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https://doi.org/10.5109/13479

出版情報：Bulletin of informatics and cybernetics. 31 (1), pp.35-46, 1999-03. Research Association of Statistical Sciences
バージョン：
権利関係：

Bulletin of Informatics and Cybernetics, Vol. 31, No. 1, 1999

BOUNDED RISK POINT ESTIMATION OF THE SCALE PARAMETER OF A NEGATIVE EXPONENTIAL DISTRIBUTION

By

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Abstract

We consider the problem of bounded risk point estimation for the scale parameter of a negative exponential distribution under a certain loss function. In this paper we propose a stopping rule and two sequential estimators for the scale parameter. The asymptotic expansions of the risk associated with the sequential estimators are obtained.

Keywords and Phrases: Bounded risk ; negative exponential distribution ; asymptotic expansion ; unifom integrability.

AMS (1991) *subject classification*: Primary 62L12.

1. Introduction

Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d)random variables with the probability density function (pdf)

$$f_{\mu,\sigma}(x) = \sigma^{-1}\exp\{-(x-\mu)/\sigma\}I(x \geq \mu), \tag{1.1}$$

where $-\infty < \mu < \infty$ and $0 < \sigma < \infty$ are two unknown parameters, and $I(A)$ denotes the indicator function of A. This paper deals with the problem of bounded risk point estimation for the scale parameter σ. Suppose that $\delta_n = \delta_n(X_1, \ldots, X_n)$ is an estimator of σ based on a sample $X_1, \ldots, X_n$ of size n and $R(\delta_n)$ is a risk associated with δ_n. We wish to estimate σ by use of the smallest sample size satisfying $R(\delta_n) \leq w$ with a preassigned constant $w > 0$. In this paper, we give an estimator δ_n and its risk $R(\delta_n)$, and consider this problem. Then for small w we can find an asymptotically optimal sample size. However, the optimal sample size contains the unknown scale parameter. Therefore, we will propose a stopping rule and give a natural sequential estimator of σ. Furthermore, we will propose another sequential estimator the risk of which is smaller than that of the natural sequential estimator. Sequential and multistage estimation

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problems for σ were reviewed in Mukhopadhyay (1988) and Ghosh, Mukhopadhyay and Sen (1997) (Section 6.6), for example.

In Section 2 we formulate the problem of bounded risk point estimation for σ and provide results. Section 3 gives their proofs.

2. Formulation and results

In this section we consider the problem of bounded risk point estimation for the scale parameter σ. Let $X_1, X_2, \dots$ be i.i.d random variables with pdf $f_{\mu,\sigma}$ given by (1.1). Set

$$T_n = \min(X_1, \dots, X_n) \quad \text{and} \quad \hat{\sigma}_n = (n-1)^{-1} \sum_{i=1}^{n} (X_i - T_n) \quad \text{for} \quad n \geq 2.$$

In the paper we use the estimator $\hat{\sigma}_n$ for σ. Define the loss function by

$$L(\hat{\sigma}_n) = (\hat{\sigma}_n - \sigma)^2 / \sigma.$$

The risk associated with $\hat{\sigma}_n$ is given by

$$R(\hat{\sigma}_n) = E\{L(\hat{\sigma}_n)\} = (n-1)^{-1}\sigma. \tag{2.1}$$

This weighted loss function is appropriate when there is a possibility that σ may be close to 0, so that small absolute error is too weak a requirement to place on the estimate of σ. Similary the above loss function makes sense if σ may be very large since in that case the requirment of small absolute error may be too stringent. Note that this loss function has the same dimension as observations. Let $w > 0$ be a preassigned number. Then we wish to estimate σ by $\hat{\sigma}_n$ under the above loss function and find the smallest sample size which will satisfy that the risk $R(\hat{\sigma}_n)$ is not greater than w. From (2.1)

$$R(\hat{\sigma}_n) \leq w$$

is equivalent to

$$n \geq (\sigma/w) + 1. \tag{2.2}$$

Let

$$n^* = \sigma/w \quad \text{and} \quad n_0 = n^* + 1. \tag{2.3}$$

For simplicity, n^* is assumed to be an integer. Then n_0 is the optimal fixed sample size in the sense that n_0 is the smallest sample size which satisfies $R(\hat{\sigma}_n) \leq w$. Unfortunately, n_0 contains the unknown parameter σ and there is no fixed sample size procedure which will satisfy that the risk $R(\hat{\sigma}_{n_0})$ with the optimal sample size n_0 is not greater than w. Thus, taking account of (2.2) we propose the following stopping rule by which the sampling is stopped :

$$N = N_w = \inf\{n \geq m : \sum_{i=1}^{n} (X_i - T_n) \leq w(n-1)^2 l(n-1)\}, \tag{2.4}$$

where $m \geq 2$ is a starting sample size and $l(x)$ is a given continuous function on $(0, \infty)$ satisfying that $l(x) > 0$ on $(0, \infty)$ and

$$l(x) = 1 + l_0 x^{-1} + o(x^{-1}) \quad \text{as} \quad x \to \infty \tag{2.5}$$

with a constant l_0. Since $P\{N < \infty\} = 1$ for each $w > 0$, we estimate σ by $\hat{\sigma}_N$. Then the risk associated with $\hat{\sigma}_N$ is given by $R(\hat{\sigma}, N) = E\{(\hat{\sigma}_N - \sigma)^2/\sigma\}$ with $\hat{\sigma} = \{\hat{\sigma}_n; n \geq 1\}$.

We shall first give asymptotic expansions of the expected sample size $E(N)$ and $R(\hat{\sigma}, N)/w$.

THEOREM 2.1. *The following results hold.*

(i) *For* $m \geq 2$,

$$E(N) = n^* + \nu - l_0 + o(1) \quad as \quad w \to 0.$$

(ii) *For* $m \geq 6$,

$$R(\hat{\sigma}, N)/w = 1 - (\nu - l_0 - 4)/n^* + o(1/n^*) \quad as \quad w \to 0,$$

where ν *is the constant given in* Lemma 3.1 (iii) *and approximately* 0.747.

REMARK 2.1. (i) If $v \equiv \nu - l_0 - 4 > 0$, then $R(\hat{\sigma}, N) < w$ for $w > 0$ sufficiently small. (ii) The larger the value of v is, the smaller $R(\hat{\sigma}, N)/w$ is, although the expected sample size becomes larger.

The estimator $\hat{\sigma}_n$ is unbiased, but we do not know whether or not $\hat{\sigma}_N$ is unbiased. The following proposition provides the bias of $\hat{\sigma}_N$.

PROPOSITION 2.1. *For* $m \geq 4$,

$$E(\hat{\sigma}_N) - \sigma = -w + o(w) \quad as \quad w \to 0.$$

According to this proposition, we consider the bias-corrected estimators

$$\hat{\sigma}^* = \{\hat{\sigma}_n^*; n \geq 1\} \quad \text{and} \quad \hat{\sigma}_n^* = \hat{\sigma}_n + w. \tag{2.6}$$

Then, the asymptotic expansion of $R(\hat{\sigma}^*, N)/w$ associated with the bias-corrected estimator $\hat{\sigma}_N^*$ is given by the following theorem.

THEOREM 2.2. *For* $m \geq 6$,

$$R(\hat{\sigma}^*, N)/w = 1 - (\nu - l_0 - 3)/n^* + o(1/n^*) \quad as \quad w \to 0.$$

We compare now the sequential procedure $(\hat{\sigma}^*, N)$ with $(\hat{\sigma}, N)$ from the viewpoint of second order asymptotic relative efficiency. From Theorems 2.1 and 2.2 we have

$$n^*\{R(\hat{\sigma}, N) - R(\hat{\sigma}^*, N)\}/w = v^* - v + o(1) \quad \text{as} \quad w \to 0,$$

where $v = \nu - l_0 - 4$ and $v^* = \nu - l_0 - 3$. Choose $l_0 < \nu - 4$ so that v and v^* are positive. Then both $R(\hat{\sigma}, N)/w$ and $R(\hat{\sigma}^*, N)/w$ are less than 1 for $w > 0$ sufficiently small and $v^* - v = 1$. The value $v^* - v > 0$ means that $n^*\{1 - R(\hat{\sigma}^*, N)/w\}$ is asymptotically $v^* - v$ greater than $n^*\{1 - R(\hat{\sigma}, N)/w\}$. Therefore, the procedure $(\hat{\sigma}^*, N)$ is asymptotically more efficient than $(\hat{\sigma}, N)$ in the above sense.

REMARK 2.2. For example, let us take $l(x) = (1 - 2x^{-1})^2 + x^{-2}$ in (2.5). Then $l_0 = -4 < \nu - 4$, and $v = \nu > 0$ and $v^* = \nu + 1 > 0$. Thus, as $w \to 0$

$$R(\hat{\sigma}, N)/w = 1 - \nu/n^* + o(1/n^*)$$

and

$$R(\hat{\sigma}^*, N)/w = 1 - (\nu + 1)/n^* + o(1/n^*).$$

In this case, we have for w sufficiently small

$$R(\hat{\sigma}, N) < w \quad \text{and} \quad R(\hat{\sigma}^*, N) < w.$$

Therefore the condition on the risk is asymptotically satisfied.

3. Proofs

In this section we shall give the proofs of the results in Section 2. Let

$$Y_{in} = (n - i + 1)(X_{n(i)} - X_{n(i-1)}) \quad \text{for} \quad i = 2, \cdots, n,$$

where $X_{n(1)} \leq X_{n(2)} \leq \cdots \leq X_{n(n)}$ are the order statistics of $X_1, \dots, X_n$. Then $Y_{2n}, \dots, Y_{nn}$ are i.i.d random variables with pdf $f_{0,\sigma}$ and $\hat{\sigma}_n = (n-1)^{-1} \sum_{i=2}^{n} Y_{in}$. Let $W_1, W_2, \dots,$ be a sequence of i.i.d random variables with pdf $f_{0,1}$ and put

$$S_n = \sum_{i=1}^{n} W_i, \quad \overline{W}_n = S_n/n \quad \text{and} \quad Z_n = \sigma W_n.$$

Define

$$Q = Q_w = \inf\{n \geq m - 1 : \; S_n \leq (n^*)^{-1} n^2 l(n) \}. \tag{3.1}$$

Then we have

$$N \stackrel{d}{=} Q + 1, \tag{3.2}$$

which means that the distribution of N is the same as that of $Q+1$. Let

$$R_w = cQ^2 l(Q) - S_Q\ , \quad c = (n^*)^{-1},$$
$$r_c = cQ^2(l(Q)-1) - R_w \quad \text{and} \quad Q^* = c^{1/2}(Q-n^*).$$

Throughout this section, M denotes a generic positive constant which is independent of w. From (2.5), the definition of Q and the results of Woodroofe (1977) we have the following lemma.

LEMMA 3.1. *The following results hold.*

(i) $Q \xrightarrow{a.s.} \infty$, $cQ \xrightarrow{a.s.} 1$ *as* $w \to 0$ *and* $cQ^2 - Q = S_Q - Q - r_c$,
where ' $\xrightarrow{a.s}$ ' *denotes almost sure convergence.*

(ii) *For any fixed* $s>0$, $E\{(cQ)^s\} = O(1)$ *and* $E\{(R_w)^s\} = O(1)$ *as* $w \to 0$.

(iii) $R_w \xrightarrow{d} H$ *and* $r_c \xrightarrow{d} l_0 - H$ *as* $w \to 0$,
where H *is the random variable given in* Theorem 2.1 *of* Woodroofe (1977),
$\nu \equiv E(H) = 1 - \sum_{n=1}^{\infty} n^{-1} E\{(S_n - 2n)^+\}$ *and* ' $\xrightarrow{d}$ ' *stands for convergence in distribution.*

(iv) $\sup_{0<w} E|Q(l(Q)-1)|^s \le M$ *for any fixed* $s>0$.

(v) *For any fixed* $\beta > 0$, $\sup_{0<w<w_0} E|r_c|^\beta \le M$ *for some* $w_0 > 0$.

(vi) $Q^* \xrightarrow{d} N(0,1)$ *as* $w \to 0$,
where $N(0,1)$ *stands for a standard normal random variable.*

(vii) *For any fixed* $s>0$, $\{|Q^*|^s;\ 0<w<w_0\}$ *is uniformly integrable for* $m > \frac{1}{2}s$.

(viii) Q^* *and* R_w *are asymptotically independent.*

(ix) $E(Q) = n^* + \nu - l_0 - 1 + o(1)$ *as* $w \to 0$ *for* $m>1$.

(x) *For any fixed* $s>0$, $E\{(cQ)^{-s}\} = O(1)$ *as* $w \to 0$ *for* $m \ge s+1$.

By Theorem 2 of Chow, Hsiung and Lai (1979) we can get the following lemma.

LEMMA 3.2. *For any fixed* $s>0$, $\{|c^{1/2}(S_Q - Q)|^s;\ 0<w<w_0\}$ *is uniformly integrable for some* $w_0>0$.

LEMMA 3.3. *For any* $x>0$ *and* n *with* $P(N=n)>0$,

$$P\{\hat{\sigma}_n \le x | N=n\} = P\{\sigma \overline{W}_{n-1} \le x | Q = n-1\}.$$

PROOF. Due to Lombard and Swanepoel (1978), $Y \equiv \{\sum_{i=2}^{n} Y_{in} : n \geq 2\}$ has the same distribution as $Z = \{\sum_{i=1}^{n-1} Z_i : n \geq 2\}$ where Y_{in} and Z_i are defined at the beginning of this section. The stopping rule N in (2.4) can be rewritten as

$$N = \inf\{n \geq m : \sum_{i=2}^{n} Y_{in} \leq w(n-1)^2 l(n-1)\}.$$

Thus from (2.3), (3.1) and (3.2) we have

$$\begin{aligned}
&P\{\hat{\sigma}_n \leq x | N = n\} = P\{\hat{\sigma}_n \leq x, N = n\}/P(N = n) \\
&= P\{(n-1)^{-1} \sum_{i=2}^{n} Y_{in} \leq x,\ \sum_{i=2}^{k} Y_{ik} > w(k-1)^2 l(k-1) \text{ for } k = 2, \ldots, n-1, \\
&\qquad \sum_{i=2}^{n} Y_{in} \leq w(n-1)^2 l(n-1)\}/P(Q = n-1) \\
&= P\{(n-1)^{-1} \sum_{i=1}^{n-1} Z_i \leq x,\ \sum_{i=1}^{k-1} Z_i > w(k-1)^2 l(k-1) \text{ for } k = 2, \ldots, n-1, \\
&\qquad \sum_{i=1}^{n-1} Z_i \leq w(n-1)^2 l(n-1)\}/P(Q = n-1) \\
&= P\{\sigma \overline{W}_{n-1} \leq x,\ S_k > (n^*)^{-1} k^2 l(k) \text{ for } k = 1, \ldots, n-2,\ S_{n-1} \leq (n^*)^{-1}(n-1)^2 \\
&\qquad l(n-1)\}/P(Q = n-1) \\
&= P\{\sigma \overline{W}_{n-1} \leq x,\ Q = n-1\}/P(Q = n-1) \\
&= P\{\sigma \overline{W}_Q \leq x | Q = n-1\},
\end{aligned}$$

which concludes the lemma.

LEMMA 3.4. *For any fixed $\beta > 0$, $\{|(S_Q - Q)/Q^{1/2}|^\beta ; 0 < w < w_0\}$ is uniformly integrable for some $w_0 > 0$ if $m > \frac{1}{2}\beta + 1$.*

PROOF. Choose $\alpha > 1$ and $p > 1$ such that $m \geq \frac{1}{2}\alpha p \beta + 1$. Let $q = p/(p-1)$. Then by Hölder's inequality and Lemmas 3.1 and 3.2 we have

$$\begin{aligned}
&E(|(S_Q - Q)/Q^{1/2}|^\beta)^\alpha = E|(cQ)^{-1/2} c^{1/2}(S_Q - Q)|^{\alpha\beta} \\
&\leq (E|cQ|^{-\alpha p \beta/2})^{1/p} (E|c^{1/2}(S_Q - Q)|^{\alpha\beta q})^{1/q} \leq M
\end{aligned}$$

for all $0 < w < w_0$, which gives the uniform integrability of $\{|(S_Q - Q)/Q^{1/2}|^\beta ; 0 < w < w_0\}$. This completes the proof.

Proof of Theorem 2.1

(i) is an immediate consequence of (3.2) and Lemma 3.1. We shall prove (ii). From (2.3), (3.2) and Lemma 3.3 we get

$$
\begin{aligned}
R(\hat{\sigma}, N) &= \sigma^{-1} E(\hat{\sigma}_N - \sigma)^2 = \sigma^{-1} E(\sigma \overline{W}_Q - \sigma)^2 \\
&= wE\{(n^*/Q^2)(S_Q - Q)^2\} \\
&= w[E\{(S_Q - Q)^2/n^*\} + n^* E\{(Q^{-2} - n^{*-2})(S_Q - Q)^2\}],
\end{aligned}
$$

which, together with Theorem 2 of Chow, Robbins and Teicher (1965) and Lemma 3.1, implies

$$E(S_Q - Q)^2 = E(Q) = n^* + \nu - l_0 - 1 + o(1) \quad \text{as} \quad w \to 0.$$

Hence

$$
\begin{aligned}
R(\hat{\sigma}, N)/w = 1 &+ (\nu - l_0 - 1)/n^* + o(1/n^*) \\
&+ (n^*)^{-1} E\{(cQ)^{-2}(1 + cQ)(1 - cQ)(S_Q - Q)^2\}.
\end{aligned}
$$

Set

$$I = (cQ)^{-2}(1 + cQ)(1 - cQ)(S_Q - Q)^2. \tag{3.3}$$

Then

$$n^*\{R(\hat{\sigma}, N)/w - 1\} = \nu - l_0 - 1 + E(I) + o(1) \quad \text{as} \quad w \to 0. \tag{3.4}$$

Since by Lemma 3.1 (i)

$$1 - cQ = -(S_Q - Q)Q^{-1} + r_c Q^{-1}, \tag{3.5}$$

we get from (3.3) that

$$
\begin{aligned}
I &= -(cQ)^{-2}(1 + cQ)(S_Q - Q)^3 Q^{-1} + (cQ)^{-2}(1 + cQ) r_c (S_Q - Q)^2 Q^{-1} \\
&\equiv -I_1 + I_2, \ \text{say}.
\end{aligned} \tag{3.6}
$$

In order to prove (ii), it is sufficient from (3.4) and (3.6) to show

$$E(I_1) = -5 + o(1) \quad \text{as} \quad w \to 0 \tag{3.7}$$

and

$$E(I_2) = 2l_0 - 2\nu + o(1) \quad \text{as} \quad w \to 0. \tag{3.8}$$

We shall first prove (3.7). It follows from (3.5) that

$$
\begin{aligned}
I_1 &= (cQ)^{-2}\{2c(S_Q - Q)^3 + (1 - cQ)(S_Q - Q)^3/Q\} \\
&= 2(cQ)^{-2}c(S_Q - Q)^3 - (cQ)^{-2}(S_Q - Q)^4/Q^2 + (cQ)^{-2}r_c(S_Q - Q)^3/Q^2 \\
&\equiv 2I_{11} - I_{12} + I_{13}, \text{ say.}
\end{aligned}
\tag{3.9}
$$

By Lemma 3.1 (i) and the result of Anscombe (1952) we get $(S_Q - Q)^4/Q^2 \xrightarrow{d} \chi_1^4$ as $w \to 0$, where χ_1^2 denotes a chi-squared random variable with one degree of freedom. Hence it follows from Lemma 3.1 that

$$
I_{12} \xrightarrow{d} \chi_1^4 \text{ and } I_{13} \xrightarrow{p} 0 \text{ as } w \to 0, \tag{3.10}
$$

where ' $\xrightarrow{p}$ ' stands for convergence in probability. Since $m > 5$, we can choose constants $\alpha > 1$ and $p > 1$ such that $m \geq 4\alpha p + 1$. Let $q = p/(p-1)$. Then by Hölder's inequality and Lemmas 3.1 and 3.2 we get

$$
\begin{aligned}
E|I_{12}|^\alpha &= E\left|(cQ)^{-4}\{c^{1/2}(S_Q - Q)\}^4\right|^\alpha \\
&\leq \left(E(cQ)^{-4\alpha p}\right)^{1/p}\left(E|c^{1/2}(S_Q - Q)|^{4\alpha q}\right)^{1/q} \leq M
\end{aligned}
$$

for all $0 < w < w_0$. This yields the uniform integrability of $\{I_{12}\ ; 0 < w < w_0\}$. Thus from (3.10)

$$
E(I_{12}) = 3 + o(1) \text{ as } w \to 0. \tag{3.11}
$$

Throughout this section we use the above method to prove uniform integrabilities of sequences of random variables, and omit their proofs because the calculations are tedious. Since $I_{13} = \{c^{1/2}(S_Q - Q)\}^3 r_c (cQ)^{-7/2} Q^{-1/2}$, it follows from Lemmas 3.1 and 3.2 that $\{I_{13}\ ; 0 < w < w_0\}$ is uniformly integrable. Thus by (3.10) we get

$$
E(I_{13}) = o(1) \text{ as } w \to 0,
$$

which, together with (3.9) and (3.11), yields

$$
E(I_1) = 2E(I_{11}) - 3 + o(1) \text{ as } w \to 0. \tag{3.12}
$$

Set

$$
J_1 = \{(cQ)^{-2} - 1\}c(S_Q - Q)^3 \text{ and } J_2 = (S_Q - Q)^3. \tag{3.13}
$$

Then

$$
E(I_{11}) = E(J_1) + cE(J_2). \tag{3.14}
$$

From Theorem 9 of Chow et al. (1965) and Lemma 3.1 (ix) we have

$$
E(J_2) = 2c^{-1} + 2(\nu - l_0 - 1) + 3E\{Q(S_Q - Q)\} + o(1). \tag{3.15}
$$

By (3.5) and Wald's equation we get

$$\begin{aligned} E\{Q(S_Q-Q)\} &= c^{-1}E\{(cQ)(S_Q-Q)\} \\ &= c^{-1}[E(S_Q-Q)+E\{(S_Q-Q)^2/Q\}-E\{r_c(S_Q-Q)/Q\}] \\ &= c^{-1}[E\{(S_Q-Q)^2/Q\}-E\{r_c(S_Q-Q)/Q\}]. \end{aligned} \tag{3.16}$$

It follows from Lemma 3.4 that $\{(S_Q-Q)^2/Q\ ; 0<w<w_0\}$ is uniformly integrable. Hence by the fact that $(S_Q-Q)^2/Q \xrightarrow{d} \chi_1^2$ as $w \to 0$, we have

$$E\{(S_Q-Q)^2/Q\} = 1+o(1) \quad \text{as} \quad w \to 0. \tag{3.17}$$

By using Lemma 3.1 and the strong law of large numbers (SLLN), we can show that $r_c(S_Q-Q)/Q \xrightarrow{p} 0$ as $w \to 0$ and that $\{r_c(S_Q-Q)/Q\ ; 0<w<w_0\}$ is uniformly integrable, which yield

$$E\{r_c(S_Q-Q)/Q\} = o(1) \quad \text{as} \quad w \to 0.$$

Thus (3.16) and (3.17) give $E\{Q(S_Q-Q)\} = c^{-1}(1+o(1))$ as $w \to 0$, which, together with (3.14) and (3.15), implies

$$E(I_{11}) = E(J_1)+5+o(1) \quad \text{as} \quad w \to 0. \tag{3.18}$$

Since $\{(cQ)^{-2}-1\}c = -\{(cQ)^{-1}+1\}c(Q-n^*)/Q$ and $c(Q-n^*) = (S_Q-Q)/Q - r_c/Q$ by Lemma 3.1 (i), we get from (3.13) that

$$\begin{aligned} J_1 &= -\{(cQ)^{-1}+1\}(S_Q-Q)^4/Q^2 + \{(cQ)^{-1}+1\}r_c(S_Q-Q)^3/Q^2 \\ &\equiv -J_{11}+J_{12}\ , \text{ say.} \end{aligned} \tag{3.19}$$

From Lemma 3.1 and the fact that $(S_Q-Q)/Q^{1/2} \xrightarrow{d} N(0,1)$ as $w \to 0$, we have

$$J_{11} \xrightarrow{d} 2\chi_1^4 \quad \text{and} \quad J_{12} \xrightarrow{p} 0 \quad \text{as} \quad w \to 0. \tag{3.20}$$

By Lemmas 3.1 and 3.4 we can show the uniform integrability of $\{J_{11}\ ; 0<w<w_0\}$, which, together with (3.20), yields

$$E(J_{11}) = 6+o(1) \quad \text{as} \quad w \to 0. \tag{3.21}$$

By using (3.20) and the uniform integrability of $\{J_{12}\ ; 0<w<w_0\}$ we have $E(J_{12}) = o(1)$ as $w \to 0$, which, together with (3.19) and (3.21), implies $E(J_1) = -6+o(1)$ as $w \to 0$. Thus from (3.12) and (3.18) we get $E(I_1) = -5+o(1)$ as $w \to 0$, which gives (3.7). Next we shall show (3.8). Lemma 3.1 (i) implies

$$(cQ)^{-2}(S_Q-Q)^2 = c^{-1}Q^{*2} + 2(cQ)^{-1}(Q-n^*)r_c + (cQ)^{-2}{r_c}^2.$$

Hence from (3.6)

$$I_2 = \{1+(cQ)^{-1}\}r_c{Q^*}^2 + 2\{1+(cQ)^{-1}\}{r_c}^2(Q-n^*)/Q + (1+cQ)(cQ)^{-2}{r_c}^3Q^{-1}$$

$$\equiv I_{21} + I_{22} + I_{23}, \text{ say.} \tag{3.22}$$

From Lemma 3.1 we have

$$I_{22} \xrightarrow{p} 0 \text{ and } I_{23} \xrightarrow{p} 0 \text{ as } w \to 0. \tag{3.23}$$

According to the fact that for $\alpha > 1$, $|I_{22}|^\alpha \le M|(cQ)^{-\frac{1}{2}}Q^*{r_c}^2|^\alpha + |(cQ)^{-\frac{3}{2}}Q^*{r_c}^2|^\alpha$ and Lemma 3.1, we obtain the uniform integrability of $\{I_{22};\ 0 < w < w_0\}$, which, together with (3.23), implies

$$E(I_{22}) = o(1) \text{ as } w \to 0. \tag{3.24}$$

By using the fact that $Q^{-1} \le 1$ we get the uniform integrability of $\{I_{23}\ ; 0 < w < w_0\}$, which, together with (3.23), yields

$$E(I_{23}) = o(1) \text{ as } w \to 0.$$

Hence from (3.22) and (3.24) we get

$$E(I_2) = E(I_{21}) + o(1) \text{ as } w \to 0. \tag{3.25}$$

Since $r_c = cQ^2(l(Q)-1) - R_w$,

$$I_{21} = \{1+(cQ)^{-1}\}{Q^*}^2cQ^2(l(Q)-1) - \{1+(cQ)^{-1}\}{Q^*}^2R_w$$

$$\equiv I_{211} - I_{212}, \text{ say.} \tag{3.26}$$

Let K be a standard normal random variable which is independent of H. Then by Lemma 3.1 (iii), (vi) and (viii) we have

$$(Q^*, R_w) \xrightarrow{d} (K, H) \text{ and } cQ^2(l(Q)-1) \xrightarrow{a.s.} l_0 \text{ as } w \to 0,$$

which imply

$$I_{211} \xrightarrow{d} 2l_0K^2 \text{ and } I_{212} \xrightarrow{d} 2K^2H \text{ as } w \to 0. \tag{3.27}$$

From Lemma 3.1 we can show the uniform integrability of $\{I_{211}; 0 < w < w_0\}$, which yields

$$E(I_{211}) = 2l_0 + o(1) \text{ as } w \to 0. \tag{3.28}$$

By the independency of K^2 and H we have

$$E(K^2H) = E(K^2)E(H) = \nu. \tag{3.29}$$

Since we can show the unifom integrability of $\{I_{212}\ ; 0 < w < w_0\}$, we have by (3.27) and (3.29) that

$$E(I_{212}) = 2\nu + o(1) \quad \text{as} \quad w \to 0,$$

which, together with (3.25), (3.26) and (3.28), implies

$$E(I_2) = 2l_0 - 2\nu + o(1) \quad \text{as} \quad w \to 0.$$

Thus (3.8) is shown. Therefore the proof of Theorem 2.1 is complete.

Proof of Proposition 2.1.

From (2.3), (3.2), Lemma 3.3 and Wald's equation we have

$$\begin{aligned} E(\hat{\sigma}_N) - \sigma &= \sum_{n=m}^{\infty} \int (x-\sigma) dP(\hat{\sigma}_n \le x | N = n) P(N = n) \\ &= \sum_{n=m}^{\infty} \int (x-\sigma) dP(\sigma \overline{W}_{n-1} \le x | Q = n-1) P(Q = n-1) \\ &= E(\sigma \overline{W}_Q - \sigma) \\ &= wE(S_Q - Q) + wn^* E\{(Q^{-1} - {n^*}^{-1})(S_Q - Q)\} \\ &= wE\{(cQ)^{-1}(1 - cQ)(S_Q - Q)\} \\ &\equiv wE(I), \quad \text{say.} \end{aligned} \tag{3.30}$$

By (3.5), Lemma 3.1 and SLLN we get

$$\begin{aligned} I &= -(cQ)^{-1}(S_Q - Q)^2/Q + (cQ)^{-1} r_c (S_Q - Q)/Q \\ &\xrightarrow{d} -\chi_1^2 \quad \text{as} \quad w \to 0. \end{aligned} \tag{3.31}$$

Since $I = -(cQ)^{-1} Q^* c^{1/2} (S_Q - Q)$ it follows from Lemmas 3.1 and 3.2 that $\{I\ ; 0 < w < w_0\}$ is uniformly integrable. Thus (3.31) implies that $E(I) = -1 + o(1)$ as $w \to 0$, which, together with (3.30), concludes the proposition. Therefore, the proof is complete.

Proof of Theorem 2.2

From (2.6) we get

$$R(\hat{\sigma}^*, N) = R(\hat{\sigma}, N) + 2w\sigma^{-1} E(\hat{\sigma}_N - \sigma) + w^2 \sigma^{-1}.$$

Thus, by using (2.3), Theorem 2.1 and Proposition 2.1 we obtain Theorem 2.2. This completes the proof.

Acknowledgements

The first author was partially supported by a Grant-in-Aid for Science Research, number 09640251, from the Ministry of Education, Science and Culture of Japan. We would like to thank the referee for his helpful comments.

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Received July 16, 1998
Revised October 19, 1998