k^* = \hat{\sigma}_k \circ (g_k)_{-1} \quad \textit{on } [k, k+1, \ldots), \quad k \ge 2. \tag{47}$$

3. Finitely Many Partition of Continuum

In this section we are concerned with partition of any given nonnegative *real* into finitely many ones.

3.1. Regular Partition Problems

We consider the *main partition problem*:

$$\text{MPP}(c) \qquad \begin{array}{ll} \text{Maximize} & x_1x_2\cdots x_t \\ \text{subject to} & \text{(i)} \quad x_1 + x_2 + \cdots + x_t = c \\ & \text{(ii)} \quad x_m \geq 0 \quad 1 \leq m \leq t \\ & \text{(iii)} \quad t \geq 1 \end{array} \tag{48}$$

and the *inverse partition problem*:

$$\text{IPP}(c) \qquad \begin{array}{ll} \text{minimize} & x_1 + x_2 + \cdots + x_t \\ \text{subject to} & \text{(i)}' \quad x_1x_2\cdots x_t = c \\ & \text{(ii)} \quad x_m \geq 0 \quad 1 \leq m \leq t \\ & \text{(iii)} \quad t \geq 1 \end{array} \tag{49}$$

where $c \geq 0$. The three constraints mean to partition a given nonnegative real c into t nonnegative ones, which is also a control variable.

Let $f(c)$, $g(c)$ be the maximum value and the minimu value, respectively. Then we have

LEMMA 3.1. *Both optimum value functions* $f, g : [0, \infty) \to [0, \infty)$ *are continuous and strictly increasing. Both go to* ∞ *as so does* c.

THEOREM 3.2.

$$f(c) = \operatorname*{Max}_{0 \leq x \leq c} [xf(c-x)] \qquad c \geq 4 \tag{50}$$

$$f(c) = c \qquad 0 \leq c \leq 4 \tag{51}$$

$$g(c) = \min_{0 < x < \infty} \left[x + g\left(\frac{c}{x}\right) \right] \qquad c \geq 4 \tag{52}$$

$$g(c) = c \qquad 0 \leq c \leq 4. \tag{53}$$

Let us define $\pi^*(c)$, $\hat{\sigma}(c)$ be the maximizer in (50) and the minimizer in (52), respectively.

Let h^{-1} be the inverse function of a continuous strictly increasing function h.

Then we have the inverse relationship between Main and Inverse Divisin Problems:

THEOREM 3.3. *(Inverse Theorem)*

$$\text{(i)} \ g = f^{-1}, \ f = g^{-1} \quad on \ [0, \infty) \tag{54}$$

$$\text{(ii)} \ \hat{\sigma} = \pi^* \circ f^{-1}, \quad \pi^* = \hat{\sigma} \circ g^{-1} \quad on \ [4, \infty). \tag{55}$$

In fact, we see that both problems have the optimal solutions as follows:

$$f(c) = \left(\tfrac{c}{n}\right)^n \qquad c \in I_n, \quad n \geq 1 \tag{56}$$

$$\pi^*(c) = \tfrac{c}{n}, \ t^*(c) = n \qquad c \in I_n, \quad n \geq 2 \tag{57}$$

where I_n is the interval

$$I_1 = [0, 4], \qquad I_n = \left[\frac{n^n}{(n-1)^{n-1}}, \ \frac{(n+1)^{n+1}}{n^n} \right] \quad n \geq 2. \tag{58}$$

$$g(c) = nc^{1/n} \qquad c \in J_n, \quad n \geq 1 \tag{59}$$

$$\hat{\sigma}(c) = c^{1/n}, \quad \hat{t}(c) = n \qquad c \in J_n, \quad n \geq 2 \tag{60}$$

where J_n is the interval

$$J_1 = [0, 4], \qquad J_n = \left[\left(\frac{n}{n-1} \right)^{(n-1)n}, \ \left(\frac{n+1}{n} \right)^{n(n+1)} \right] \quad n \geq 2. \tag{61}$$

We note that

$$\lim_{c \to \infty} \frac{c}{t^*(c)} = e, \qquad \lim_{c \to \infty} c^{1/\hat{t}(c)} = e. \tag{62}$$

3.2. Fixed Partition Problems

In this subsection we are concerned with partition into any given total number n. For any $n \in N$, we consider the *main partition problem* and the *inverse partition problem*:

$$\text{MPP}(c; n) \qquad \begin{array}{ll} \text{Maximize} & x_1 x_2 \cdots x_n \\ \text{subject to} & \text{(i)} \quad x_1 + x_2 + \cdots + x_n = c \\ & \text{(ii)} \quad x_i \geq 0 \quad 1 \leq i \leq n \end{array} \tag{63}$$

$$\text{IPP}(c; n) \qquad \begin{array}{ll} \text{minimize} & x_1 + x_2 + \cdots + x_n \\ \text{subject to} & \text{(i)}' \quad x_1 x_2 \cdots x_n \geq c \\ & \text{(ii)} \quad x_i \geq 0 \quad 1 \leq i \leq n \end{array} \tag{64}$$

where $c \in [0, \infty)$, $n \in N$. Let $f_n(c)$, $g_n(c)$ be the maximum value and minimum value, respectively. Then

LEMMA 3.4. *Both optimum value functions $f_n, g_n : [0, \infty) \to [0, \infty)$ are continuous and strictly increasing. They go to ∞ as so does c.*

THEOREM 3.5.

$$f_n(c) = \operatorname*{Max}_{0 \leq x \leq c} [x f_{n-1}(c - x)] \qquad c \in [0, \infty), \quad n \geq 2 \tag{65}$$

$$f_1(c) = c \qquad c \in [0, \infty) \tag{66}$$

$$g_n(c) = \min_{0 < x < \infty} \left[x + g_{n-1}\left(\frac{c}{x} \right) \right] \qquad c \in [0, \infty), \quad n \geq 2 \tag{67}$$

$$g_1(c) = c \qquad c \in [0, \infty). \tag{68}$$

We denote the maximizer in (65) by $\pi_n^*(c)$ and the minimizer in (67) by $\hat{\sigma}_n(c)$, respectively.

We have also the envelopping relation and the inverse relations as follows:

THEOREM 3.6.

$$f(c) = \operatorname*{Max}_{n\geq 1} f_n(c) \qquad c \in [0, \infty) \tag{69}$$

$$g(c) = \min_{n\geq 1} g_n(c) \qquad c \in [0, \infty). \tag{70}$$

THEOREM 3.7. *(Inverse Theorem (Iwamoto(1977a,1977b,1977c))*

$$\text{(i)}\ g_n = f_n^{-1},\quad f_n = g_n^{-1} \quad on\ [0, \infty),\ n \geq 1 \tag{71}$$

$$\text{(ii)}\ \hat{\sigma}_n = \pi_n^* \circ f_n^{-1},\quad \pi_n^* = \hat{\sigma}_n \circ g_n^{-1} \quad on\ [0, \infty),\ n \geq 2. \tag{72}$$

In fact, the MPP$(c;n)$ has the following optimal solutions:

$$f_n(c) = \left(\frac{c}{n}\right)^n \qquad c \in [0, \infty),\quad n \geq 1 \tag{73}$$

$$\pi_n^*(c) = \frac{c}{n} \qquad c \in [0, \infty),\quad n \geq 2. \tag{74}$$

The IPP$(c;n)$ has the optimal solutions:

$$g_n(c) = nc^{1/n} \qquad c \in [0, \infty),\quad n \geq 1 \tag{75}$$

$$\hat{\sigma}_n(c) = c^{1/n} \qquad c \in [0, \infty),\quad n \geq 2. \tag{76}$$

4. Continuously Many Partition of Continuum

In this section we are concened with a continuous partition of any nonnegative real into any continuously many ones (see Iwamoto(1983), Iwamoto and Wang(1983,1986)).

4.1. Regular Partition Problems

As regular *continuous partition*, we consider the *main partition problem*:

$$\text{MCPP}(c) \qquad \begin{array}{ll} \text{Maximize} & \displaystyle\int_0^t \log x(s)ds \\ \text{subject to} & \text{(i)}\ \displaystyle\int_0^t x(s)ds = c \\ & \text{(ii)}\ \ x(s) \geq 0 \quad 0 \leq s \leq t \\ & \text{(iii)}\ \ t \geq 0 \end{array} \tag{77}$$

and the *inverse partition problem*

$$\text{ICPP}(c) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\int_0^t x(s)ds \\ \text{subject to} & \text{(i)}'\ \displaystyle\int_0^t \log x(s)ds = c \\ & \text{(ii)}\ \ x(s) \geq 0 \quad 0 \leq s \leq t \\ & \text{(iii)}\ \ t \geq 0 \end{array} \tag{78}$$

where $c \geq 0$.

Let $f(c)$, $g(c)$ be the maximum value and the minimum value, respectively. First we have the strict monotonicity of optimum value functions $f(\cdot)$, $g(\cdot)$ as follows:

LEMMA 4.1. *The optimum value functions* $f, g : [0, \infty) \to [0, \infty)$ *are differentiable and strictly increasing. Both go to* ∞ *as so does* c.

THEOREM 4.2. *Both the optimum value functions* $f = f(c)$, $g = g(c)$ *satisfy the Bellman equations:*

$$0 = \operatorname*{Max}_{x \geq 0}[\log x - f'x], \quad f(0) = 0 \tag{79}$$

$$0 = \min_{x \geq 0}[x - g' \log x], \quad g(0) = 0, \tag{80}$$

respectively.

Letting $\pi^*(c)$ be the maximizer of (79) and $\hat{\sigma}(c)$ be the minimizer of (80), we have

$$\log f' + 1 = 0, \quad \text{for } x^* = \pi^*(c) = 1/f'(c) \tag{81}$$

$$\log g' - 1 = 0, \quad \text{for } \hat{x} = \hat{\sigma}(c) = g'(c), \tag{82}$$

respectively. Thus we have optimum value functions, optimizer and optimum continuous partition with optimum length as follows:

$$f(c) = \frac{c}{e}, \quad \pi^*(c) = e; \quad x^*(s) = e, \quad t^*(c) = \frac{c}{e} \tag{83}$$

$$g(c) = ec, \quad \hat{\sigma}(c) = e; \quad \hat{x}(s) = e, \quad \hat{t}(c) = c. \tag{84}$$

Furthermore, we have the inverse relationship between Main and Inverse Partition Problems:

THEOREM 4.3. *(Inverse Theorem)*

$$\text{(i)} \quad g = f^{-1}, \ f = g^{-1} \tag{85}$$

$$\text{(ii)} \quad \hat{\sigma} = \pi^* \circ f^{-1}, \quad \pi^* = \hat{\sigma} \circ g^{-1} \tag{86}$$

$$\text{(iii)} \quad \hat{t} = t^* \circ f^{-1}, \quad t^* = \hat{t} \circ g^{-1}. \tag{87}$$

We remark that

$$\frac{c}{t^*(c)} = e, \qquad c^{1/\hat{t}(\log c)} = e. \tag{88}$$

4.2. Fixed Partition Problems

For any given real number $t \geq 0$, we consider the pair of maximization problem and minimization problem (Iwamoto and Wang(1983,1986)):

$$\text{MCPP}(c;t) \qquad \begin{array}{ll} \text{Maximize} & \displaystyle\int_0^t \log x(s)ds \\ \text{subject to} & \text{(i)} \quad \displaystyle\int_0^t x(s)ds = c \\ & \text{(ii)} \quad x(s) \geq 0 \quad 0 \leq s \leq t \end{array} \tag{89}$$

and

$$\text{ICPP}(c;t) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\int_0^t x(s)ds \\ \text{subject to} & \text{(i)}' \quad \displaystyle\int_0^t \log x(s)ds = c \\ & \text{(ii)} \quad x(s) \geq 0 \quad 0 \leq s \leq t \end{array} \tag{90}$$

where $c \geq 0$.

Let $f(t,c)$, $g(t,c)$ be the maximum value and the minimum value, respectively. First we have the strict monotonicity of optimum value functions $f(t,\cdot)$, $g(t,\cdot)$ as follows:

LEMMA 4.4. *The optimum value functions $f = f(t,c), g = g(t,c)$ are continuously differentiable. For each $t \geq 0$, both functions $f(t,\cdot), g(t,\cdot) : [0, \infty) \to [0, \infty)$ are strictly increasing. Both go to ∞ as so does c.*

THEOREM 4.5. *Both the optimum value functions $f = f(t,c)$, $g = g(t,c)$ satisfy the Bellman equations:*

$$f_t = \operatorname*{Max}_{x \geq 0}[\log x - f_c x], \quad f(0,c) = 0 \tag{91}$$

$$g_t = \min_{x \geq 0}[x - g_c \log x], \quad g(0,c) = 0\,, \tag{92}$$

respectively.

Letting $\pi^*(t,c)$ be the maximizer of (91) and $\hat{\sigma}(t,c)$ be the minimizer of (92), we have

$$f_t + \log f_c + 1 = 0, \quad \text{for } x^* = \pi^*(t,c) = 1/f_c(t,c) \tag{93}$$

$$g_t = g_c(1 - \log g_c), \quad \text{for } \hat{x} = \hat{\sigma}(t,c) = g_c(t,c)\,, \tag{94}$$

respectively. Thus we have optimum value function, optimizer and optimum partition number as follows:

$$f(t,c) = t \log \frac{c}{t}, \quad \pi^*(t,c) = \frac{c}{t}; \quad x^*(s) = \frac{c}{t} \tag{95}$$

$$g(t,c) = te^{c/t}, \quad \hat{\sigma}(t,c) = e^{c/t}; \quad \hat{x}(s) = e^{c/t}. \tag{96}$$

Furthermore, we have an envelopping relation between the regular problem and the fixed problems:

THEOREM 4.6. *(Envelopping Theorem)*

$$f(c) \quad = \quad \operatorname*{Max}_{t \geq 0} f(c,t) \qquad c \in [0, \infty) \tag{97}$$

$$g(c) \quad = \quad \min_{t \geq 0} g(c,t) \qquad c \in [0, \infty). \tag{98}$$

By letting $h_t(c) := h(t,c)$, we have the inverse relationship between Main and Inverse Partition Problems:

THEOREM 4.7. *(Inverse Theorem)*

$$\text{(i)} \quad g_t = f_t^{-1}, \ f_t = g_t^{-1} \tag{99}$$

$$\text{(ii)} \quad \hat{\sigma}_t = \pi_t^* \circ f_t^{-1}, \quad \pi_t^* = \hat{\sigma}_t \circ g_t^{-1}. \tag{100}$$

5. Optimal Solution Tables and Figures

In this section we specify optimal solutions in tables. Optimal value function, optimal policy and optimal partition are illustrated. Further the forementioned inverse relation are also shown in tables. The specification verifies that all the results in Inverese Theorems are valid.

Table 1. Optimal Solution for MPP

given quantity n	maximum value $f(n)$	optimal policy $\pi^*(n)$	optimal partition $n =$
1	1	–	$1 = 1$
2	2	–	$2 = 2$
3	3	–	$3 = 3$
4	4	–	$4 = 4 = 2 + 2$
5	6	2,3	$5 = 3 + 2$
6	9	3	$6 = 3 + 3$
7	12	2,3,4	$7 = 3 + 2 + 2 = 3 + 4$
8	18	2,3	$8 = 3 + 3 + 2$
9	27	3	$9 = 3 + 3 + 3$
10	36	2,3,4	$10 = 3 + 3 + 2 + 2 = 3 + 3 + 4$
11	54	2,3	$11 = 3 + 3 + 3 + 2$
12	81	3	$12 = 3 + 3 + 3 + 3$
13	108	2,3,4	$13 = 3 + 3 + 3 + 2 + 2 = 3 + 3 + 3 + 4$
14	162	2,3	$14 = 3 + 3 + 3 + 3 + 2$
15	243	3	$15 = 3 + 3 + 3 + 3 + 3$
16	324	2,3,4	$16 = 3 + 3 + 3 + 3 + 2 + 2 = 3 + 3 + 3 + 3 + 4$
17	486	2,3	$17 = 3 + 3 + 3 + 3 + 3 + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3m$	3^m	3	$3m = 3 + \cdots + 3$
$3m + 1$	$4 \cdot 3^{m-1}$	2,3,4	$3m + 1 = 3 + \cdots + 3 + 2 + 2 = 3 + \cdots + 3 + 4$
$3m + 2$	$2 \cdot 3^m$	2,3	$3m + 2 = 3 + \cdots + 3 + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.1 Optimal Solution for IPP

given quantity n	minimum value $g(n)$	optimal policy $\hat{\sigma}(n)$	optimal partition $\geq n(\ = n$ or $\ > n)$
1	1	–	$1 = 1$
2	2	–	$\mathit{2} = \mathit{2}$
3	3	–	$\mathit{3} = \mathit{3}$
4	4	–	$\mathit{2}^{\mathit{2}} = \mathit{4} = \mathit{4}$
5	5	–	$3 \cdot 2 > \mathit{5} = \mathit{5}$
6	5	2,3	$\mathit{3} \cdot \mathit{2} = \mathit{6}$
7	6	2,3,4	$3^2 > 2^3 > 7$
8	6	2,3,4	$3^2 > \mathit{2}^{\mathit{3}} = \mathit{8}$
9	6	3	$\mathit{3}^{\mathit{2}} = \mathit{9}$
10	7	2,3,4,5	$3 \cdot 2^2 > \mathit{5} \cdot \mathit{2} = \mathit{10}$
11	7	2,3,4	$3 \cdot 2^2 > 11$
12	7	2,3,4	$\mathit{3} \cdot \mathit{2}^{\mathit{2}} = \mathit{12}$
13	8	2,3,4,5	$3^2 \cdot 2 > 2^4 > 5 \cdot 3 > 13$
14	8	2,3,4,5	$3^2 \cdot 2 > 2^4 > 5 \cdot 3 > 14$
15	8	2,3,4,5	$3^2 \cdot 2 > 2^4 > \mathit{5} \cdot \mathit{3} = \mathit{15}$
16	8	2,3,4	$3^2 \cdot 2 > \mathit{2}^{\mathit{4}} = \mathit{16}$
17	8	2,3	$3^2 \cdot 2 > 17$
18	8	2,3	$\mathit{3}^{\mathit{2}} \cdot \mathit{2} = \mathit{18}$
19	9	2,3,4,5	$3^3 > 3 \cdot 2^3 > 5 \cdot 2^2 > 19$
20	9	2,3,4,5	$3^3 > 3 \cdot 2^3 > \mathit{5} \cdot \mathit{2}^{\mathit{2}} = \mathit{20}$
21	9	2,3,4	$3^3 > 3 \cdot 2^3 > 21$
22	9	2,3,4	$3^3 > 3 \cdot 2^3 > 22$
23	9	2,3,4	$3^3 > 3 \cdot 2^3 > 23$
24	9	2,3,4	$3^3 > \mathit{3} \cdot \mathit{2}^{\mathit{3}} = \mathit{24}$
25	9	3	$3^3 > 25$
26	9	3	$3^3 > 26$
27	9	3	$\mathit{3}^{\mathit{3}} = \mathit{27}$
28	10	2,3,4,5	$3^2 \cdot 2^2 > 2^5 > 5 \cdot 3 \cdot 2 > 28$
29	10	2,3,4,5	$3^2 \cdot 2^2 > 2^5 > 5 \cdot 3 \cdot 2 > 29$
30	10	2,3,4,5	$3^2 \cdot 2^2 > 2^5 > \mathit{5} \cdot \mathit{3} \cdot \mathit{2} = \mathit{30}$
31	10	2,3,4	$3^2 \cdot 2^2 > 2^5 > 31$
32	10	2,3,4	$3^2 \cdot 2^2 > \mathit{2}^{\mathit{5}} = \mathit{32}$
33	10	2,3,4	$3^2 \cdot 2^2 > 33$
34	10	2,3,4	$3^2 \cdot 2^2 > 34$
35	10	2,3,4	$3^2 \cdot 2^2 > 35$
36	10	2,3,4	$\mathit{3}^{\mathit{2}} \cdot \mathit{2}^{\mathit{2}} = \mathit{36}$ (continued)

Table 2.2 Optimal Solution for IPP

given quantity n	minimum value $g(n)$	optimal policy $\hat{\sigma}(n)$	optimal partition $\geq n(\;= n$ or $\;> n)$
36	10	2,3,4	$3^2 \cdot 2^2 = 36$
37	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 5 \cdot 2^3 > 37$
38	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 5 \cdot 2^3 > 38$
39	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 5 \cdot 2^3 > 39$
40	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 5 \cdot 2^3 = 40$
41	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 41$
42	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 42$
43	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 43$
44	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 > 44$
45	11	2,3,4,5	$3^3 \cdot 2 > 3 \cdot 2^4 > 5 \cdot 3^2 = 45$
46	11	2,3,4	$3^3 \cdot 2 > 3 \cdot 2^4 > 46$
47	11	2,3,4	$3^3 \cdot 2 > 3 \cdot 2^4 > 47$
48	11	2,3,4	$3^3 \cdot 2 > 3 \cdot 2^4 = 48$
49	11	2,3	$3^3 \cdot 2 > 49$
50	11	2,3	$3^3 \cdot 2 > 50$
⋮	⋮	⋮	⋮
53	11	2,3	$3^3 \cdot 2 > 53$
54	11	2,3	$3^3 \cdot 2 = 54$
55	12	2,3,4,5	$3^4 > 3^2 \cdot 2^3 > 2^6 > 5 \cdot 3 \cdot 2^2 > 55$
56	12	2,3,4,5	$3^4 > 3^2 \cdot 2^3 > 2^6 > 5 \cdot 3 \cdot 2^2 > 56$
⋮	⋮	⋮	⋮
59	12	2,3,4,5	$3^4 > 3^2 \cdot 2^3 > 2^6 > 5 \cdot 3 \cdot 2^2 > 59$
60	12	2,3,4,5	$3^4 > 3^2 \cdot 2^3 > 2^6 > 5 \cdot 3 \cdot 2^2 = 60$
61	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 2^6 > 61$
⋮	⋮	⋮	⋮ (continued)

Table 2.3 Optimal Solution for IPP

given quantity n	minimum value $g(n)$	optimal policy $\hat{\sigma}(n)$	optimal partition $\geq n(\; = n$ or $\; > n)$
⋮	⋮	⋮	⋮
60	12	2,3,4,5	$3^4 > 3^2 \cdot 2^3 > 2^6 > \mathit{5 \cdot 3 \cdot 2^2 = 60}$
61	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 2^6 > 61$
62	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 2^6 > 62$
63	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 2^6 > 63$
64	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > \mathit{2^6 = 64}$
65	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 65$
66	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 66$
⋮	⋮	⋮	⋮
71	12	2,3,4	$3^4 > 3^2 \cdot 2^3 > 71$
72	12	2,3,4	$3^4 > \mathit{3^2 \cdot 2^3 = 72}$
73	12	3	$3^4 > 73$
74	12	3	$3^4 > 74$
⋮	⋮	⋮	⋮
80	12	3	$3^4 > 80$
81	12	3	$\mathit{3^4 = 81}$
82	13	2,3,4,5	$3^3 \cdot 2^2 > 3 \cdot 2^5 > 5 \cdot 3^2 \cdot 2 > 82$
83	13	2,3,4,5	$3^3 \cdot 2^2 > 3 \cdot 2^5 > 5 \cdot 3^2 \cdot 2 > 83$
⋮	⋮	⋮	⋮
89	13	2,3,4,5	$3^3 \cdot 2^2 > 3 \cdot 2^5 > 5 \cdot 3^2 \cdot 2 > 89$
90	13	2,3,4,5	$3^3 \cdot 2^2 > 3 \cdot 2^5 > \mathit{5 \cdot 3^2 \cdot 2 = 90}$
91	13	2,3,4	$3^3 \cdot 2^2 > 3 \cdot 2^5 > 91$
92	13	2,3,4	$3^3 \cdot 2^2 > 3 \cdot 2^5 > 92$
⋮	⋮	⋮	⋮
95	13	2,3,4	$3^3 \cdot 2^2 > 3 \cdot 2^5 > 95$
96	13	2,3,4	$3^3 \cdot 2^2 > \mathit{3 \cdot 2^5 = 96}$
97	13	2,3,4	$3^3 \cdot 2^2 > 97$
98	13	2,3,4	$3^3 \cdot 2^2 > 98$
⋮	⋮	⋮	⋮
107	13	2,3,4	$3^3 \cdot 2^2 > 107$
108	13	2,3,4	$\mathit{3^3 \cdot 2^2 = 108}$
⋮	⋮	⋮	⋮ (continued)

Table 2.4 Optimal Solution for IPP

given quantity n	minimum value $g(n)$	optimal policy $\hat{\sigma}(n)$	optimal partition $\geq n(\ = n$ or $\ > n)$
108	13	2,3,4	$3^3 \cdot 2^2 = 108$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^m + 1$	$3m+1$	2,3,4,5	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2$ $> 3^{m-5} \cdot 2^8 > 3^m + 1$
$3^m + 2$	$3m+1$	2,3,4,5	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2$ $> 3^{m-5} \cdot 2^8 > 3^m + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{m-5} \cdot 2^8$	$3m+1$	2,3,4,5	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2$ $> 3^{m-5} \cdot 2^8 = 3^{m-5} \cdot 2^8$
$3^{m-5} \cdot 2^8 + 1$	$3m+1$	2,3,4,5	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2$ $> 3^{m-5} \cdot 2^8 + 1$
$3^{m-5} \cdot 2^8 + 2$	$3m+1$	2,3,4,5	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2$ $> 3^{m-5} \cdot 2^8 + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$5 \cdot 3^{m-2} \cdot 2$	$3m+1$	2,3,4,5	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2$ $= 5 \cdot 3^{m-2} \cdot 2$
$5 \cdot 3^{m-2} \cdot 2 + 1$	$3m+1$	2,3,4	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 + 1$
$5 \cdot 3^{m-2} \cdot 2 + 2$	$3m+1$	2,3,4	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 > 5 \cdot 3^{m-2} \cdot 2 + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{m-3} \cdot 2^5$	$3m+1$	2,3,4	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 = 3^{m-3} \cdot 2^5$
$3^{m-3} \cdot 2^5 + 1$	$3m+1$	2,3,4	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 + 1$
$3^{m-3} \cdot 2^5 + 2$	$3m+1$	2,3,4	$3^{m-1} \cdot 2^2 > 3^{m-3} \cdot 2^5 + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{m-1} \cdot 2^2$	$3m+1$	2,3,4	$3^{m-1} \cdot 2^2 = 3^{m-1} \cdot 2^2$
$3^{m-1} \cdot 2^2 + 1$	$3m+2$	2,3,4,5	$3^m \cdot 2 > 3^{m-2} \cdot 2^4 > 5 \cdot 3^{m-1} > 3^{m-4} \cdot 2^7$ $> 5 \cdot 3^{m-3} \cdot 2^3 > 3^{m-6} \cdot 2^{10} > 3^{m-1} \cdot 2^2 + 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$ (continued)

Table 2.5 Optimal Solution for IPP

given quantity n	minimum value $g(n)$	optimal policy $\hat{\sigma}(n)$	optimal partition $\geq n(\ = n$ or $\ > n)$
$3^{m-1}\cdot 2^2+1$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3 > 3^{m-6}\cdot 2^{10} > 3^{m-1}\cdot 2^2+1$
$3^{m-1}\cdot 2^2+2$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3 > 3^{m-6}\cdot 2^{10} > 3^{m-1}\cdot 2^2+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{m-6}\cdot 2^{10}$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3 > 3^{m-6}\cdot 2^{10} = 3^{m-6}\cdot 2^{10}$
$3^{m-6}\cdot 2^{10}+1$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3 > 3^{m-6}\cdot 2^{10}+1$
$3^{m-6}\cdot 2^{10}+2$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3 > 3^{m-6}\cdot 2^{10}+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$5\cdot 3^{m-3}\cdot 2^3$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3 = 5\cdot 3^{m-3}\cdot 2^3$
$5\cdot 3^{m-3}\cdot 2^3+1$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3+1$
$5\cdot 3^{m-3}\cdot 2^3+2$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 > 5\cdot 3^{m-3}\cdot 2^3+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{m-4}\cdot 2^7$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7 = 3^{m-4}\cdot 2^7$
$3^{m-4}\cdot 2^7+1$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7+1$
$3^{m-4}\cdot 2^7+2$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} > 3^{m-4}\cdot 2^7+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$5\cdot 3^{m-1}$	$3m+2$	2,3,4,5	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1} = 5\cdot 3^{m-1}$
$5\cdot 3^{m-1}+1$	$3m+2$	2,3,4	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1}+1$
$5\cdot 3^{m-1}+2$	$3m+2$	2,3,4	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 > 5\cdot 3^{m-1}+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{m-2}\cdot 2^4$	$3m+2$	2,3,4	$3^m\cdot 2 > 3^{m-2}\cdot 2^4 = 3^{m-2}\cdot 2^4$
$3^{m-2}\cdot 2^4+1$	$3m+2$	2,3	$3^m\cdot 2 > 3^{m-2}\cdot 2^4+1$
$3^{m-2}\cdot 2^4+2$	$3m+2$	2,3	$3^m\cdot 2 > 3^{m-2}\cdot 2^4+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^m\cdot 2$	$3m+2$	2,3	$3^m\cdot 2 = 3^m\cdot 2$
$3^m\cdot 2+1$	$\vdots$	$\vdots$	$\vdots$ (continued)

Table 2.6 Optimal Solution for IPP

given quantity n	minimum value $g(n)$	optimal policy $\hat{\sigma}(n)$	optimal partition $\geq n(\ = n$ or $\ > n)$
⋮	⋮	⋮	⋮
$3^m \cdot 2$	$3m+2$	2,3	$3^m \cdot 2 = 3^m \cdot 2$
$3^m \cdot 2 + 1$	$3m+3$	2,3,4,5	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 > 3^{m-5} \cdot 2^9 > 3^m \cdot 2 + 1$
$3^m \cdot 2 + 2$	$3m+3$	2,3,4,5	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 > 3^{m-5} \cdot 2^9 > 3^m \cdot 2 + 2$
⋮	⋮	⋮	⋮
$3^{m-5} \cdot 2^9$	$3m+3$	2,3,4,5	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 > 3^{m-5} \cdot 2^9 = 3^{m-5} \cdot 2^9$
$3^{m-5} \cdot 2^9 + 1$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 > 3^{m-5} \cdot 2^9 + 1$
$3^{m-5} \cdot 2^9 + 2$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 > 3^{m-5} \cdot 2^9 + 2$
⋮	⋮	⋮	⋮
$5 \cdot 3^{m-2} \cdot 2^2$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $5 \cdot 3^{m-2} \cdot 2^2 = 5 \cdot 3^{m-2} \cdot 2^2$
$5 \cdot 3^{m-2} \cdot 2^2 + 1$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 + 1$
$5 \cdot 3^{m-2} \cdot 2^2 + 2$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $> 5 \cdot 3^{m-2} \cdot 2^2 + 2$
⋮	⋮	⋮	⋮
$3^{m-3} \cdot 2^6$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6$ $= 3^{m-3} \cdot 2^6$
$3^{m-3} \cdot 2^6 + 1$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6 + 1$
$3^{m-3} \cdot 2^6 + 2$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 > 3^{m-3} \cdot 2^6 + 2$
⋮	⋮	⋮	⋮
$3^{m-1} \cdot 2^3$	$3m+3$	2,3,4	$3^{m+1} > 3^{m-1} \cdot 2^3 = 3^{m-1} \cdot 2^3$
$3^{m-1} \cdot 2^3 + 1$	$3m+3$	3	$3^{m+1} > 3^{m-1} \cdot 2^3 + 1$
$3^{m-1} \cdot 2^3 + 2$	$3m+3$	3	$3^{m+1} > 3^{m-1} \cdot 2^3 + 2$
⋮	⋮	⋮	⋮
3^{m+1}	$3m+3$	3	$3^{m+1} = 3^{m+1}$
⋮	⋮	⋮	⋮

Table 3.1 Upper-inverse Solution for MPP

given quantity n	u-inverse value $f^{-1}(n)$	composite policy $\pi^*(f^{-1}(n))$	resulting partition $n =$
1	1	–	–
2	2	–	–
3	3	–	–
4	4	–	–
5	5	2,3	$5 = 3 + 2$
6	5	2,3	$5 = 3 + 2$
7	6	3	$6 = 3 + 3$
8	6	3	$6 = 3 + 3$
9	6	3	$6 = 3 + 3$
10	7	2,3,4	$7 = 3 + 2 + 2 = 3 + 4$
11	7	2,3,4	$7 = 3 + 2 + 2 = 3 + 4$
12	7	2,3,4	$7 = 3 + 2 + 2 = 3 + 4$
13	8	2,3	$8 = 3 + 3 + 2$
14	8	2,3	$8 = 3 + 3 + 2$
⋮	⋮	⋮	⋮
18	8	2,3	$8 = 3 + 3 + 2$
19	9	3	$9 = 3 + 3 + 3$
20	9	3	$9 = 3 + 3 + 3$
⋮	⋮	⋮	⋮
27	9	3	$9 = 3 + 3 + 3$
28	10	2,3,4	$10 = 3 + 3 + 2 + 2 = 3 + 3 + 4$
29	10	2,3,4	$10 = 3 + 3 + 2 + 2 = 3 + 3 + 4$
⋮	⋮	⋮	⋮
36	10	2,3,4	$10 = 3 + 3 + 2 + 2 = 3 + 3 + 4$
37	11	2,3	$11 = 3 + 3 + 3 + 2$
38	11	2,3	$11 = 3 + 3 + 3 + 2$
⋮	⋮	⋮	⋮
54	11	2,3	$11 = 3 + 3 + 3 + 2$
55	12	3	$12 = 3 + 3 + 3 + 3$
56	12	3	$12 = 3 + 3 + 3 + 3$
⋮	⋮	⋮	⋮
81	12	3	$12 = 3 + 3 + 3 + 3$
⋮	⋮	⋮	⋮

Table 3.2 Upper-inverse Solution for MPP

given quantity n	u-inverse value $f^{-1}(n)$	composite policy $\pi^*(f^{-1}(n))$	resulting partition $n =$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
54	11	2,3	$11 = 3+3+3+2$
55	12	3	$12 = 3+3+3+3$
56	12	3	$12 = 3+3+3+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
81	12	3	$12 = 3+3+3+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2 \cdot 3^{m-1}+1$	$3m$	3	$3m = 3+\cdots 3$
$2 \cdot 3^{m-1}+2$	$3m$	3	$3m = 3+\cdots 3$
$\vdots$	$\vdots$	$\vdots$	(m-times)
3^m	$3m$	3	$3m = 3+\cdots 3$
3^m+1	$3m+1$	2,3,4	$3m+1 = 3+\cdots 3+2+2 = 3+\cdots 3+4$
3^m+2	$3m+1$	2,3,4	$3m+1 = 3+\cdots 3+2+2 = 3+\cdots 3+4$
$\vdots$	$\vdots$	$\vdots$	($m-1$-times)
$4 \cdot 3^{m-1}$	$3m+1$	2,3,4	$3m+1 = 3+\cdots 3+2+2 = 3+\cdots 3+4$
$4 \cdot 3^{m-1}+1$	$3m+2$	2,3	$3m+2 = 3+\cdots 3+2$
$4 \cdot 3^{m-1}+2$	$3m+2$	2,3	$3m+2 = 3+\cdots 3+2$
$\vdots$	$\vdots$	$\vdots$	(m-times)
$2 \cdot 3^m$	$3m+2$	2,3	$3m+2 = 3+\cdots 3+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4 Lower-inverse Solution for IPP

given quantity n	l-inverse value $g_{-1}(n)$	composite policy $\hat{\sigma}(g_{-1}(n))$	resulting partition $\geq n(\ =n$ or $\ >n)$
1	1	–	–
2	2	–	–
3	3	–	–
4	4	–	–
5	6	2,3	–
6	9	3	$3\cdot 3=9$
7	12	2,3,4	$3\cdot 2\cdot 2=3\cdot 4=12$
8	18	2,3	$3\cdot 3\cdot 2=18$
9	27	3	$3\cdot 3\cdot 3=27$
10	36	2,3,4	$3\cdot 3\cdot 2\cdot 2=3\cdot 3\cdot 4=36$
11	54	2,3	$3\cdot 3\cdot 3\cdot 2=54$
12	81	3	$3\cdot 3\cdot 3\cdot 3=81$
13	108	2,3,4	$3\cdot 3\cdot 3\cdot 2\cdot 2=3\cdot 3\cdot 3\cdot 4=108$
14	162	2,3	$3\cdot 3\cdot 3\cdot 3\cdot 2=162$
15	243	3	$3\cdot 3\cdot 3\cdot 3=243$
16	324	2,3,4	$3\cdot 3\cdot 3\cdot 3\cdot 2\cdot 2=3\cdot 3\cdot 3\cdot 3\cdot 4=324$
17	486	2,3	$3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 2=486$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3m$	3^m	3	$3\cdot 3\cdots 3=3^m$
$3m+1$	$4\cdot 3^{m-1}$	2,3,4	$3\cdot 3\cdots 3\cdot 2\cdot 2=3\cdot 3\cdots 3\cdot 4=4\cdot 3^{m-1}$
$3m+2$	$2\cdot 3^m$	2,3	$3\cdot 3\cdots 3\cdot 2=2\cdot 3^m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

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