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A REPRESENTATION THEOREM FOR RELATION ALGEBRAS: CONCEPTS OF SCALAR RELATIONS AND POINT RELATIONS

By

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Abstract

This paper provides a proof of a representation theorem for homogeneous relation algebras by using concepts of scalar relations and point relations.

Keywords : L-relations, relation algebras, scalar relations, point relations, representation theorem.

1. Introduction

Just after Zadeh's work on fuzzy sets in 1965, Goguen (1967) generalized the concepts of fuzzy sets and relations to taking values on arbitrary lattices, and also stressed the importance of relations as follows:
The importance of relations is almost self-evident. Science is, in a sense, the discovery of relations between observables. Zadeh has shown the study of relations to be equivalent to the general study of systems (a system is a relation between an input space and an output space).

The modern algebraic study of (binary) relations, namely relational calculus, was begun by Tarski; see Maddux (1991) for details of the history of the study of Boolean relation algebras. Tarski (1941) also proposed a formalisation of Boolean relation algebras and their representation problem. Schmidt and Ströhlein (1985), (1993) gave a simple proof of a representation theorem for Boolean relation algebras satisfying the (so-called) Tarski rule and a point axiom. Dedekind categories (Olivier and Serrato (1995)) (or allegories (Freyd and Scedrov (1990))) provide a categorical framework for relational calculus. Relational calculus is a very useful framework for the study of mathematics (Kawahara (1995), Tarski and Givant (1987)) and theoretical computer science (Schmidt and Ströhlein (1993), Bird and de Moor (1997)) and also a useful tool for applications. Some element-free formalisations of fuzzy relations and proofs of representation theorems are given in Kawahara and Furusawa (1995), Kawahara, Furusawa and Mori (1996), and Furusawa (1997).

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In this paper we consider relation algebras, which may not be Boolean, and provide their representation theorem. Relation algebras in the sense of this paper are equivalent to Dedekind categories (or allegories) with just one object. Kawahara, Furusawa, and Mori (1996) proved a representation theorem for Dedekind categories, showing that a Dedekind category with a unit object satisfying the strict point axiom is equivalent to a subcategory of the category of L-relations (where L is the lattice of all endomorphisms on the unit object). A unit object is an abstraction of singleton (or one-point) sets, and, following Goguen (1967), L-relations in Kawahara, Furusawa and Mori (1996) are set-functions with values on a fixed complete distributive lattice L, that is, functions $R : X \times Y \to L$. The discussion in this paper does not assume the existence of a unit object, and L-relations in this paper are homogeneous relations on a set X, that is, functions $R : X \times X \to L$. This study is the first step to prove a representation theorem for Dedekind categories without unit objects.

To prove a representation theorem for relation algebras, we use concepts of scalar relations and point relations. The concept of scalar relations is an original one, which is defined in section 3 as a relation included in the identity relation and which commutes with the greatest relation with respect to composition. In the case of L-relations, scalar relations can be represented as scalar matrices. We use the concept of scalar relations to define a new concept of crisp relations different from that in Kawahara and Furusawa (1995), Kawahawa, Furusawa and Mori (1996), and Furusawa (1997). Also the set of all scalar relations is a complete distributive lattice, which is a sublattice of the relation algebra, and scalar relations represent membership values. The concept of point relations was introduced by Schmidt and Ströhlein (1985), (1993) in the context of applications of (Boolean) relation algebras to theories of graphs and programs, and it played an important role in proofs of representation theorems in Schmidt and Ströhlein (1985), Kawahara and Furusawa (1995), and Kawahara, Furusawa and Mori (1996). In this paper we define a "strict" point axiom by using our concepts of scalar relations and point relations, and then we prove our representation theorem for relation algebras.

2. L-Relations

Let $L = (L, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$ be a fixed complete distributive lattice with least element $\mathbf{0}$ and greatest element $\mathbf{1}$. Complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras. Elements of the complete distributive lattice L will be denoted by $l, l', l'', \cdots$. The supremum (least upper bound) and the infimum (greatest lower bound) of a family $\{l_\lambda\}_\lambda$ in L will be denoted by $\vee_\lambda l_\lambda$ and $\wedge_\lambda l_\lambda$, respectively.

An L-relation R on a set X is a function $R : X \times X \to L$. For $x, y \in X$ the value $R(x, y) \in L$ means the degree to which x and y are related under R. Throughout this section, all L-relations are those on a fixed set X. The set of all L-relations on X will be denoted by $\mathbf{Rel}_L(X)$. An L-relation R is contained in an L-relation S, written $R \subseteq S$, if $R(x, y) \leq S(x, y)$ for all $x, y \in X$. The empty (zero) relation O_X and the universal relation ∇_X are L-relations with $O_X(x, y) = \mathbf{0}$ and $\nabla_X(x, y) = \mathbf{1}$ for all $x, y \in X$, respectively. It is trivial that $\sqsubseteq$ is a complete distributive lattice, and $O_X \sqsubseteq R \sqsubseteq \nabla_X$ for all L-relations R. We denote the least upper bound and the greatest lower bound of a family $\{R_\lambda\}_\lambda$ by $\sqcup_\lambda R_\lambda$ and $\sqcap_\lambda R_\lambda$ respectively. We then have:

$$(\sqcup_\lambda R_\lambda)(x,y) = \vee_\lambda R_\lambda(x,y)$$

and

$$(\sqcap_\lambda R_\lambda)(x,y) = \wedge_\lambda R_\lambda(x,y)$$

for all $x, y \in X$. The composite $RS (= S \circ R)$ of an L-relation R followed by an L-relation S is defined by

$$(RS)(x,y) = \vee_{z\in X}[R(x,z) \wedge S(z,y)]$$

for all $x, y \in X$. This composition of L-relations is called sup-inf composition. The associativity $(RS)T = R(ST)$ holds for all L-relations R, S, and T. The identity relation id_X is an L-relation such that $\mathrm{id}_X(x,y) = \mathbf{1}$ if $x = y$ and $\mathrm{id}_X(x,y) = \mathbf{0}$ otherwise. The unit laws $R = R\mathrm{id}_X$ and $\mathrm{id}_X R = R$ and the zero law $RO_X = O_X R = O_X$ hold for all R. The converse (or transpose) $R^\cup$ of an L-relation R is defined by

$$R^\cup(x,y) = R(y,x)$$

for all $x, y \in X$. An L-relation R is called nonzero if $R \neq O_X$. An L-relation R is called **crisp** if $R(x,y) = \mathbf{0}$ or $R(x,y) = \mathbf{1}$ for all $x, y \in X$.

It is now obvious (Goguen (1967)) that L-relations together with the operations defined above satisfy all axioms stated in the next section. Maybe only R4(Dedekind formula) is not obvious; it will be proved in the following:

PROPOSITION 2.1. *Let R, S, T be L-relations on X. Then $RS \sqcap T \sqsubseteq R(S \sqcap R^\cup T)$ (Dedekind formula).*

PROOF. With $R^\cup(z,x) \wedge T(x,y) \leq (R^\cup T)(z,y)$, the Dedekind formula follows from

$$\begin{aligned}
(RS \sqcap T)(x,y) &= \vee_z[R(x,z) \wedge S(z,y)] \wedge T(x,y) \\
&= \vee_z[R(x,z) \wedge S(z,y) \wedge T(x,y)] \\
&= \vee_z[R(x,z) \wedge S(z,y) \wedge R^\cup(z,x) \wedge T(x,y)] \\
&\leq \vee_z[R(x,z) \wedge S(z,y) \wedge (R^\cup T)(z,y)] \\
&= \vee_z[R(x,z) \wedge (S \sqcap R^\cup T)(z,y)] \\
&= [R(S \sqcap R^\cup T)](x,y)
\end{aligned}$$

for all $x, y \in X$. □

This formula is called "modular law" in Freyd and Scedrov (1990), Bird and de Moor (1997); in Schmidt and Ströhlein (1985), (1993) the Dedekind rule is given as $QR \sqcap S \sqsubseteq (Q \sqcap SR^\cup)(R \sqcap Q^\cup S)$. Since these formulae are equivalent, we use the name "Dedekind formula" in this paper.

In this paper we will call an L-relation R the **scalar**, if there is an $l \in L$ such $R(x,y) = l$ if $x = y$ and $R(x,y) = \mathbf{0}$ otherwise. Now we denote the set of all such scalar relations by $\mathcal{S}$. Then it is clear that the tuple $(\mathcal{S}, \subseteq, \cup, \cap, O_X, \mathrm{id}_X)$ is a complete distributive lattice with least element O_X and greatest element id_X, and also that is isomorphic to L. Moreover, scalar L-relations can be characterized algebraically:

PROPOSITION 2.2. *R is a scalar L-relation if and only if $R \subseteq \mathrm{id}_X$ and $R\nabla_X = \nabla_X R$.*

PROOF. Remark that $R\nabla_X(x,y) = \vee_{z\in X}[R(x,z)\wedge\nabla_X(z,y)] = R(x,x)\wedge\nabla_X(x,y) = R(x,x)$ for all $x,y \in X$ if $R \subseteq \mathrm{id}_X$. (Similarly $\nabla_X R(x,y) = \nabla_X(x,y) \wedge R(y,y)$.) Now assume that R is a scalar L-relation. Then it is trivial that $R \subseteq \mathrm{id}$. Thus $R\nabla_X(x,y) = R(x,x) \wedge \nabla_X(x,y) = \nabla_X(x,y) \wedge R(y,y) = \nabla_X R(x,y)$. Next assume that $R \subseteq \mathrm{id}$ and $R\nabla_X = \nabla_X R$. By $R \subseteq \mathrm{id}_X$ we obtain $R(x,y) = \mathbf{0}$ if $x \neq y$. Also $R(x,x) = R\nabla_X(x,y) = \nabla_X R(x,y) = R(y,y)$ for all $x,y \in X$. Therefore R is a scalar L-relation. □

3. Axioms of Relation Algebras

This section provides the axioms R1-R4 of relation algebras and lists some basic properties of relation algebras. A relation algebra $\mathcal{R}$, which will be defined below, is an algebraic structure over a nonempty set $\mathcal{R}$ of elements called "relations". Originally, relation algebras were formalized by Tarski (1941) as complete *Boolean algebras* with composition and converse. But in this paper, relation algebras are only complete *distributive lattices* with composition and converse. In other words, relation algebras (in this paper) are complete Dedekind categories (Olivier and Serrato (1995), Kawahara, Furusawa and Mori (1996)) or complete distributive allegories (Freyd and Scedrov (1990)) with just one object. Elements of $\mathcal{R}$ are denoted by Greek letters such as $\alpha, \beta, \cdots$. The composite $\alpha;\beta$ of a relation α followed by a relation β will be written by $\alpha\beta$, unless confusion is possible.

DEFINITION 3.1. A relation algebra $\mathcal{R} = (\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, ;, {}^\sharp, O, \nabla, \mathrm{id})$ is an algebraic structure over a nonempty set $\mathcal{R}$ satisfying the following:
R1. [Complete Distributive Lattice] The tuple $(\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, O, \nabla)$ is a complete distributive lattice.
R2. [Involutive Monoid] The tuple $(\mathcal{R}, ;, {}^\sharp, \mathrm{id}, O)$ is an involutive monoid with unit element id and zero element O. That is,
(a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, (b) $\alpha\mathrm{id} = \mathrm{id}\alpha = \alpha$, (c) $\alpha O = O\alpha = O$, (d) $(\alpha^\sharp)^\sharp = \alpha$, (e) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$,
(f) If $\alpha \sqsubseteq \beta$, then $\alpha^\sharp \sqsubseteq \beta^\sharp$.
R3. [Distributive Law] $\alpha(\sqcup_\lambda \beta_\lambda) = \sqcup_\lambda \alpha\beta_\lambda$.
R4. [Dedekind Formula] $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$. □

It is clear that every algebra $\mathbf{Rel}_L(X) = (\mathbf{Rel}_L(X), \subseteq, \cup, \cap, \circ, {}^\cup, O_X, \nabla_X, \mathrm{id}_X)$ of L-relations is a relation algebra. Let $\mathcal{R} = (\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, ;, {}^\sharp, O, \nabla, \mathrm{id})$ be a relation algebra. A relation algebra $\mathcal{R}$ with $\nabla = O$ is trivial and not worth mentioning. Throughout the rest of the paper all discussions will assume a fixed relation algebra $\mathcal{R}$ with $\nabla \neq O$. A relation α is nonzero if $\alpha \neq O$.

PROPOSITION 3.2. *Let α, β, β' be relations. Then the following hold:*

(a) *If $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta'$ and $\beta\alpha \sqsubseteq \beta'\alpha$.*

(b) *$O^\sharp = O$, $\nabla^\sharp = \nabla$ and $\mathrm{id}^\sharp = \mathrm{id}$.*

(c) *$(\alpha \sqcup \beta)^\sharp = \alpha^\sharp \sqcup \beta^\sharp$ and $(\alpha \sqcap \beta)^\sharp = \alpha^\sharp \sqcap \beta^\sharp$.*

(d) *If $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}$, then $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.*

(e) *If $\alpha \sqsubseteq \mathrm{id}$ and $\beta \sqsubseteq \mathrm{id}$, then $\alpha^\sharp = \alpha\alpha = \alpha$ and $\alpha\beta = \alpha \sqcap \beta$.*

(f) *If $\beta \sqsubseteq \mathrm{id}$ and $\beta' \sqsubseteq \mathrm{id}$, then $\alpha(\beta \sqcap \beta') = \alpha\beta \sqcap \alpha\beta'$.*

PROOF. (a) If $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta \sqcup \alpha\beta' = \alpha(\beta \sqcup \beta') = \alpha\beta'$ by R3. (b) $O^\sharp \sqsubseteq O^{\sharp\sharp} = O$ since $O \sqsubseteq O^\sharp$, and $\nabla = \nabla^{\sharp\sharp} \sqsubseteq \nabla^\sharp$ since $\nabla^\sharp \sqsubseteq \nabla$, and $\mathrm{id}^\sharp = \mathrm{id}^\sharp\mathrm{id} = \mathrm{id}^\sharp\mathrm{id}^{\sharp\sharp} = (\mathrm{id}^\sharp\mathrm{id})^\sharp = \mathrm{id}^{\sharp\sharp} = \mathrm{id}$. (c) First note that $\alpha^\sharp \sqcup \beta^\sharp \sqsubseteq (\alpha \sqcup \beta)^\sharp$. Hence $\alpha \sqcup \beta = \alpha^{\sharp\sharp} \sqcup \beta^{\sharp\sharp} \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^\sharp$ and $(\alpha \sqcup \beta)^\sharp \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^{\sharp\sharp} = \alpha^\sharp \sqcup \beta^\sharp$. (d) If $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}$, then $\alpha\beta \sqcap \alpha\beta' \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\alpha\beta') \sqsubseteq \alpha(\beta \sqcap \mathrm{id}\beta') = \alpha(\beta \sqcap \beta')$ by R4. (e) Assume that $\alpha \sqsubseteq \mathrm{id}$ and $\beta \sqsubseteq \mathrm{id}$. Then $\alpha = \alpha \sqcap \nabla \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp\nabla) \sqsubseteq \mathrm{id} \sqcap \alpha^\sharp\nabla \sqsubseteq \alpha^\sharp(\alpha \sqcap \nabla) \sqsubseteq \alpha^\sharp$ by R4. Similarly it can be shown that $\alpha^\sharp \sqsubseteq \alpha$ holds. Also $\alpha\alpha \sqsubseteq \alpha$ is trivial by (a), and $\alpha = \alpha \sqcap \nabla \sqsubseteq \alpha(\alpha \sqcap \alpha^\sharp\nabla) \sqsubseteq \alpha\alpha$ by R4. Moreover, since $\alpha\beta \sqsubseteq \beta$, $\alpha\beta = \alpha\beta \sqcap \beta \sqsubseteq \alpha\beta$ and $\alpha \sqcap \beta \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp\beta) \sqsubseteq \alpha\beta$ by R4. (f) If $\beta \sqsubseteq \mathrm{id}$ and $\beta' \sqsubseteq \mathrm{id}$, then $\alpha\beta \sqcap \alpha\beta' \sqsubseteq (\alpha \sqcap \alpha\beta'\beta^\sharp)\beta \sqsubseteq \alpha\beta'\beta = \alpha(\beta \sqcap \beta')$ by R4 and (e). □

Note that $\alpha(\sqcap_\lambda \beta_\lambda) \sqsubseteq \sqcap_\lambda(\alpha\beta_\lambda)$ and $\nabla\nabla = \nabla$ hold immediately by the last proposition 3.2(a).

The concepts of scalar relations and crisp relations in relation algebras are defined by the following:

DEFINITION 3.3. Let $\mathcal{R}$ be a relation algebra.

(a) A relation k is called **scalar** if and only if $k \sqsubseteq \mathrm{id}$ and $k\nabla = \nabla k$.

(b) A relation α is called **crisp** if for all nonzero scalar relations k and all relations β, $k\beta \sqsubseteq \alpha$ implies $\beta \sqsubseteq \alpha$. □

It is trivial that O and id are scalar relations, and that ∇ is crisp (but O and id are not necessarily crisp).

The concept of crisp relations has been defined in Kawahara, Furusawa and Mori (1996) on the assumption of the existence of a unit object. The concept of the crispness can also be found in Kawahara and Furusawa (1995) and Furusawa (1997), where it is defined via semi-scalar multiplication. In this paper we need neither a unit object, nor semi-scalar multiplication. Instead we used the concept of scalar relations to define crisp relations.

Next we provide some basic properties of scalar relations and crisp relations.

PROPOSITION 3.4. *Let k be a scalar relation and α, β relations. Then the following holds:*

(a) $k\alpha = \alpha \sqcap k\nabla$ *and* $\alpha k = \alpha \sqcap \nabla k$. *In particular,* $k = \mathrm{id} \sqcap k\nabla$.

(b) $k\alpha = \alpha k$.

(c) $(k \sqcap k')\alpha = \alpha(k \sqcap k')$, $(k \sqcup k')\alpha = \alpha(k \sqcup k')$.

(d) *If* $k\nabla \sqsubseteq k'\nabla$, *then* $k \sqsubseteq k'$.

(e) *If* α *and* β *are crisp, then so is* $\alpha \sqcap \beta$.

PROOF. (a) Since $k \sqsubseteq \mathrm{id}$ and $\alpha \sqsubseteq \nabla$, $k\alpha \sqsubseteq \alpha \sqcap k\nabla = k(k^\sharp\alpha \sqcap \nabla) = kk^\sharp\alpha = k\alpha$ by R4 and 3.2(e). Similarly it can be shown that $\alpha k = \alpha \sqcap \nabla k$. (b) From (a) it holds that $k\alpha = \alpha \sqcap k\nabla = \alpha \sqcap \nabla k = \alpha k$. (c) $(k \sqcap k')\alpha = (kk')\alpha = \alpha(kk') = \alpha(k \sqcap k')$ by 3.2(e) and (b). $(k \sqcup k')\alpha = k\alpha \sqcup k'\alpha = \alpha k \sqcup \alpha k' = \alpha(k \sqcup k')$ by R3 and (b). (d) Assume that $k\nabla \sqsubseteq k'\nabla$. Then $k = \mathrm{id} \sqcap k\nabla \sqsubseteq \mathrm{id} \sqcap k'\nabla = k'$ by (a). (e) If $k\gamma \sqsubseteq \alpha \sqcap \beta$, then $k\gamma \sqsubseteq \alpha$ and $k\gamma \sqsubseteq \beta$ by R1. Since α and β are crisp, $\gamma \sqsubseteq \alpha$ and $\gamma \sqsubseteq \beta$. Thus $\gamma \sqsubseteq \alpha \sqcap \beta$ by R1. □

In addition to the definition of crisp relations, scalar relations also play an important role in other respects. Let us denote the set of all scalar relations by L. Then L is closed under the operations supremum $\sqcup$ and infimum $\sqcap$ by proposition 3.4(c) and axiom R1. So the tuple $(L, \sqsubseteq, \sqcap, \sqcup, O, \mathrm{id})$ is a complete distributive lattice, and it is a sublattice of the relation algebra $\mathcal{R}$ with the least element O and the greatest element id.

4. Strict Point Axiom

This section introduces a new concept of point relations and a strict point axiom. A concept of point relations was introduced in Schmidt and Ströhlein (1985), (1993) to give a simple proof of a representation theorem for Boolean relation algebras and apply such algebras to computer science. Kawahara and Furusawa (1995) made the concept more strict to prove a representation theorem for fuzzy relation algebras. The concept of point relations is defined in this paper in the spirit of Kawahara and Furusawa (1995), but we have to pay attention to the difference between the notions of crisp relations in Kawahara and Furusawa (1995) and in this paper.

Before define the concept of point relations, we describe properties of relations which correspond to the vector relations in Schmidt and Ströhlein (1985), (1993).

PROPOSITION 4.1. *Let* α *be a crisp relation such that* $\nabla\alpha = \alpha$. *Then the following three conditions are equivalent :* (a) $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$, (b) $\nabla = \alpha\alpha^\sharp$, (c) $\nabla = \alpha\nabla$.

PROOF. (a)⇒(b) If $\mathrm{id} \sqsubseteq \alpha\alpha^\sharp$, then $\nabla = \nabla\mathrm{id} \sqsubseteq \nabla\alpha\alpha^\sharp = \alpha\alpha^\sharp$. (b)⇒(c) If $\nabla = \alpha\alpha^\sharp$, then $\nabla = \alpha\alpha^\sharp \sqsubseteq \alpha\nabla$. (c)⇒(a) If $\nabla = \alpha\nabla$, then $\mathrm{id} = \mathrm{id} \sqcap \nabla = \mathrm{id} \sqcap \alpha\nabla \sqsubseteq \alpha(\alpha^\sharp\mathrm{id} \sqcap \nabla) = \alpha\alpha^\sharp$. □

The concept of point relations in relation algebras is defined as follows:

DEFINITION 4.2. A **point relation** x is a crisp relation such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\mathrm{id} \sqsubseteq xx^\sharp$ and $\nabla x = x$. (Point relations will be denoted by lower case Roman letters such as $x, y, z, \cdots$.) The set of all point relations is denoted by X. □

Note that a point relation x is nonzero from its totality $\mathrm{id} \sqsubseteq xx^\sharp$. For point relations x and y, the relation $x^\sharp y$ is nonzero since $y \sqsubseteq x(x^\sharp y)$ by the totality $\mathrm{id} \sqsubseteq xx^\sharp$ of x.

PROPOSITION 4.3. *Let x, x_0, y, y_0 be point relations and k a nonzero scalar. Then the following holds:*

(a) *If $kx \sqsubseteq y$, then $x = y$.*

(b) *If $kx^\sharp y \sqsubseteq x_0^\sharp y_0$, then $x = x_0$ and $y = y_0$.*

PROOF. (a) Since y is crisp, it holds that $x \sqsubseteq y$. Using $\mathrm{id} \sqsubseteq xx^\sharp$, $x^\sharp \sqsubseteq y^\sharp$ and $y^\sharp y \sqsubseteq \mathrm{id}$ we have $y \sqsubseteq xx^\sharp y \sqsubseteq xy^\sharp y \sqsubseteq x$. (b) Assume that $kx^\sharp y \sqsubseteq x_0^\sharp y_0$. Then $ky = k\nabla y = k\nabla x^\sharp y = \nabla kx^\sharp y \sqsubseteq \nabla x_0^\sharp y_0 = y_0$ by 4.1 and so $y = y_0$ by (a). Similarly $x = x_0$. □

By making use of our last definition of point relations in relation algebras, we add the following axiom:

DEFINITION 4.4. A relation algebra $\mathcal{R}$ satisfies the **strict point axiom** iff:
R5. (a) For each nonzero relation α there are a nonzero scalar relation k and two point relations x and y such that $x\alpha y^\sharp = k\nabla$.
(b) $\sqcup_{x\in X} x^\sharp x = \mathrm{id}$. □

Note that the condition (b) of the strict point axiom R5 is equivalent to $\sqcup_{x \in X} x = \nabla$. In what follows we assume that the fixed relation algebra $\mathcal{R}$ satisfies the strict point axiom R5.

PROPOSITION 4.5. *Let α be a relation, x and y point relations. Then the following holds:*

(a) *If α is a nonzero relation, then there exist a nonzero scalar relation k and point relations x and y such that $kx^\sharp y \sqsubseteq \alpha$.*

(b) *If $x \neq y$, then $x \sqcap y = O$ and $xy^\sharp = O$.*

(c) *$x\alpha y^\sharp = k\nabla$ if and only if $\alpha \sqcap x^\sharp y = k(x^\sharp y)$.*

(d) *If $\alpha \sqsubseteq x^\sharp y$, then there exists a scalar relation k such that $\alpha = kx^\sharp y$.*

PROOF. (a) If $\alpha \neq O$, then then there exist a nonzero scalar relation k and point relations x and y such that $x\alpha y^\sharp = k\nabla$ by the strict point axiom R5. Since x and y are point relations, $kx^\sharp y = kx^\sharp \nabla y = x^\sharp k\nabla y = x^\sharp x\alpha y^\sharp y \sqsubseteq \alpha$ by 3.4(b). (b) Assume that $x \neq y$ and $x \sqcap y \neq O$. Then there exist a nonzero scalar relation k and point relations x_0 and y_0 such that $kx_0^\sharp y_0 \sqsubseteq x \sqcap y$ by (a). From 3.4(e) $x \sqcap y$ is crisp, so it holds that $x_0^\sharp y_0 \sqsubseteq x \sqcap y$.

Thus $y_0 = \nabla x_0^\sharp y_0 \sqsubseteq \nabla(x \sqcap y) \sqsubseteq \nabla x \sqcap \nabla y = x \sqcap y$ by 4.1. Therefore $x = y_0 = y$ by R1 and 4.3(a). Finally, if $x \sqcap y = O$, then $xy^\sharp = xy^\sharp \sqcap \nabla \sqsubseteq (x \sqcap \nabla y)y^\sharp = (x \sqcap y)y^\sharp = O$. (c) Assume that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$. Then it holds that $x\alpha y^\sharp = x\alpha y^\sharp \sqcap \nabla = x\alpha y^\sharp \sqcap (xx^\sharp)(yy^\sharp) = x(\alpha \sqcap x^\sharp y)y^\sharp = x[k(x^\sharp y)]y^\sharp = k(xx^\sharp)(yy^\sharp) = k\nabla$ by 4.1, 3.2(d) and 3.4(b). Next assume that $x\alpha y^\sharp = k\nabla$. Then $\alpha \sqcap x^\sharp y \sqsubseteq x^\sharp(x\alpha y^\sharp \sqcap \mathrm{id})y \sqsubseteq x^\sharp x\alpha y^\sharp y = x^\sharp(k\nabla)y = k(x^\sharp y)$ by R4, 4.1 and 3.4(b). Conversely, $k(x^\sharp y) = k(x^\sharp \nabla y) = x^\sharp k \nabla y = x^\sharp(x\alpha y^\sharp)y \sqsubseteq \alpha$ by 3.4(b). Thus $k(x^\sharp y) \sqsubseteq \alpha \sqcap x^\sharp y$. (d) It is trivial that if $\alpha = O$ then $\alpha = O(x^\sharp y)$. Next assume that $\alpha \neq O$. Then, by the strict point axiom and (c), there are a nonzero scalar relation k and point relations x_0, y_0 such that $\alpha \sqcap x_0^\sharp y_0 = k(x_0^\sharp y_0)$. Hence $k(x_0^\sharp y_0) \sqsubseteq \alpha \sqsubseteq x^\sharp y$, and so $x = x_0$ and $y = y_0$ by 4.3(b), which implies $\alpha = k(x^\sharp y)$. □

By (d) of the last proposition, for every relation α and for every two point relations x, y there exists a scalar relation k such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, and so $x\alpha y^\sharp = k\nabla$ by (c) of the last proposition. Also, by proposition 3.4(d), such a scalar relation k is unique. For a relation α and point relations x, y, we define $\psi(\alpha)(x, y)$ to be the unique scalar relation k with $x\alpha y^\sharp = k\nabla$. Thus, by proposition 3.4(d), $\psi(\alpha)(x, y)$ is the unique scalar relation such that $x\alpha y^\sharp = \psi(\alpha)(x, y)\nabla$. Therefore $\psi(\alpha)$ defines an L-relation on the set X of all point relations in $\mathcal{R}$ since the set L of all scalar relations is a complete distributive lattice.

5. Representation Theorem

First we prove a representation theorem for relation algebras satisfying the strict point axiom R5. The representation problem of Boolean relation algebras was proposed by Tarski (1941) and investigated for a long time, see Schmidt and Ströhlein (1985), (1993) and Maddux (1991) for more details on the history of the investigation of the representation theorem for Boolean relation algebras. Also Kawahara and Furusawa (1995) proved an algebraic representation theorem of fuzzy relations, and Kawahara, Furusawa and Mori (1996) proved such theorems for Dedekind categories (or allegories) and Zadeh categories. The following theorem also is a representation theorem for Dedekind categories with just one object.

THEOREM 5.1 REPRESENTATION THEOREM. *Let $\mathcal{R}$ be a relation algebra satisfying the strict point axiom. Then every relation α has a unique representation*

$$\alpha = \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x, y)y \ .$$

PROOF. Since $\mathrm{id} = \sqcup_{x \in X} x^\sharp x$ and $\mathrm{id} = \sqcup_{y \in X} y^\sharp y$ by the strict point axiom, we have

$$\begin{aligned} \alpha &= \mathrm{id}\alpha\mathrm{id} \\ &= (\sqcup_{x \in X} x^\sharp x)\alpha(\sqcup_{y \in X} y^\sharp y) \\ &= \sqcup_{x,y \in X} x^\sharp x\alpha y^\sharp y \\ &= \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x, y)\nabla y \\ &= \sqcup_{x,y \in X} x^\sharp \psi(\alpha)(x, y)y \ . \end{aligned}$$

Finally we show the uniqueness of the representation. Assume that $\alpha = \sqcup_{x,y\in X} x^\sharp k_{x,y} y$. Then for all $x_0, y_0 \in X$ we have $\psi(\alpha)(x_0, y_0)\nabla = x_0 \alpha y_0^\sharp = \sqcup_{x,y\in X} x_0 x^\sharp k_{x,y} y y_0^\sharp = k_{x,y}$ by 4.5(b). □

From the last theorem we can deduce the next property of the function $\psi : \mathcal{R} \to \mathbf{Rel}_L(X)$.

COROLLARY 5.2. *For every relation algebra $\mathcal{R}$ satisfying the strict point axiom, the function $\psi : \mathcal{R} \to \mathbf{Rel}_L(X)$ is bijective.*

PROOF. If $\psi(\alpha) = \psi(\beta)$, then by the last theorem we have

$$\alpha = \sqcup_{x,y\in X} x^\sharp \psi(\alpha)(x,y)\nabla y = \sqcup_{x,y\in X} x^\sharp \psi(\beta)(x,y)\nabla y = \beta \ ,$$

which shows that ψ is injective. Given an L-relation $R \in \mathbf{Rel}_L(X)$, we set $\alpha_R = \sqcup_{x,y\in X} x^\sharp R(x,y)\nabla y$. Then by the uniqueness of the representation in the last theorem we have $R(x,y) = \psi(\alpha_R)(x,y)$, which shows that ψ is surjective. □

The following proposition shows that $\psi : \mathcal{R} \to \mathbf{Rel}_L(X)$ preserves all operations of L-fuzzy relations, that is, ψ is a homomorphism of relation algebras from $\mathcal{R}$ to $\mathbf{Rel}_L(X)$.

PROPOSITION 5.3. *Let α, β be relations. Then the following holds:*

(a) $\psi(O) = O_X$, $\psi(\nabla) = \nabla_X$ *and* $\psi(\mathrm{id}) = \mathrm{id}_X$.

(b) *If $\alpha \sqsubseteq \beta$, then $\psi(\alpha) \sqsubseteq \psi(\beta)$.*

(c) $\psi(\alpha \sqcup \beta) = \psi(\alpha) \cup \psi(\beta)$,

(d) $\psi(\alpha \sqcap \beta) = \psi(\alpha) \cap \psi(\beta)$.

(e) $\psi(\alpha^\sharp) = \psi(\alpha)^\cup$.

(f) $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$.

PROOF. (a) The first follows from $\psi(O)(x,y)\nabla = xOy^\sharp = O\nabla$, the second follows from $\psi(\nabla)(x,y)\nabla = x\nabla y^\sharp = \mathrm{id}\nabla$ by 4.1. Remarking $\psi(\mathrm{id})(x,y)\nabla = x\mathrm{id}y^\sharp = xy^\sharp$, the last follows from $\psi(\mathrm{id})(x,y)\nabla = \mathrm{id}\nabla$ if $x = y$ and $\psi(\mathrm{id})(x,y)\nabla = O\nabla$, otherwise by 4.1 and 4.5(b).
(b) If $\alpha \sqsubseteq \beta$, then $\psi(\alpha)(x,y)\nabla = x\alpha y^\sharp \sqsubseteq x\beta y^\sharp = \psi(\beta)(x,y)\nabla$.
(c) It follows from

$$\begin{aligned}
\psi(\alpha \sqcup \beta)(x,y)\nabla &= x(\alpha \sqcup \beta)y^\sharp \\
&= x\alpha y^\sharp \sqcup x\beta y^\sharp \\
&= \psi(\alpha)(x,y)\nabla \sqcup \psi(\beta)(x,y)\nabla \\
&= [\psi(\alpha)(x,y) \sqcup \psi(\beta)(x,y)]\nabla \\
&= [\psi(\alpha) \cup \psi(\beta)](x,y)\nabla \ .
\end{aligned}$$

(d) It follows from

$$\begin{aligned}\psi(\alpha \sqcap \beta)(x,y)\nabla &= x(\alpha \sqcap \beta)y^\sharp \\ &= x\alpha y^\sharp \sqcap x\beta y^\sharp \\ &= \psi(\alpha)(x,y)\nabla \sqcap \psi(\beta)(x,y)\nabla \\ &= [\psi(\alpha)(x,y) \sqcap \psi(\beta)(x,y)]\nabla \\ &= [\psi(\alpha) \cap \psi(\beta)](x,y)\nabla \ ,\end{aligned}$$

by 3.2(d) and 3.2(f) since x and y are point relations and $\psi(\alpha)(x,y), \psi(\beta)(x,y) \sqsubseteq \mathrm{id}$.
(e) It follows from $\psi(\alpha^\sharp)(x,y)\nabla = x\alpha^\sharp y^\sharp = (y\alpha x^\sharp)^\sharp = (\psi(\alpha)(y,x)\nabla)^\sharp = \psi(\alpha)(y,x)\nabla = \psi(\alpha)^\cup(x,y)\nabla$ since $\psi(\alpha)(y,x)$ is a scalar relation.
(f) It follows from

$$\begin{aligned}\psi(\alpha\beta)(x,y)\nabla &= x(\alpha\beta)y^\sharp \\ &= x\alpha \mathrm{id} \beta y^\sharp \\ &= x\alpha(\sqcup_{z\in X} z^\sharp z)\beta y^\sharp \\ &= \sqcup_{z\in X} x\alpha z^\sharp z\beta y^\sharp \\ &= \sqcup_{z\in X} \psi(\alpha)(x,z)\nabla\psi(\beta)(z,y)\nabla \\ &= \sqcup_{z\in X} \psi(\alpha)(x,z)\psi(\beta)(z,y)\nabla \\ &= \sqcup_{z\in X} [\psi(\alpha)(x,z) \sqcap \psi(\beta)(z,y)]\nabla \\ &= (\psi(\alpha)\psi(\beta))(x,y)\nabla \ ,\end{aligned}$$

since $\psi(\alpha)(x,z)$ and $\psi(\beta)(z,y)$ are scalar relations. □

It is now obvious that ψ^{-1} is a function and is a homomorphism of algebras of L-relations from $\mathbf{Rel}_L(X)$ to $\mathcal{R}$. Thus we have the following corollary:

COROLLARY 5.4 ISOMORPHISM THEOREM. *Every relation algebra $\mathcal{R}$ satisfying the strict point axiom is isomorphic to the algebra $\mathbf{Rel}_L(X)$ of L-relations on the set X of all point relations of $\mathcal{R}$, where L is the distributive lattice of scalar relations in $\mathcal{R}$.* □

6. Conclusion

In this paper we proved a representation theorem for homogeneous relation algebras $\mathcal{R}$ satisfying the strict point axiom, which can be considered as Dedekind categories with just one object, using concepts of scalar relations and point relations. In Kawahara, Furusawa and Mori (1996) the representation theorem for Dedekind category was proved without using the concept of scalar relations, but with using the assumption of the existence of the unit object. It is shown in this paper by defining the new algebraic concept of scalar relations that such a representation theorem can be proved without assuming the existence of a unit object.

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