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A COMPUTATIONAL VERIFICATION METHOD OF SOLUTION WITH UNIQUENESS FOR OBSTACLE PROBLEMS

By

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Abstract

A numerical method for automatic proof of the existence of solutions for variational inequalities is proposed. It is based on the infinite dimensional fixed point theorem and computable error estimates for finite element approximations of the original problems. Particularly, in this paper, we consider the method to prove the uniqueness of solution for obstacle problem. Further, some numerical examples are presented.

1. Introduction

In the author's previous work (Ryoo and Nakao (1998)), we proposed a numerical method for automatic proof of the existence of solutions for variational inequalities by computer. The basic approach of this method consists of the fixed point formulation of variational inequalities and the construction of the function set, in computer, satisfying the validation condition of infinite dimensional fixed point theorem, i.e., Schauder's fixed point theorem. In order to realize the verification, we used the finite element approximations and computable a priori error estimates as well as the method of interval arithmetic. In this paper, we consider the method to verify the local uniqueness of solution for obstacle problem. This method can be applied, at least theoretically, to general variational inequalities.

In the following section, we describe the obstacle problem considered in this paper and give the fixed point formulation to prove the existence of the solutions. In Section 3, we introduce a computational verification condition. In Section 4, we present a computer algorithm, based upon the idea in the previous section, to construct the set satisfying the verification conditions by Schauder's fixed point theorem. Also we consider about the method to prove the uniqueness of solution for obstacle problem. Some numerical examples are illustrated in the last section.

2. Problem and the fixed point formulation

Let Ω be a bounded convex domain in R^n, $1 \leq n \leq 2$, with piecewise smooth boundary $\partial\Omega$. We set $V \equiv H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ and

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$$a(w,v)=\int_\Omega \nabla w\cdot\nabla v dx,$$

where

$$\nabla w\cdot\nabla v=\frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1}+\frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2}.$$

We define $K=\{v\in V : v\geq 0 \text{ a.e. on } \Omega\}$.

First, we note that, for any $g\in L^2(\Omega)$, the problem:

$$a(w,\psi-w)\geq(g,\psi-w),\quad \forall\psi\in K,\quad w\in K, \tag{2.1}$$

has a unique solution $w\in V\cap H^2(\Omega)$, and the estimate

$$|w|_{H^2(\Omega)}\leq\|g\|_{L^2(\Omega)} \tag{2.2}$$

holds (see Glowinski (1984) in detail), where $|w|_{H^2}$ implies the semi-norm of w on $H^2(\Omega)$ defined by

$$|w|^2_{H^2(\Omega)}\equiv\sum_{i,j=1}^{2}\|\frac{\partial^2 w}{\partial x_i\partial x_j}\|^2_{L^2(\Omega)}.$$

Now consider the following variational inequality;

$$\begin{cases}\text{Find } w\in K \quad \text{such that}\\ a(w,v-w)\geq(f(w),v-w),\forall v\in K.\end{cases} \tag{2.3}$$

Here, f is assumed to satisfy the following hypotheses :

A1. f is the continuous map from V to $L^2(\Omega)$.

A2. For each bounded subset $W\in V$, $f(W)$ is also bounded set in $L^2(\Omega)$.

We adopt $(\nabla\phi,\nabla\psi)$ as the inner product on V, where $(\cdot,\cdot)$ denotes the L^2- inner product on Ω. Hence, the associated norm is defined by $\|\phi\|_V=\|\nabla\phi\|_{L^2(\Omega)}$.

We now take an appropriate finite dimensional subspace V_h of V for $0<h<1$. Usually, V_h is taken to be a finite element subspace with mesh size h. We then define K_h, an approximation of K, by

$$K_h=V_h\cap K=\{v_h|v_h\in V_h,\quad v_h\geq 0 \text{ on } \overline{\Omega}\}.$$

We now define $v=P_K(w)$, the projection of $w\in V$ into K, as the solution of the following problem:

$$v\in K:\quad a(v,\zeta-v)\geq a(w,\zeta-v),\quad \forall\zeta\in K. \tag{2.4}$$

Also define $v_h=P_{K_h}(w)$, the projection of w into K_h, as follows:

$$v_h\in K_h:\quad a(v_h,\zeta-v_h)\geq a(w,\zeta-v_h),\quad \forall\zeta\in K_h. \tag{2.5}$$

Now, as one of the approximation properties of K_h, assume that

A3. For each $w \in V \cap H^2(\Omega)$, there exists a positive constant C, independent of h, such that

$$\|w - P_{K_h} w\|_V \le Ch|w|_{H^2(\Omega)}. \tag{2.6}$$

Here, C has to be numerically determined.

To verify the existence of a solution of (2.3) in a computer, we use the fixed point formulation. For each $w \in V$, by the Riesz representation theorem for the Hilbert space, there exists a unique $F(w) \in V$ such that

$$(\nabla F(w), \nabla v) = (f(w), v), \quad \forall v \in V. \tag{2.7}$$

That is,

$$\begin{cases} \exists F(w) \in V \text{ such that} \\ -\Delta F(w) = f(w) \text{ in } \Omega, \\ \quad F(w) = 0 \text{ on } \partial\Omega. \end{cases}$$

Then the map $F : V \longrightarrow V$ is a compact operator.
By (2.7), problem (2.3) is equivalent to finding $w \in V$ such that

$$a(w, v - w) \ge a(F(w), v - w), \qquad \forall v \in K. \tag{2.8}$$

By using (2.4) and (2.8), we now have the following fixed point problem for the compact operator $P_K F$.

$$\text{Find } \exists w \in V \text{ such that } w = P_K F(w). \tag{2.9}$$

3. Verification condition

In order to deal with the functions and equations in the space V of infinite dimension in a computer, we introduce two concepts, rounding and rounding error.

Now we define the dual cone of K_h by

$$K_h^* = \{w \in V : a(w, v) \le 0, \quad \forall v \in K_h\}.$$

Note that K_h^* is also closed convex cone in V with vertex at 0 which is the only point common to K_h and K_h^*. From (2.4) it follows that K_h^* is the set of points whose projections into K_h is 0. We need some additional lemma which is from Rodrigues (1987).

LEMMA 3.1. *Any $w \in V$ can be uniquely decomposed into the sum of two orthogonal elements. That is,*

$$w = P_{K_h} w \bigoplus (I - P_{K_h}) w = P_{K_h} w \bigoplus P_{K_h^*} w.$$

Here, $\bigoplus$ denotes the sum of two orthogonal elements in the sense of V.

For any $w \in V$, we now define the rounding $R(P_K F(w)) \in K_h$ as the solution of the following problem:

$$a(R(P_K F(w)), v_h - R(P_K F(w))) \ge (f(w), v_h - R(P_K F(w))), \quad \forall v_h \in K_h.$$

Next, for any subset $W \subset V$, we define the rounding $R(P_K FW) \subset K_h$ by the projection of V onto K_h, that is,

$$R(P_K FW) = \{w_h \in K_h : w_h = R(P_K F(w)), \ \ w \in W\}.$$

Moreover, for $W \subset V$ we define $RE(P_K FW)$, the rounding error of $P_K FW$, as a subset of K_h^*, i.e.,

$$RE(P_K FW) = \{v \in K_h^* : \|v\|_V \le Ch\|f(W)\|_{L^2}\}, \tag{3.1}$$

where

$$\|f(W)\|_{L^2} \equiv \sup_{w \in W} \|f(w)\|_{L^2}.$$

Here, C is the same positive constant as in (2.6). By using the approximation property A3 of K_h, we have

$$P_K F(w) - R(P_K F(w)) \in RE(P_K F(w)), \quad \forall w \in W.$$

Therefore, we have the following verification condition by Schauder's fixed point theorem.

LEMMA 3.2. *If there exists a nonempty, bounded, convex, and closed subset $W \subset K$ such that*

$$R(P_K FW) \oplus RE(P_K FW) \subset W, \tag{3.2}$$

then there exists a solution of $w = P_K F(w)$ in W.

4. Verification procedures with uniqueness

In order to find a set W satisfying the (3.2), we use iterative procedures, that is, the sequential iteration (see Ryoo and Nakao (1998)) or Newton-like method (see Nakao and Ryoo (1998)). In this paper, we describe only the former case for simplicity. In this section, we propose a computer algorithm to construct the set W which satisfies the verification condition (3.2).

Now, we consider the following approximate problem corresponding to (2.1) as

$$a(w_h, v_h - w_h) \ge (g, v_h - w_h), \ \ \forall v_h \in K_h, \ \ w_h \in K_h. \tag{4.1}$$

Since bilinear form $a(\,\cdot\,,\,\cdot\,)$ is symmetric, (4.1) is actually equivalent to the quadratic programming problem:

$$\min_{v \in K_h} \left[\frac{1}{2}a(v,v) - (g,v)\right]. \tag{4.2}$$

Let $\{\phi_j\}_{j=1\cdots M}$ be a basis of V_h with linear functions such that $\phi_j(x) \ge 0, \ \ \forall x \in \Omega$ and satisfying

$$\phi_j(x_i) = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases}$$

where x_i is a node of the finite element mesh.

Then (4.2) reduces to the following vector form:

$$\min_{z\geq 0}\left[\frac{1}{2}z'Dz - P'z\right], \tag{4.3}$$

where $z \geq 0$ means componentwise inequality. Here, $D = (d_{ij})$ with $d_{ij} = (\nabla\phi_i, \nabla\phi_j)$, $1 \leq i, j \leq M$, and z is the coefficient vector with $\{\phi_j\}$ corresponding to the function v in (4.2). Also, $P \equiv \Big((g, \phi_j)\Big)_{1\leq j\leq M}$.

By the well-known result (see, for example, Bazaraa and Shetty 1979), a vector $z = (z_j) \in R^M$ with $z \geq 0$ is an optimal solution to (4.2) if and only if there exists $u = (u_j) \in R^M$ such that

$$\begin{cases} u - Dz = -P, \\ uz = 0, \quad u \geq 0, \quad z \geq 0. \end{cases} \tag{4.4}$$

Here, uz means inner product on R^M.

By the fact that $u, z \geq 0$, we have $z_j = 0$ or $u_j = 0$ for $1 \leq j \leq M$. Thus we have the following system of nonlinear equations:

$$\begin{cases} u - Dz = -P, \\ u_j z_j = 0, \quad 1 \leq j \leq M, \end{cases} \tag{4.5}$$

in $2M$ unknowns u, z.

Now let A_j be intervals on R^1, $1 \leq j \leq M$, and let $\sum_{j=1}^M A_j\phi_j$ be a linear combination of $\{\phi_j\}$, i.e., an element of the power set 2^{V_h} in the following sense:

$$\sum_{j=1}^M A_j\phi_j = \{\sum_{j=1}^M a_j\phi_j; a_j \in A_j, \quad 1 \leq j \leq M\}.$$

Now, we denote the set of all nonnegative real numbers by R^+ and for any $\alpha \in R^+$, we set

$$[\alpha] \equiv \{\phi \in K_h^*; \quad \|\phi\|_V \leq \alpha\}. \tag{4.6}$$

Let denote all the set of linear combinations of $\{\phi_j\}$ with interval coefficients by $\mathcal{D}$, i.e.,

$$\mathcal{D} \equiv \{\sum_{j=1}^M A_j\phi_j | A_j; \text{ interval in } R^1, \quad 1 \leq j \leq M\}.$$

For $(w, \alpha) \in \mathcal{D} \times R^+$, setting $W := w \bigoplus [\alpha]$ and $g = f(W)$ in (2.1), we consider the following nonlinear system:

$$\begin{cases} u - Dz = -(f(W), \phi_j), \quad 1 \leq j \leq M, \\ u_j z_j = 0, \quad 1 \leq j \leq M. \end{cases} \tag{4.7}$$

Here, $(f(W), \phi_j)$ is evaluated as interval Y_i such that $\{(f(w), \phi_j) | w \in W\} \subset Y_i$.

In order to solve (4.7) with guaranteed accuracy, we use the following theorem which is given in Rump (1983).

THEOREM 4.1. *Let $\Phi : R^n \to R^n$ be a function with continuous first derivative and let $R \in R^{n\times n}$ (real $n \times n$ matrix), $\tilde{x} \in R^n$. Denote the Jacobian matrix of Φ by $\Phi' \in R^{n\times n}$ and for $X \in IR^n$ (real interval vectors with n components) define $\Phi'(X) := \cap\{Y \in IR^n : \Phi'(x) \in Y$ for all $x \in X\}$. If for some $X \in IR^n$ with $0 \in X$*

$$-R \cdot \Phi(\tilde{x}) + \{I - R \cdot \Phi'(\tilde{x} + X)\} \cdot X \subseteq \overset{\circ}{X},$$

then there exists an $\hat{x} \in \tilde{x} + \overset{\circ}{X}$ with $\Phi(\hat{x}) = 0$.

Using the solution of (4.7), we define the map $T : V \to \mathcal{D}$ by

$$T(W) = \widetilde{W}, \tag{4.8}$$

where $\widetilde{W}$ stands for the set of verified solutions obtained by Theorem 4.1; that is, an interval vector, of the nonlinear system (4.7) with interval coefficients in the right hands sides.

We now consider the fully automatic computer generation of the set W satisfying Lemma 3.2. First, we generate the following iteration sequence $\{(w_h^{(n)}, \alpha_n)\}$ for $n = 0, 1, 2, \cdots$, where $(w_h^{(n)}, \alpha_n) \in \mathcal{D} \times R^+$ with $w_h^{(n)} \subset K_h$ and set $W^{(n)} := w_h^{(n)} \bigoplus [\alpha_n]$.

For $i = 0$, we choose an appropriate initial value $w_h^{(0)} \in K_h$ and $\alpha_0 \in R^+$. Usually, $w_h^{(0)}$ will be determined as

$$a(w_h^{(0)}, v_h - w_h^{(0)}) \geq (f(w_h^{(0)}), v_h - w_h^{(0)}), \forall v_h \in K_h, w_h^{(0)} \in K_h \tag{4.9}$$

which corresponds to the Galerkin approximation for (2.3). And the standard selection for α_0 will be $\alpha_0 = 0$. For $n \geq 1$, first for a given $0 < \delta \ll 1$, we define the $\delta - inflation$ of $(w_h^{(n-1)}, \alpha_{n-1})$ by

$$\begin{cases} \widetilde{w}_h^{(n-1)} = w_h^{(n-1)} + \sum_{j=1}^{M} [-1, 1]\delta\phi_j, \\ \widetilde{\alpha}_{n-1} = \alpha_{n-1} + \delta. \end{cases}$$

Next, for the set $\widetilde{W}^{(n-1)} = \widetilde{w}_h^{(n-1)} \bigoplus [\widetilde{\alpha}_{n-1}]$, we define $w_h^{(n)} \in \mathcal{D}$ and $\alpha_n \in R^+$ by

$$\begin{cases} w_h^{(n)} = T(\widetilde{W}^{(n-1)}), \\ \alpha_n = Ch \sup_{w \in \widetilde{W}^{(n-1)}} \|f(w)\|_{L^2(\Omega)}. \end{cases} \tag{4.10}$$

Here, C is the constant defined in (2.6).

Now we have the following verification condition on a computer.

THEOREM 4.2. *If for an integer N, two relationships*

$$w_h^{(N)} \subset \widetilde{w}_h^{(N-1)} \text{ and } \alpha_N < \widetilde{\alpha}_{N-1} \tag{4.11}$$

hold, then there exists a solution w of (2.3) in $w_h^{(N)} \bigoplus [\alpha_N]$. Here, the first term of (4.11) means the inclusion in the sense of each coefficient interval of $w_h^{(N)}$ and $\widetilde{w}_h^{(N-1)}$

We omit the proof of this theorem, for it is quite similar to that of the corresponding theorem in Nakao (1992).

Although the verification method in the above enables us to find a solution in the set $w_h^{(N)} \bigoplus [\alpha_N]$, it is impossible to assure uniqueness of the solution in the same set. We now present a technique including the verification of uniqueness under the following additional assumption.

A4. Suppose that there exists a $\lambda < 1$ such that

$$\|P_K F(w_1) - P_K F(w_2)\|_V \leq \lambda \|w_1 - w_2\|_V, \quad \forall w_1, w_2 \in W.$$

By using above assumption and Banach fixed point theorem, we have the following direct consequence which sharpen the Theorem 4.2.

THEOREM 4.3. *Assume that $P_K F$ satisfies* A4 *on a set W. If for an integer N, two relationships*

$$w_h^{(N)} \subset \widetilde{w}_h^{(N-1)} \text{ and } \alpha_N < \widetilde{\alpha}_{N-1} \tag{4.12}$$

hold, then there is one and only one solution w of (2.3) *in $w_h^{(N)} \bigoplus [\alpha_N]$.*

5. Numerical examples

In this section, we present some numerical examples for verification according to the procedures described in the previous section. We consider the case

$$f(w) = aw + b. \tag{5.1}$$

Here, we assume that $a, b \in L^\infty(\Omega)$. Frist, in order to validate A4, we need some properties for $P_K F$. Let $\mathcal{L}(V)$ be the set of bounded linear operators from V to V.

We consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

As well-known, the first eigenvalue λ_1 of (5.2) is equivalent to the the following problem

$$\min_{w \in V} \frac{\|w\|_V^2}{\|w\|_{L^2}^2} = \lambda_1. \tag{5.3}$$

Hence, we obtain

$$\forall w \in V, \quad \frac{\|w\|_V}{\|w\|_{L^2}} \geq \sqrt{\lambda_1}.$$

Furthermore, by well-known results, it follows that

$$\lambda_1 = \frac{1}{2\pi^2} \tag{5.4}$$

for the unit square in two dimensional case. In one dimensional case, we can take λ_1 as

$$\lambda_1 = \frac{1}{\pi^2}. \tag{5.5}$$

And, by (2.7), we have $F(w_1) - F(w_2) = (-\Delta)^{-1}(a(w_1 - w_2))$.

Now, setting $Aw := (-\Delta)^{-1}aw$, consider the following inequality:

$$\begin{aligned}
&||P_K F(w_1) - P_K F(w_2)||_V \\
&\leq ||F(w_1) - F(w_2)||_V \\
&\leq ||A(w_1 - w_2)||_V \\
&\leq ||A||_{\mathcal{L}(V)}||w_1 - w_2||_V .
\end{aligned}$$

Here, we used the fact that $||P_K||_{\mathcal{L}(V)} \leq 1$. Further, we obtain

$$\begin{aligned}
||A||_{\mathcal{L}(V)} &= \sup_{w\neq 0\in V} \frac{||Aw||_V}{||w||_V} = \sup_{w\in V} \frac{(\nabla Aw, \nabla Aw)}{||w||_V ||Aw||_V} \\
&= \sup_{w\in V} \frac{(aw, Aw)}{||w||_V ||Aw||_V} \leq ||a||_{L^\infty} \sup_{w\in V} \frac{||w||_{L^2}||Aw||_{L^2}}{||w||_V ||Aw||_V}.
\end{aligned}$$

Hence, by using (5.3) and (5.4), we have

$$||A||_{\mathcal{L}(V)} \leq \frac{||a||_{L^\infty}}{2\pi^2}$$

for the unit square in two dimensional case. Similarly, for one dimensional case, we obtain

$$||A||_{\mathcal{L}(V)} \leq \frac{||a||_{L^\infty}}{\pi^2}.$$

Therefore, we have the following results.

THEOREM 5.1. *If the function* a *in* (5.1)*satisfies*

$$\frac{||a||_{L^\infty}}{2\pi^2} < 1 \ (n=2) \quad or \quad \frac{||a||_{L^\infty}}{\pi^2} < 1 \ (n=1),$$

then the assumption A4 *holds.*

In what follows we consider the one dimensional case. Then, we can estimate constant C in Section 2 as below.

Let $\Omega = (0,1)$ and let $g \in L^2(\Omega)$. For a positive integer M, set $h = \frac{1}{M}$; we consider $x_i = i \cdot h$ for $i = 0, 1, 2, \cdots M$, that is, a uniform partition of Ω and set $e_i = (x_{i-1}, x_i)$, $\quad i = 1, 2, \cdots M$. We then approximate V and K by

$$V_h = \{v_h \in C^0(\Omega); \quad v_h(0) = v_h(1) = 0, \quad v_h|_{e_i} \in P_1, i = 1, 2, 3 \cdots M\},$$

where P_1 is the space of polynomials of degree ≤ 1, and

$$K_h = \{v_h \in V_h; \quad v_h(x) \geq 0, \quad \forall x \in \Omega\},$$

respectively.

Regarding the approximation error $||w_h - w||_V$ and $||w_h - w||_{L^2(\Omega)}$, we have the following lemma (see Ryoo and Nakao (1998)).

LEMMA 5.2. *Let* w *and* w_h *be solutions of*(2.1) *and*(4.1), *respectively. If* $g \in L^2(\Omega)$, *then we have*

$$||w_h - w||_V \leq \frac{\sqrt{5}}{\pi} h ||g||_{L^2(\Omega)},$$

$$\|w_h - w\|_{L^2(\Omega)} \le \frac{\sqrt{5}}{\pi}\Big(\frac{1}{\pi} + \frac{4\sqrt{2}}{3}\Big) h^2 \|g\|_{L^2(\Omega)}.$$

Hence, we may take $C = \frac{\sqrt{5}}{\pi}$ in (2.6). We now present some computed results of verification with uniqueness.

Example 1. We consider

$$f(w) = w \sin x - \sin x \sin 2\pi x + 4\pi^2 \sin 2\pi x.$$

We choose the basis $\{\phi_i\}_{i=1}^{M}$ of V_h as usual hat functions.

The execution conditions are as follows.

Numbers of elements $= 100$
$\dim V_h = 99$
Extension parameters : $\delta = 10^{-3}$
Initial values : $w_h^{(0)} =$ Galerkin approximation (4.9), $\alpha_0 = 0$,
the outline of $w_h^{(0)}$ is displayed in Figure 1.

In this case, it would be deduced that there exists a free boundary around $x = 0.717172$.

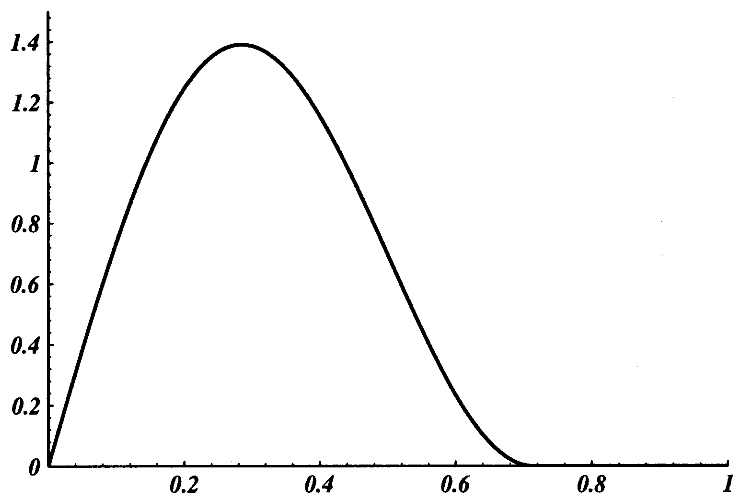

Figure 1: Approximate solution $w_h^{(0)}$

The results are as follows:

Iteration numbers for verification : $N = 3$
L^2 – error bound : 0.001952
Maximum width of coefficient intervals in $\{w_h^{(N)}\} = 0.001707$.

Example 2. Next we consider

$$f(w) = Kw + 8\cos 2\pi x,$$

where K is a constant. The basis $\{\phi_i\}_{i=1}^{M}$ of V_h is the same as in Example 1.

The execution conditions are as follows.

Numbers of elements $= 100$
$\dim V_h = 99$
$K = 2$
Extension parameters : $\delta = 10^{-3}$
Initial values : $w_h^{(0)} =$ Galerkin approximation (4.9), $\alpha_0 = 0$,
the outline of $w_h^{(0)}$ is displayed in Figure 2.

From the Figure 2, two free boundary points can be located around $x = 0.367347$ and $x = 0.632653$.

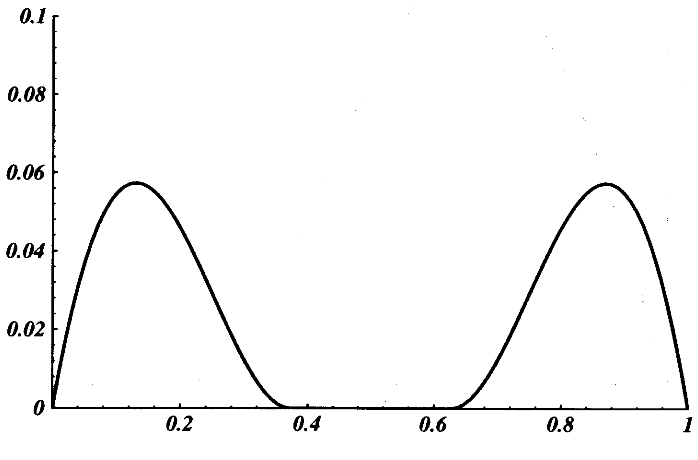

Figure 2: Approximate solution $w_h^{(0)}$

The results are as follows:

Iteration numbers for verification : $N = 3$
L^2 – error bound : 0.000052
Maximum width of coefficient intervals in $\{w_h^{(N)}\} = 0.000127$.

REMARK. In the above calculations, we used usual computer arithmetic with double precision instead of strict interval computions (e.q., ACRITH-XSC, PASCAL-XSC,C-XSC etc.). But, from our experiences, the order of magnitude of roundoff error is , in general, under 10^{-10}. Therefore, it is almost negligible compared with the truncation error which amounts to $10^{-3} \sim 10^{-2}$.

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