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BOUNDED RISK POINT ESTIMATION OF THE SCALE PARAMETER OF A NEGATIVE EXPONENTIAL DISTRIBUTION

By

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Abstract

We consider the problem of bounded risk point estimation for the scale parameter of a negative exponential distribution under a certain loss function. In this paper we propose a stopping rule and two sequential estimators for the scale parameter. The asymptotic expansions of the risk associated with the sequential estimators are obtained.

Keywords and Phrases: Bounded risk ; negative exponential distribution ; asymptotic expansion ; uniform integrability.


1. Introduction

Let \( X_1, X_2, \ldots \) be independent and identically distributed (i.i.d) random variables with the probability density function (pdf)

\[
f_{\mu,\sigma}(x) = \sigma^{-1} \exp\left\{-(x - \mu)/\sigma\right\} I(x \geq \mu),
\]

where \(-\infty < \mu < \infty\) and \(0 < \sigma < \infty\) are two unknown parameters, and \(I(A)\) denotes the indicator function of \(A\). This paper deals with the problem of bounded risk point estimation for the scale parameter \(\sigma\). Suppose that \(\delta_n = \delta_n(X_1, \ldots, X_n)\) is an estimator of \(\sigma\) based on a sample \(X_1, \ldots, X_n\) of size \(n\) and \(R(\delta_n)\) is a risk associated with \(\delta_n\). We wish to estimate \(\sigma\) by use of the smallest sample size satisfying \(R(\delta_n) \leq w\) with a preassigned constant \(w > 0\). In this paper, we give an estimator \(\delta_n\) and its risk \(R(\delta_n)\), and consider this problem. Then for small \(w\) we can find an asymptotically optimal sample size. However, the optimal sample size contains the unknown scale parameter. Therefore, we will propose a stopping rule and give a natural sequential estimator of \(\sigma\). Furthermore, we will propose another sequential estimator the risk of which is smaller than that of the natural sequential estimator. Sequential and multistage estimation

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problems for \( \sigma \) were reviewed in Mukhopadhyay (1988) and Ghosh, Mukhopadhyay and Sen (1997) (Section 6.6), for example.

In Section 2 we formulate the problem of bounded risk point estimation for \( \sigma \) and provide results. Section 3 gives their proofs.

2. Formulation and results

In this section we consider the problem of bounded risk point estimation for the scale parameter \( \sigma \). Let \( X_1, X_2, \ldots \) be i.i.d random variables with pdf \( f_{\mu, \sigma} \) given by (1.1). Set

\[
T_n = \min(X_1, \ldots, X_n) \quad \text{and} \quad \hat{\sigma}_n = (n - 1)^{-1} \sum_{i=1}^{n} (X_i - T_n) \quad \text{for} \quad n \geq 2.
\]

In the paper we use the estimator \( \hat{\sigma}_n \) for \( \sigma \). Define the loss function by

\[
L(\hat{\sigma}_n) = (\hat{\sigma}_n - \sigma)^2 / \sigma.
\]

The risk associated with \( \hat{\sigma}_n \) is given by

\[
R(\hat{\sigma}_n) = E\{L(\hat{\sigma}_n)\} = (n - 1)^{-1} \sigma.
\] (2.1)

This weighted loss function is appropriate when there is a possibility that \( \sigma \) may be close to 0, so that small absolute error is too weak a requirement to place on the estimate of \( \sigma \). Similarly the above loss function makes sense if \( \sigma \) may be very large since in that case the requirement of small absolute error may be too stringent. Note that this loss function has the same dimension as observations. Let \( w > 0 \) be a preassigned number. Then we wish to estimate \( \sigma \) by \( \hat{\sigma}_n \) under the above loss function and find the smallest sample size which will satisfy that the risk \( R(\hat{\sigma}_n) \) is not greater than \( w \). From (2.1)

\[
R(\hat{\sigma}_n) \leq w
\]

is equivalent to

\[
n \geq (\sigma / w) + 1.
\] (2.2)

Let

\[
n^* = \sigma / w \quad \text{and} \quad n_0 = n^* + 1.
\] (2.3)

For simplicity, \( n^* \) is assumed to be an integer. Then \( n_0 \) is the optimal fixed sample size in the sense that \( n_0 \) is the smallest sample size which satisfies \( R(\hat{\sigma}_n) \leq w \). Unfortunately, \( n_0 \) contains the unknown parameter \( \sigma \) and there is no fixed sample size procedure which will satisfy that the risk \( R(\hat{\sigma}_{n_0}) \) with the optimal sample size \( n_0 \) is not greater than \( w \). Thus, taking account of (2.2) we propose the following stopping rule by which the sampling is stopped:

\[
N = N_w = \inf\{n \geq m : \sum_{i=1}^{n} (X_i - T_n) \leq w(n - 1)^2 l(n - 1)\},
\] (2.4)
where \( m \geq 2 \) is a starting sample size and \( l(x) \) is a given continuous function on \((0, \infty)\) satisfying that \( l(x) > 0 \) on \((0, \infty)\) and

\[
l(x) = 1 + l_0x^{-1} + o(x^{-1}) \quad \text{as} \quad x \to \infty
\]  

(2.5)

with a constant \( l_0 \). Since \( P\{N < \infty\} = 1 \) for each \( w > 0 \), we estimate \( \sigma \) by \( \hat{\sigma}_N \). Then the risk associated with \( \hat{\sigma}_N \) is given by \( R(\hat{\sigma}, N) = E\{(\hat{\sigma}_N - \sigma)^2/\sigma\} \) with \( \hat{\sigma} = \{\hat{\sigma}_n; n \geq 1\} \).

We shall first give asymptotic expansions of the expected sample size \( E(N) \) and \( R(\hat{\sigma}, N)/w \).

**THEOREM 2.1.** The following results hold.

(i) For \( m \geq 2 \),

\[
E(N) = n^* + \nu - l_0 + o(1) \quad \text{as} \quad w \to 0.
\]

(ii) For \( m \geq 6 \),

\[
R(\hat{\sigma}, N)/w = 1 - (\nu - l_0 - 4)/n^* + o(1/n^*) \quad \text{as} \quad w \to 0,
\]

where \( \nu \) is the constant given in Lemma 3.1 (iii) and approximately 0.747.

**REMARK 2.1.** (i) If \( \nu - l_0 - 4 > 0 \), then \( R(\hat{\sigma}, N) < w \) for \( w > 0 \) sufficiently small. (ii) The larger the value of \( \nu \) is, the smaller \( R(\hat{\sigma}, N)/w \) is, although the expected sample size becomes larger.

The estimator \( \hat{\sigma}_n \) is unbiased, but we do not know whether or not \( \hat{\sigma}_N \) is unbiased. The following proposition provides the bias of \( \hat{\sigma}_N \).

**PROPOSITION 2.1.** For \( m \geq 4 \),

\[
E(\hat{\sigma}_N) - \sigma = -w + o(w) \quad \text{as} \quad w \to 0.
\]

According to this proposition, we consider the bias-corrected estimators

\[
\hat{\sigma}^* = \{\hat{\sigma}^*_n; n \geq 1\} \quad \text{and} \quad \hat{\sigma}^*_n = \hat{\sigma}_n + w.
\]  

(2.6)

Then, the asymptotic expansion of \( R(\hat{\sigma}^*, N)/w \) associated with the bias-corrected estimator \( \hat{\sigma}^*_N \) is given by the following theorem.

**THEOREM 2.2.** For \( m \geq 6 \),

\[
R(\hat{\sigma}^*, N)/w = 1 - (\nu - l_0 - 3)/n^* + o(1/n^*) \quad \text{as} \quad w \to 0.
\]
We compare now the sequential procedure \((\hat{\sigma}^*, N)\) with \((\hat{\sigma}, N)\) from the viewpoint of second order asymptotic relative efficiency. From Theorems 2.1 and 2.2 we have

\[ n^* \{R(\hat{\sigma}, N) - R(\hat{\sigma}^*, N)\}/w = v^* - v + o(1) \quad \text{as} \quad w \to 0, \]

where \(v = \nu - l_0 - 4\) and \(v^* = \nu - l_0 - 3\). Choose \(l_0 < \nu - 4\) so that \(v\) and \(v^*\) are positive. Then both \(R(\hat{\sigma}, N)/w\) and \(R(\hat{\sigma}^*, N)/w\) are less than 1 for \(w > 0\) sufficiently small and \(v^* - v = 1\). The value \(v^* - v > 0\) means that \(n^* \{1 - R(\hat{\sigma}^*, N)/w\}\) is asymptotically \(v^* - v\) greater than \(n^* \{1 - R(\hat{\sigma}, N)/w\}\). Therefore, the procedure \((\hat{\sigma}^*, N)\) is asymptotically more efficient than \((\hat{\sigma}, N)\) in the above sense.

**Remark 2.2.** For example, let us take \(l(x) = (1 - 2x^{-1})^2 + x^{-2}\) in (2.5). Then \(l_0 = -4 < \nu - 4\), and \(v = \nu > 0\) and \(v^* = \nu + 1 > 0\). Thus, as \(w \to 0\)

\[ R(\hat{\sigma}, N)/w = 1 - \nu/n^* + o(1/n^*) \]

and

\[ R(\hat{\sigma}^*, N)/w = 1 - (\nu + 1)/n^* + o(1/n^*). \]

In this case, we have for \(w\) sufficiently small

\[ R(\hat{\sigma}, N) < w \quad \text{and} \quad R(\hat{\sigma}^*, N) < w. \]

Therefore the condition on the risk is asymptotically satisfied.

### 3. Proofs

In this section we shall give the proofs of the results in Section 2. Let

\[ Y_{in} = (n - i + 1)(X_{n(i)} - X_{n(i-1)}) \quad \text{for} \quad i = 2, \ldots, n, \]

where \(X_{n(1)} \leq X_{n(2)} \leq \cdots \leq X_{n(n)}\) are the order statistics of \(X_1, \ldots, X_n\). Then \(Y_{2n}, \ldots, Y_{nn}\) are i.i.d random variables with pdf \(f_{0,\sigma}\) and \(\hat{\sigma}_n = (n - 1)^{-1} \sum_{i=2}^{n} Y_{in}\). Let \(W_1, W_2, \ldots, \) be a sequence of i.i.d random variables with pdf \(f_{0,1}\) and put

\[ S_n = \sum_{i=1}^{n} W_i, \quad \overline{W}_n = S_n/n \quad \text{and} \quad Z_n = \sigma W_n. \]

Define

\[ Q = Q_w = \inf\{n \geq m - 1 : S_n \leq (n^*)^{-1} n^2 l(n)\}. \quad (3.1) \]

Then we have

\[ N \overset{d}{=} Q + 1, \quad (3.2) \]
which means that the distribution of \( N \) is the same as that of \( Q + 1 \). Let

\[
R_w = cQ^2 l(Q) - S_Q, \quad c = (n^*)^{-1},
\]

\[
r_c = cQ^2 (l(Q) - 1) - R_w \quad \text{and} \quad Q^* = c^{1/2} (Q - n^*).
\]

Throughout this section, \( M \) denotes a generic positive constant which is independent of \( w \). From (2.5), the definition of \( Q \) and the results of Woodroofe (1977) we have the following lemma.

**LEMMA 3.1.** The following results hold.

(i) \( Q \xrightarrow{a.s.} \infty, \ cQ \xrightarrow{a.s.} 1 \) as \( w \to 0 \) and \( cQ^2 - Q = S_Q - Q - \tau_c \),

\[ a.a \]

where ' \( \xrightarrow{a.s.} \) ' denotes almost sure convergence.

(ii) For any fixed \( s > 0 \), \( E\{ (cQ)^s \} = O(1) \) and \( E\{ (R_w)^s \} = O(1) \) as \( w \to 0 \).

(iii) \( R_w \xrightarrow{d} H \) and \( \tau_c \xrightarrow{d} l_0 - H \) as \( w \to 0 \),

where \( H \) is the random variable given in Theorem 2.1 of Woodroofe (1977),

\[ \nu \equiv E(H) = 1 - \sum_{n=1}^{\infty} n^{-1} E\{ (S_n - 2n)^+ \} \]

and ' \( \xrightarrow{d} \) ' stands for convergence in distribution.

(iv) \( \sup_{0 < w} E|Q(l(Q) - 1)|^s \leq M \) for any fixed \( s > 0 \).

(v) For any fixed \( \beta > 0 \), \( \sup_{0 < w < w_0} E|\tau_c|^\beta \leq M \) for some \( w_0 > 0 \).

(vi) \( Q^* \xrightarrow{d} N(0,1) \) as \( w \to 0 \),

where \( N(0,1) \) stands for a standard normal random variable.

(vii) For any fixed \( s > 0 \), \( \{ Q^*|^s ; \ 0 < w < w_0 \} \) is uniformly integrable for \( m > \frac{1}{2}s \).

(viii) \( Q^* \) and \( R_w \) are asymptotically independent.

(ix) \( E(Q) = n^* + \nu - l_0 - 1 + o(1) \) as \( w \to 0 \) for \( m > 1 \).

(x) For any fixed \( s > 0 \), \( E\{ (cQ)^{-s} \} = O(1) \) as \( w \to 0 \) for \( m \geq s + 1 \).

By Theorem 2 of Chow, Hsiung and Lai (1979) we can get the following lemma.

**LEMMA 3.2.** For any fixed \( s > 0 \), \( \{ |c^{1/2} (S_Q - Q)|^s ; \ 0 < w < w_0 \} \) is uniformly integrable for some \( w_0 > 0 \).

**LEMMA 3.3.** For any \( x > 0 \) and \( n \) with \( P(N = n) > 0 \),

\[ P\{ \hat{\sigma}_n \leq x | N = n \} = P\{ \sigma \tilde{W}_{n-1} \leq x | Q = n - 1 \}. \]
PROOF. Due to Lombard and Swanepoel (1978), $Y = \left\{ \sum_{i=2}^{n} Y_{in} : n \geq 2 \right\}$ has the same distribution as $Z = \left\{ \sum_{i=1}^{n} Z_i : n \geq 2 \right\}$ where $Y_{in}$ and $Z_i$ are defined at the beginning of this section. The stopping rule $N$ in (2.4) can be rewritten as

$$N = \inf\{n \geq m : \sum_{i=2}^{n} Y_{in} \leq w(n-1)^2 l(n-1)\}.$$ 

Thus from (2.3), (3.1) and (3.2) we have

$$P\{\sigma_{n} \leq x | N = n\} = P\{\sigma_{n} \leq x, N = n\}/P(N = n)$$

$$= P\{(n-1)^{-1} \sum_{i=2}^{n} Y_{in} \leq x, \sum_{i=2}^{k} Y_{ik} > w(k-1)^2 l(k-1) \text{ for } k = 2, \ldots, n-1,$$

$$\sum_{i=2}^{n} Y_{in} \leq w(n-1)^2 l(n-1)\}/P(Q = n-1)$$

$$= P\{(n-1)^{-1} \sum_{i=1}^{n-1} Z_i \leq x, \sum_{i=1}^{k-1} Z_i > w(k-1)^2 l(k-1) \text{ for } k = 2, \ldots, n-1,$$

$$\sum_{i=1}^{n-1} Z_i \leq w(n-1)^2 l(n-1)\}/P(Q = n-1)$$

$$= P\{\sigma \overline{W}_{n-1} \leq x, Q > (n*)^{-1} k^2 l(k) \text{ for } k = 1, \ldots, n-2, S_{n-1} \leq (n*)^{-1} (n-1)^2 l(n-1)\}/P(Q = n-1)$$

$$= P\{\sigma \overline{W}_{Q} \leq x | Q = n-1\},$$

which concludes the lemma.

**LEMMA 3.4.** For any fixed $\beta > 0$, $\{(S_Q - Q)/Q^{1/2} \beta : 0 < w < w_0\}$ is uniformly integrable for some $w_0 > 0$ if $m > \frac{1}{2}\alpha \beta + 1$.

**PROOF.** Choose $\alpha > 1$ and $p > 1$ such that $m \geq \frac{1}{2} \alpha p \beta + 1$. Let $q = p/(p-1)$. Then by Hölder’s inequality and Lemmas 3.1 and 3.2 we have

$$E((S_Q - Q)/Q^{1/2} \beta)^{\alpha} = E((cQ)^{-1/2} c^{1/2}(S_Q - Q))^{\alpha \beta}$$

$$\leq (E|cQ|^{-\alpha p \beta /2})^{1/p} (E|c^{1/2}(S_Q - Q)|^{\alpha \beta q})^{1/q} \leq M$$

for all $0 < w < w_0$, which gives the uniform integrability of $\{(S_Q - Q)/Q^{1/2} \beta ; 0 < w < w_0\}$. This completes the proof.
Proof of Theorem 2.1

(i) is an immediate consequence of (3.2) and Lemma 3.1. We shall prove (ii). From (2.3), (3.2) and Lemma 3.3 we get

\[ R(\hat{\sigma}, N) = \sigma^{-1}E(\hat{\sigma}N - \sigma)^2 = \sigma^{-1}E(\sigma \bar{W} - \sigma)^2 \]
\[ = wE\{n^*/Q^2(S_Q - Q)^2\} \]
\[ = w\{E\{(S_Q - Q)^2/n^*\} + n^*E\{(Q^{-2} - n^{-2})(S_Q - Q)^2\}\} \]

which, together with Theorem 2 of Chow, Robbins and Teicher (1965) and Lemma 3.1, implies

\[ E(S_Q - Q)^2 = E(Q) = n^* + \nu - l_0 - 1 + o(1) \text{ as } w \to 0. \]

Hence

\[ R(\hat{\sigma}, N)/w = 1 + (\nu - l_0 - 1)/n^* + o(1/n^*) \]
\[ + (n^*)^{-1}E\{(cQ)^{-2}(1 + cQ)(1 - cQ)(S_Q - Q)^2\}. \]

Set

\[ I = (cQ)^{-2}(1 + cQ)(1 - cQ)(S_Q - Q)^2. \] (3.3)

Then

\[ n^*\{R(\hat{\sigma}, N)/w - 1\} = \nu - l_0 - 1 + E(I) + o(1) \text{ as } w \to 0. \] (3.4)

Since by Lemma 3.1 (i)

\[ 1 - cQ = -(S_Q - Q)Q^{-1} + r_c Q^{-1}, \] (3.5)

we get from (3.3) that

\[ I = -(cQ)^{-2}(1 + cQ)(S_Q - Q)^3Q^{-1} + (cQ)^{-2}(1 + cQ)r_c(S_Q - Q)^2Q^{-1} \]
\[ \equiv -I_1 + I_2, \text{ say.} \] (3.6)

In order to prove (ii), it is sufficient from (3.4) and (3.6) to show

\[ E(I_1) = -5 + o(1) \text{ as } w \to 0 \] (3.7)

and

\[ E(I_2) = 2l_0 - 2\nu + o(1) \text{ as } w \to 0. \] (3.8)

We shall first prove (3.7). It follows from (3.5) that
\[ I_1 = (cQ)^{-2} \{2c(S_Q - Q)^3 + (1 - cQ)(S_Q - Q)^3/Q \} \]
\[ = 2(cQ)^{-2}c(S_Q - Q)^3 - (cQ)^{-2}(S_Q - Q)^4/Q^2 + (cQ)^{-2}r_\varepsilon(S_Q - Q)^3/Q^2 \]
\[ \equiv 2I_{11} - I_{12} + I_{13}, \text{ say.} \] (3.9)

By Lemma 3.1 (i) and the result of Anscombe (1952) we get \((S_Q - Q)^4/Q^2 \overset{\mathcal{D}}{\to} \chi_1^4\) as \(w \to 0\), where \(\chi_1^2\) denotes a chi-squared random variable with one degree of freedom. Hence it follows from Lemma 3.1 that

\[ I_{12} \overset{\mathcal{D}}{\to} \chi_1^4 \text{ and } I_{13} \overset{\mathbb{P}}{\to} 0 \text{ as } w \to 0, \] (3.10)

where \(\overset{\mathcal{D}}{\to}\) stands for convergence in probability. Since \(m > 5\), we can choose constants \(\alpha > 1\) and \(p > 1\) such that \(m \geq 4\alpha p + 1\). Let \(q = p/(p - 1)\). Then by Hölder's inequality and Lemmas 3.1 and 3.2 we get

\[ E|I_{12}|^\alpha = E\left|(cQ)^{-4}\{c^{1/2}(S_Q - Q)\}^4\right|^\alpha \]
\[ \leq (E(cQ)^{-4\alpha p})^{1/p} (E|c^{1/2}(S_Q - Q)|^{4\alpha q})^{1/q} \leq M \]

for all \(0 < w < w_0\). This yields the uniform integrability of \(\{I_{12} ; 0 < w < w_0\}\). Thus from (3.10)

\[ E(I_{12}) = 3 + o(1) \text{ as } w \to 0. \] (3.11)

Throughout this section we use the above method to prove uniform integrabilities of sequences of random variables, and omit their proofs because the calculations are tedious. Since \(I_{13} = \{c^{1/2}(S_Q - Q)\}^2r_\varepsilon(cQ)^{-7/2}Q^{-1/2}\), it follows from Lemmas 3.1 and 3.2 that \(\{I_{13} ; 0 < w < w_0\}\) is uniformly integrable. Thus by (3.10) we get

\[ E(I_{13}) = o(1) \text{ as } w \to 0, \]

which, together with (3.9) and (3.11), yields

\[ E(I_1) = 2E(I_{11}) - 3 + o(1) \text{ as } w \to 0. \] (3.12)

Set

\[ J_1 = \{(cQ)^{-2} - 1\}c(S_Q - Q)^3 \text{ and } J_2 = (S_Q - Q)^3. \] (3.13)

Then

\[ E(I_{11}) = E(J_1) + cE(J_2). \] (3.14)

From Theorem 9 of Chow et al. (1965) and Lemma 3.1 (ix) we have

\[ E(J_2) = 2c^{-1} + 2(\nu - l_0 - 1) + 3E\{Q(S_Q - Q)\} + o(1). \] (3.15)
By (3.5) and Wald's equation we get

\[ E\{Q(S_Q - Q)\} = c^{-1}E\{(cQ)(S_Q - Q)\} \]

\[ = c^{-1}[E(S_Q - Q) + E\{(S_Q - Q)^2/Q\} - E\{r_c(S_Q - Q)/Q\}] \]

\[ = c^{-1}[E\{(S_Q - Q)^2/Q\} - E\{r_c(S_Q - Q)/Q\}]. \] (3.16)

It follows from Lemma 3.4 that \( \{(S_Q - Q)^2/Q ; 0 < w < w_0\} \) is uniformly integrable. Hence by the fact that \( (S_Q - Q)^2/Q \overset{d}{\to} \chi_1^2 \) as \( w \to 0 \), we have

\[ E\{(S_Q - Q)^2/Q\} = 1 + o(1) \quad \text{as} \quad w \to 0. \] (3.17)

By using Lemma 3.1 and the strong law of large numbers (SLLN), we can show that \( r_c(S_Q - Q)/Q \overset{P}{\to} 0 \) as \( w \to 0 \) and that \( \{r_c(S_Q - Q)/Q ; 0 < w < w_0\} \) is uniformly integrable, which yield

\[ E\{r_c(S_Q - Q)/Q\} = o(1) \quad \text{as} \quad w \to 0. \]

Thus (3.16) and (3.17) give \( E\{Q(S_Q - Q)\} = c^{-1}(1 + o(1)) \) as \( w \to 0 \), which, together with (3.14) and (3.15), implies

\[ E(I_{11}) = E(J_1) + 5 + o(1) \quad \text{as} \quad w \to 0. \] (3.18)

Since \( \{(cQ)^{-2} - 1\}c = -\{(cQ)^{-1} + 1\}c(Q - n^*)/Q \) and \( c(Q - n^*) = (S_Q - Q)/Q - r_c/Q \) by Lemma 3.1 (i), we get from (3.13) that

\[ J_1 = \{(cQ)^{-1} + 1\}(S_Q - Q)^4/Q^2 + \{(cQ)^{-1} + 1\}r_c(S_Q - Q)^3/Q^2 \]

\[ = -J_{11} + J_{12}, \quad \text{say.} \] (3.19)

From Lemma 3.1 and the fact that \( (S_Q - Q)/Q^{1/2} \overset{d}{\to} N(0, 1) \) as \( w \to 0 \), we have

\[ J_{11} \overset{d}{\to} 2\chi_1^4 \quad \text{and} \quad J_{12} \overset{P}{\to} 0 \quad \text{as} \quad w \to 0. \] (3.20)

By Lemmas 3.1 and 3.4 we can show the uniform integrability of \( \{J_{11} ; 0 < w < w_0\} \), which, together with (3.20), yields

\[ E(J_{11}) = 6 + o(1) \quad \text{as} \quad w \to 0. \] (3.21)

By using (3.20) and the uniform integrability of \( \{J_{12} ; 0 < w < w_0\} \) we have \( E(J_{12}) = o(1) \) as \( w \to 0 \), which, together with (3.19) and (3.21), implies \( E(J_1) = -6 + o(1) \) as \( w \to 0 \). Thus from (3.12) and (3.18) we get \( E(I_1) = -5 + o(1) \) as \( w \to 0 \), which gives (3.7). Next we shall show (3.8). Lemma 3.1 (i) implies

\[ (cQ)^{-2}(S_Q - Q)^2 = c^{-1}Q^*^2 + 2(cQ)^{-1}(Q - n^*)r_c + (cQ)^{-2}r_c^2. \]
Hence from (3.6)
\[ I_2 = \{1 + (cQ)^{-1}\}r_eQ^* - 2\{1 + (cQ)^{-1}\}r_e^2(Q - n^*)/Q + (1 + cQ)(cQ)^{-2}r_e^3Q^{-1} \]
\[ \equiv I_{21} + I_{22} + I_{23}, \text{ say.} \] (3.22)

From Lemma 3.1 we have
\[ I_{22} \overset{p}{\rightarrow} 0 \text{ and } I_{23} \overset{p}{\rightarrow} 0 \text{ as } w \to 0. \] (3.23)

According to the fact that for \( \alpha > 1, |I_{22}|^{\alpha} \leq M|cQ|^{-1/2}Q^*r_e^2|^{\alpha} + |cQ|^{-1/2}Q^*r_e^2|^{\alpha} \) and Lemma 3.1, we obtain the uniform integrability of \( \{I_{22}; 0 < w < w_0\} \), which, together with (3.23), implies
\[ E(I_{22}) = o(1) \text{ as } w \to 0. \] (3.24)

By using the fact that \( Q^{-1} \leq 1 \) we get the uniform integrability of \( \{I_{23}; 0 < w < w_0\} \), which, together with (3.23), yields
\[ E(I_{23}) = o(1) \text{ as } w \to 0. \]

Hence from (3.22) and (3.24) we get
\[ E(I_2) = E(I_{21}) + o(1) \text{ as } w \to 0. \] (3.25)

Since \( r_e = cQ^2(l(Q) - 1) - R_w \),
\[ I_{21} = \{1 + (cQ)^{-1}\}Q^*cQ^2(l(Q) - 1) - \{1 + (cQ)^{-1}\}Q^*R_w \]
\[ \equiv I_{211} - I_{212}, \text{ say.} \] (3.26)

Let \( K \) be a standard normal random variable which is independent of \( H \). Then by Lemma 3.1 (iii), (vi) and (viii) we have
\[ (Q^*, R_w) \overset{d}{\rightarrow} (K, H) \text{ and } cQ^2(l(Q) - 1) \overset{a.s.}{\rightarrow} l_0 \text{ as } w \to 0, \]
which imply
\[ I_{211} \overset{d}{\rightarrow} 2l_0K^2 \text{ and } I_{212} \overset{d}{\rightarrow} 2K^2H \text{ as } w \to 0. \] (3.27)

From Lemma 3.1 we can show the uniform integrability of \( \{I_{211}; 0 < w < w_0\} \), which yields
\[ E(I_{211}) = 2l_0 + o(1) \text{ as } w \to 0. \] (3.28)

By the independency of \( K^2 \) and \( H \) we have
\[ E(K^2H) = E(K^2)E(H) = \nu. \] (3.29)
Since we can show the uniform integrability of \( \{I_{212} : 0 < w < w_0\} \), we have by (3.27) and (3.29) that

\[
E(I_{212}) = 2\nu + o(1) \quad \text{as} \quad w \to 0,
\]

which, together with (3.25), (3.26) and (3.28), implies

\[
E(I_2) = 2\nu_0 - 2\nu + o(1) \quad \text{as} \quad w \to 0.
\]

Thus (3.8) is shown. Therefore the proof of Theorem 2.1 is complete.

**Proof of Proposition 2.1.**

From (2.3), (3.2), Lemma 3.3 and Wald's equation we have

\[
E(\hat{\sigma}_N) - \sigma = \sum_{n=m}^{\infty} \int (x - \sigma) dP(\hat{\sigma}_n \leq x | N = n) P(N = n)
\]

\[
= \sum_{n=m}^{\infty} \int (x - \sigma) dP(\sigma \bar{W}_{n-1} \leq x | Q = n - 1) P(Q = n - 1)
\]

\[
= E(\sigma \bar{W}_Q - \sigma)
\]

\[
= wE(S_Q - Q) + wn^* E\{(Q^{-1} - n^{*^{-1}})(S_Q - Q)\}
\]

\[
= wE\{(cQ)^{-1} (1 - cQ)(S_Q - Q)\}
\]

\[
\equiv wE(I), \quad \text{say.} \quad (3.30)
\]

By (3.5), Lemma 3.1 and SLLN we get

\[
I = -(cQ)^{-1} (S_Q - Q)^2/Q + (cQ)^{-1} r_c (S_Q - Q)/Q
\]

\[
\overset{d}{\to} -\chi_1^2 \quad \text{as} \quad w \to 0. \quad (3.31)
\]

Since \( I = -(cQ)^{-1} Q^{c^{1/2}}(S_Q - Q) \) it follows from Lemmas 3.1 and 3.2 that \( \{I : 0 < w < w_0\} \) is uniformly integrable. Thus (3.31) implies that \( E(I) = -1 + o(1) \) as \( w \to 0 \), which, together with (3.30), concludes the proposition. Therefore, the proof is complete.

**Proof of Theorem 2.2**

From (2.6) we get

\[
R(\hat{\sigma}^*, N) = R(\hat{\sigma}, N) + 2w\sigma^{-1}E(\hat{\sigma}_N - \sigma) + w^2\sigma^{-1}.
\]

Thus, by using (2.3), Theorem 2.1 and Proposition 2.1 we obtain Theorem 2.2. This completes the proof.
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