ASYMPTOTIC PROPERTIES OF JACKKNIFE SKEWNESS ESTIMATORS AND EDGEWORTH EXPANSIONS

Maesono, Yoshihiko
Faculty of Economics, Kyushu University

https://doi.org/10.5109/13469

出版情報：Bulletin of informatics and cybernetics. 30 (1), pp.51-68, 1998-03. Research Association of Statistical Sciences

Bulletin of Informatics and Cybernetics, Vol. 30, No. 1, 1998

ASYMPTOTIC PROPERTIES OF JACKKNIFE SKEWNESS ESTIMATORS AND EDGEWORTH EXPANSIONS

By

Yoshihiko MAESONO*

Abstract

It has been found in simulation studies that jackknife estimators of skewness have downward biases. In this paper we obtain asymptotic representations of the jackknife skewness estimators for U-statistics with remainder term $o_p(n^{-1})$ and discuss the biases theoretically. Using the asymptotic representations, we also obtain Edgeworth expansions with remainder term $o(n^{-1/2})$.

Key words and Phrases. H-decomposition, jackknife skewness estimator, third central moment, U-statistics, variance estimator.

1. Introduction

Let $X_1, \cdots, X_n$ be independently and identically distributed random variables with distribution function F and $T_n = T_n(X_1, \cdots, X_n)$ be a statistic related to the parameter θ, such as estimator, test statistic, etc. The skewness of T_n is defined as

$$\kappa_3 = \frac{\sqrt{n}E[T_n - E(T_n)]^3}{\{Var(T_n)\}^{3/2}},$$

which describes in some degree the asymmetry of its distribution about its expectation. And the skewness is a coefficient of $n^{-1/2}$ term in an Edgeworth expansion of the distribution of T_n. So, the estimator of the skewness plays an important role when obtaining an approximate upper α quantile or constructing a confidence interval based on the Edgeworth expansion. Beran (1984), and Hinkley and Wei (1984) have discussed the jackknife estimation of the skewness. The simulation studies by Beran (1984), Schemper (1987), and Tu and Zhang (1992) show that the jackknife skewness estimators have large downward biases. And Beran (1984) further has found that the biases in skewness estimators have a significant impact on the accuracy of the jackknifed Edgeworth approximation and the correctness of confidence intervals based on this approximation. In this paper we will obtain asymptotic representations of the jackknife skewness estimators and discuss the biases theoretically.

* Faculty of Economics, Kyushu University, Fukuoka 812-8581, Japan
e-mail:maesono@en.kyushu-u.ac.jp

It will be possible to study the skewness estimator for a general statistic under the conditions of van Zwet (1984), which ensure that higher order terms are negligible. But the study will be very complicated and difficult. So, since many statistics in common use are members of U-statistics or approximated by them, we will consider the skewness of U-statistics.

Let $h(x_1, \cdots, x_r)$ be a real valued function which is symmetric in its arguments. For $n \geq r$ let us define U-statistic by

$$U_n = \binom{n}{r}^{-1} \sum_{C_{n,r}} h(X_{i_1}, \cdots, X_{i_r})$$

where $\sum_{C_{n,r}}$ indicates that the summation is taken over all integers $i_1, \cdots, i_r$ satisfying $1 \leq i_1 < \cdots < i_r \leq n$. For a standardized U_n, Hoeffding (1948) proved the asymptotic normality

$$\lim_{n \to \infty} P\{\sigma_n^{-1}(U_n - \theta) \leq x\} = \Phi(x)$$

where $\theta = E[h(X_1, \cdots, X_r)]$, $\sigma_n^2 = Var(U_n)$ and $\Phi(x)$ is a distribution function of the standard normal. Thus we can construct an asymptotic confidence interval of θ as

$$U_n - \hat{\sigma}_n z_{\alpha/2} \leq \theta \leq U_n + \hat{\sigma}_n z_{\alpha/2} \tag{1}$$

where $\hat{\sigma}_n^2$ is an estimator of the variance σ_n^2 and $z_{\alpha/2}$ is an upper $\alpha/2$ level quantile of the standard normal distribution. Further Callaert, Janssen and Veraverbeke (1980), and Bickel, Goetze and van Zwet (1986) obtained an Edgeworth expansion for the distribution of U-statistic. The Edgeworth expansion $H_n(x)$ is given by

$$H_n(x) = \Phi(x) - n^{-1/2}\phi(x)\frac{\kappa_3}{6}(x^2 - 1) \tag{2}$$

where $\phi(x)$ is a density function of the standard normal and κ_3 is a skewness of U_n

$$\kappa_3 = \frac{\sqrt{n}E(U_n - \theta)^3}{(\sigma_n^2)^{3/2}} = \frac{n^2 E(U_n - \theta)^3}{(n\sigma_n^2)^{3/2}}.$$

They showed that

$$\sup_x |P\{\sigma_n^{-1}(U_n - \theta) \leq x\} - H_n(x)| = o(n^{-1/2}).$$

Thus we can construct another confidence interval

$$U_n - \hat{\sigma}_n z_{\alpha/2} - \frac{\hat{\kappa}_3 \hat{\sigma}_n}{6\sqrt{n}}(z_{\alpha/2}^2 - 1) \leq \theta \leq U_n + \hat{\sigma}_n z_{\alpha/2} - \frac{\hat{\kappa}_3 \hat{\sigma}_n}{6\sqrt{n}}(z_{\alpha/2}^2 - 1) \tag{3}$$

where $\hat{\kappa}_3$ is an estimator of κ_3. But, as pointed out by Hall (1992, Chap.3), both convergence rates of coverage probabilities in (1) and (3) are $O(n^{-1/2})$. Thus we cannot improve the convergence rates. To improve the rates, we have to consider the confidence interval based on the Edgeworth expansion of a studentized U-statistic

$$S_n = (U_n - \theta)/\hat{\sigma}_n.$$

Maesono (1995a) has obtained the Edgeworth expansion of the studentized U-statistic substituting a jackknife estimator $\hat{\sigma}_n^2$. The expansion is similar to (2) and include the skewness κ_3^* of S_n. From Maesono (1994, 1995b), we can calculate the skewness κ_3^* and then we can obtain an estimator $\hat{\kappa}_3^*$. In this paper we will discuss asymptotic properties of jackknife estimators $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ theoretically.

Let $U_n^{(i)}$ denote U-statistic computed from a sample of $n-1$ points with X_i left out and $U_n^{(i,j)}$ computed from a sample of $n-2$ points with X_i and X_j left out. The jackknife estimator $\hat{\kappa}_3$ of the skewness κ_3 of the standardized U-statistic is given by

$$\hat{\kappa}_3 = \frac{\hat{\mu}_n}{(n\hat{\sigma}_n^2)^{3/2}} \tag{4}$$

where

$$\hat{\sigma}_n^2 = \frac{n-1}{n}\sum_{i=1}^{n}[U_n^{(i)} - U_n]^2 \tag{5}$$

and

$$\begin{aligned}\hat{\mu}_n &= -\frac{(n-1)^3}{n}\sum_{i=1}^{n}(U_n^{(i)} - U_n)^3 \\ &+\frac{3(n-1)^2}{n}\sum_{i\neq j}(U_n^{(i)} - U_n)(U_n^{(j)} - U_n)[nU_n - (n-1)(U_n^{(i)} + U_n^{(j)}) + U_n^{(i,j)}].\end{aligned} \tag{6}$$

And the jackknife estimator $\hat{\kappa}_3^*$ of the skewness of S_n is given by

$$\hat{\kappa}_3^* = \frac{\hat{\nu}_n}{(n\hat{\sigma}_n^2)^{3/2}} \tag{7}$$

where

$$\begin{aligned}\hat{\nu}_n &= \frac{2(n-1)^3}{n}\sum_{i=1}^{n}(U_n^{(i)} - U_n)^3 \\ &-\frac{3(n-1)^2}{n}\sum_{i\neq j}(U_n^{(i)} - U_n)(U_n^{(j)} - U_n)[nU_n - (n-1)(U_n^{(i)} + U_n^{(j)}) + U_n^{(i,j)}].\end{aligned} \tag{8}$$

The properties of the jackknife variance estimator $\hat{\sigma}_n^2$ defined by (5) are precisely studied. Arvesen (1969) has obtained the exact representation of $\hat{\sigma}_n^2$, which is very complicated, and Efron and Stein (1982) have showed that $\hat{\sigma}_n^2$ has a positive bias. Further Maesono (1994) has obtained an asymptotic representation and an Edgeworth expansion with remainder term $o(n^{-1/2})$. Also the bias reduction for the jackknife variance estimator has been studied by Hinkley (1978), and Efron and Stein (1982). For the jackknife estimator of the third central moment, some properties have been studied. Using an adjustment of the coefficient of the estimator, Tu and Gross (1994) discussed the bias reduction of $\hat{\mu}_n$ and showed the effectiveness by simulation. There are also some another simulation studies for $\hat{\mu}_n$. Recently Maesono (1995c) has obtained an asymptotic representation and an Edgeworth expansion of $\hat{\mu}_n$. Hinkley and Wei (1984) discussed the properties of the estimator $\hat{\kappa}_3^*$ by simulation.

In this paper the asymptotic representations and the Edgeworth expansions of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ are established, and the biases of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ are studied theoretically. In Section 2, we discuss the asymptotic representations of $\hat{\sigma}_n^2$, $\hat{\mu}_n$ and $\hat{\nu}_n$. In Section 3, the asymptotic representations and the biases of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ are established and their Edgeworth expansions are obtained. Finally, in the case of variance estimation, we study the biases of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ in Section 4.

It is desirable to study asymptotic mean square errors of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$. But to calculate the errors, we should obtain more precise representations of the estimators. So, it may be studied in the future. Hereafter for the sake of simplicity, we will consider the kernel of degree 2. The generalization to the kernel with arbitrary degree will be obtained with notational complications and tedious calculations.

2. Preliminaries

At first we prepare the H-decomposition of U-statistic. The H-decomposition or $ANOVA$-decomposition is a basic tool of the analysis of variance, the jackknife inference, etc.(see Appendix). Under the assumption that $E|h(X_1, X_2)| < \infty$, let us define

$$g_1(x) = E[h(x, X_2)] - \theta, \qquad g_2(x, y) = h(x, y) - \theta - g_1(x) - g_1(y)$$

and

$$A_1 = \sum_{i=1}^{n} g_1(X_i), \quad A_2 = \sum_{C_{n,2}} g_2(X_i, X_j).$$

Then we have

$$U_n - \theta = \frac{2}{n} A_1 + \frac{2}{n(n-1)} A_2.$$

Note that

$$E[g_2(X_1, X_2)|X_1] = 0 \ \ a.s.$$

So, if one of $\{i_1, i_2\}$ is not contained in $\{j_1, \cdots, j_m\}$, for m-variate function α which satisfies $E|\alpha g_2| < \infty$, we get

$$E[g_k(X_{i_1}, X_{i_2})\alpha(X_{j_1}, \cdots, X_{j_m})] = 0.$$

Using this equation we have the variance σ_n^2 of U_n (see Lee (1990, p.31))

$$\sigma_n^2 = \frac{4}{n}\xi_1^2 + \frac{2}{n(n-1)}\xi_2^2 \tag{9}$$

where

$$\xi_1^2 = E[g_1^2(X_1)] \quad and \quad \xi_2^2 = E[g_2^2(X_1, X_2)].$$

To discuss asymptotic properties of a statistic, it is convenient to obtain an asymptotic representation with remainder term $o_p(n^{-1})$ which means

$$P\{|o_p(n^{-1})| \geq n^{-1}(\log n)^{-1}\} = o(n^{-1}).$$

Let $\tilde{T}, R$ and $T = \tilde{T} + R$ be random variables, $H(.)$ be a bounded function, and γ be a positive constant. Then

$$\begin{aligned}\sup_x |P\{T \leq x\} - H(x)| &\leq \sup_x |P\{\tilde{T} \leq x\} - H(x)| + P\{|R| \geq \gamma\} \\ &+ \max\{H(x-\gamma) - H(x), H(x+\gamma) - H(x)\}.\end{aligned}$$

So, $o_p(n^{-1})$ is very useful for discussing the Edgeworth expansion and other asymptotic properties. It follows from Markov's inequality that if

$$E|R|^\beta = O(n^{-1-\beta-\gamma}) \quad \textit{for some} \quad \beta \geq 1 \;\; \textit{and} \;\; \gamma > 0, \tag{10}$$

we have

$$P\{|R| \geq n^{-1}(\log n)^{-1}\} = o(n^{-1}). \tag{11}$$

It is trivial that $cn^{-1-\gamma} = o_p(n^{-1})$ for constant c and $\gamma > 0$.

Let us define

$$\mu_n = n^2 E(U_n - \theta)^3.$$

Then the third central moment of U_n is given by μ_n/n^2 and $\hat{\mu}_n$ defined by (6) is a jackknife estimator of μ_n. From Maesono (1995c), we have an asymptotic representation of $\hat{\mu}_n$. Let us define

$$e_1 = E[g_1^3(X_1)], \qquad e_2 = E[g_1(X_1)g_1(X_2)g_2(X_1, X_2)],$$
$$e_3 = E[g_1(X_1)g_2^2(X_1, X_2)], \qquad e_4 = E[g_2(X_1, X_2)g_2(X_1, X_3)g_2(X_2, X_3)],$$

$$\begin{aligned}\lambda_1(x) &= 4\{g_1^3(x) - e_1\} + 24\{g_1(x)E[g_1(X_2)g_2(x, X_2)] - e_2\} \\ &+12E[g_1^2(X_2)g_2(x, X_2)] - 12\xi_1^2 g_1(x) + 24E[g_1(X_2)g_2(x, X_3)g_2(X_2, X_3)],\end{aligned}$$

$$\begin{aligned}\lambda_2(x, y) = {} & 24\{g_1(x)g_1(y)g_2(x, y) + e_2 \\ & \quad -E[(g_1(x)g_2(x, X_2) + g_1(y)g_2(y, X_2))g_1(X_2)]\} \\ & -12\{g_1^2(x)g_1(y) + g_1^2(y)g_1(x) - \xi_1^2 g_1(x) - \xi_1^2 g_1(y)\} \\ & +12\{[g_1^2(x) + g_1^2(y)]g_2(x, y) - E[g_1^2(X_2)\{g_2(x, X_2) + g_2(y, X_2)\}]\} \\ & -24\xi_1^2 g_2(x, y) - 48E[(g_1(x)g_2(y, X_3) + g_1(y)g_2(x, X_3))g_1(X_3)] \\ & +24E[g_1(X_3)g_2(x, X_3)g_2(y, X_3)] \\ & +24\{E[(g_1(x) + g_1(y))g_2(x, X_3)g_2(y, X_3) \\ & \qquad +g_1(X_3)g_2(x, y)(g_2(x, X_3) + g_2(y, X_3))] \\ & \quad -2E[(g_2(x, X_3) + g_2(y, X_3))g_1(X_2)g_2(X_2, X_3)]\} \\ & +24E[g_2(x, X_3)g_2(y, X_4)g_2(X_3, X_4)]\end{aligned}$$

and

$$\delta = -3e_1 - 6e_2 + 6e_3 + 2e_4.$$

Since the product of U-statistics is a linear combination of U-statistics, applying the H-decomposition and the moment evaluation (19) in Appendix repeatedly Maesono (1995c) obtained the following lemma.

LEMMA 2.1. *If* $E|h(X_1,X_2)|^{6+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *an asymptotic representation of* $\hat{\mu}_n$ *defined by (6)*

$$\hat{\mu}_n = \mu_n + \frac{2}{n}\sum_{i=1}^{n}\lambda_1(X_i) + \frac{2}{n(n-1)}\sum_{C_{n,2}}\lambda_2(X_i,X_j) + \frac{8\delta}{n} + o_p(n^{-1}).$$

PROOF. See Maesono (1995c).

It is easy to see that $E[\lambda_1(X_1)] = E[\lambda_2(X_1,X_2)] = 0$ and $E[\lambda_2(X_1,X_2)|X_1] = 0$ *a.s.* From the moment evaluation (19) in Appendix, we can show that if $E|\lambda_2(X_1,X_2)|^q < \infty$ for $q \geq 2$,

$$E|\sum_{C_{n,2}}\lambda_2(X_i,X_j)|^q \leq cn^q. \tag{12}$$

Maesono (1995b) has studied the bias δ in the case of variance estimation.

Maesono (1994) has also obtained asymptotic representation of $n\hat{\sigma}_n^2$ as follows.

LEMMA 2.2. *If* $E|h(X_1,X_2)|^{4+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *an asymptotic representation of the jackknife variance estimator* $n\hat{\sigma}_n^2$ *defined by (5) is given by*

$$n\hat{\sigma}_n^2 = n\sigma_n^2 + \frac{2}{n}\sum_{i=1}^{n}f_1(X_i) + \frac{2}{n(n-1)}\sum_{C_{n,2}}f_2(X_i,X_j) + \frac{2\xi_2^2}{n} + o_p(n^{-1})$$

where

$$f_1(x) = 2[g_1^2(x) - \xi_1^2] + 4E[g_1(X_2)g_2(x,X_2)]$$

and

$$\begin{aligned} f_2(x,y) &= -4g_1(x)g_1(y) + 4E[g_2(x,X_3)g_2(y,X_3)] \\ &+ 4g_2(x,y)\{g_1(x)+g_1(y)\} - 4E[\{g_2(x,X_3)+g_2(y,X_3)\}g_1(X_3)] \end{aligned}$$

PROOF. See Maesono (1994).

It is easy to see that $E[f_1(X_1)] = E[f_2(X_1,X_2)] = 0$ and $E[f_2(X_1,X_2)|X_1] = 0$ *a.s.* And using H-decomposition, we can show that if $E|f_2(X_1,X_2)|^q < \infty$ for $q \geq 2$,

$$E|\sum_{C_{n,2}}f_2(X_i,X_j)|^q \leq cn^q. \tag{13}$$

As pointed out by Efron and Stein (1981), $n\hat{\sigma}_n^2$ has a positive bias $2\xi_2^2/n$. Replacing $n\hat{\sigma}_n^2 - n\sigma_n^2$ by a U-statistic with degree 2 and additional n^{-1} term, we can study the asymptotic properties of the jackknife estimator of the variance.

Next we will consider the skewness κ_3^* of the studentized U-statistic S_n. Maesono (1994) has proved an asymptotic representation of S_n. Let us define

$$\tau = \frac{3E[f_1^2(X_1)]}{2\xi_1^4} - \frac{\xi_2^2}{2\xi_1^2}, \qquad \rho = E[f_1(X_1)g_1(X_1)],$$

$$\begin{aligned} a_1(x) &= \tau g_1(x) - \frac{1}{\xi_1^2}\{(f_1(x)g_1(x) - \rho) \\ &\quad +(E[f_2(x,X_2)g_1(X_2)] - \frac{3\rho}{\xi_1^2}f_1(x)) + E[g_2(x,X_2)f_1(X_2)]\}, \\ a_2(x,y) &= g_2(x,y) - \frac{1}{\xi_1^2}[f_1(x)g_1(y) + f_1(y)g_1(x)] \end{aligned}$$

and

$$\begin{aligned} a_3(x,y,z) &= -\frac{1}{\xi_1^2}\{f_1(x)g_2(y,z) + f_1(y)g_2(x,z) + f_1(z)g_2(x,y) \\ &\quad +g_1(x)[f_2(y,z) - \frac{3}{\xi_1^2}f_1(y)f_1(z)] + g_1(y)[f_2(x,z) - \frac{3}{\xi_1^2}f_1(x)f_1(z)] \\ &\quad +g_1(z)[f_2(x,y) - \frac{3}{\xi_1^2}f_1(x)f_1(y)]\} \end{aligned}$$

Then we have the following lemma.

LEMMA 2.3. *If $E|h(X_1,\cdots,X_r)|^9 < \infty$ and $\xi_1^2 > 0$, for the studentized U-statistic $S_n = (U_n - \theta)/\hat{\sigma}_n$, we have*

$$S_n = \sqrt{n}U_n^* - \frac{\rho}{\sqrt{n}\xi_1^3} + o_p(n^{-1})$$

where

$$\begin{aligned} U_n^* &= \frac{1}{n\xi_1}\sum_{i=1}^{n}\{g_1(X_i) + \frac{a_1(X_i)}{n}\} \\ &+ \frac{2}{n(n-1)\xi_1}\sum_{C_{n,2}} a_2(X_i,X_j) + \frac{2}{n(n-1)(n-2)\xi_1}\sum_{C_{n,3}} a_3(X_i,X_j,X_k). \end{aligned}$$

PROOF. See Maesono (1994).

Since S_n is an asymptotic U-statistic, the skewness $\kappa_3^* = n^2E(U_n^*)^3$ follows from Maesono (1995b). Let us define

$$\begin{aligned} &e_5 = E[g_1^4(X_1)], \qquad e_6 = E[g_1^2(X_1)g_1(X_2)g_2(X_1,X_2)], \\ &e_7 = E[g_1(X_1)g_1(X_2)g_2(X_1,X_3)g_2(X_2,X_3)], \\ &e_8 = E[g_1^5(X_1)], \qquad e_9 = E[g_1^2(X_1)g_1^2(X_2)g_2(X_1,X_2)], \\ &e_{10} = E[g_1^3(X_1)g_1(X_2)g_2(X_1,X_2)], \\ &e_{11} = E[g_1^2(X_1)g_1(X_2)g_2(X_1,X_3)g_2(X_2,X_3)], \\ &e_{12} = E[g_1(X_1)g_1(X_2)g_1(X_3)g_2(X_1,X_2)g_2(X_2,X_3)] \end{aligned}$$

and

$$e_{13} = E[g_1(X_1)g_1(X_2)g_2(X_2,X_3)g_2(X_1,X_4)g_2(X_3,X_4)].$$

From direct computations, we have an asymptotic skewness $\hat{\kappa}_3^*$.

LEMMA 2.4. *If $E|h(X_1,\cdots,X_r)|^9 < \infty$ and $\xi_1^2 > 0$, we have*

$$\begin{aligned}
\hat{\kappa}_3^* &= n^2E(U_n^*)^3 = \frac{1}{\xi_1^3}(-2e_1 - 3e_2) \\
&+ n^{-1}\{\frac{1}{\xi_1^3}(-\frac{39}{8}e_1 - \frac{3}{2}e_2 - 3e_3 - 2e_4) \\
&\quad +\frac{1}{\xi_1^5}[-3\xi_2^2(e_1+2e_2) - \frac{3}{4}e_8 + \frac{3}{2}e_9 - 3e_{10} - 3e_{11} - 9e_{12} - 6e_{13}] \\
&\quad +\frac{1}{\xi_1^7}[3e_1(\frac{e_5}{8} + \frac{9}{2}e_6 + \frac{11}{2}e_7) + 3e_2(\frac{e_5}{2} + 10e_6 + 12e_7)] \\
&\quad -\frac{5}{2\xi_1^9}(e_1+e_2)^3\} + O(n^{-2}). \qquad (14)
\end{aligned}$$

PROOF. See Appendix.

Using Lemma 4 and 5 in Maesono (1995c), we can obtain the asymptotic representation of $\hat{\nu}_n$. Let us define

$$\begin{aligned}
\lambda_1^*(x) &= -8\{g_1^3(x) - e_1\} - 24\{g_1(x)E[g_1(X_2)g_2(x,X_2)] - e_2\} \\
&-24E[g_1^2(X_2)g_2(x,X_2)] + 24\xi_1^2 g_1(x) - 24E[g_1(X_2)g_2(x,X_3)g_2(X_2,X_3)],
\end{aligned}$$

$$\begin{aligned}
\lambda_2^*(x,y) = &-24\{g_1(x)g_1(y)g_2(x,y) + e_2 \\
&\quad -E[(g_1(x)g_2(x,X_2) + g_1(y)g_2(y,X_2))g_1(X_2)]\} \\
&+24\{g_1^2(x)g_1(y) + g_1^2(y)g_1(x) - \xi_1^2 g_1(x) - \xi_1^2 g_1(y)\} \\
&-24\{[g_1^2(x) + g_1^2(y)]g_2(x,y) - E[g_1^2(X_2)\{g_2(x,X_2) + g_2(y,X_2)\}]\} \\
&+48\xi_1^2 g_2(x,y) + 72E[(g_1(x)g_2(y,X_3) + g_1(y)g_2(x,X_3))g_1(X_3)] \\
&-48E[g_1(X_3)g_2(x,X_3)g_2(y,X_3)] \\
&-24\{E[(g_1(x) + g_1(y))g_2(x,X_3)g_2(y,X_3) \\
&\qquad +g_1(X_3)g_2(x,y)(g_2(x,X_3) + g_2(y,X_3))] \\
&\qquad -2E[(g_2(x,X_3) + g_2(y,X_3))g_1(X_2)g_2(X_2,X_3)]\} \\
&-24E[g_2(x,X_3)g_2(y,X_4)g_2(X_3,X_4)]
\end{aligned}$$

and

$$\delta^* = 6e_1 + 12e_2 - 12e_3 - 3e_4.$$

Similarly as $\hat{\mu}_n$, we can easily obtain a representation of $\hat{\nu}_n$.

LEMMA 2.5. *If* $E|h(X_1, X_2)|^{6+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *an asymptotic representation of* $\hat{\nu}_n$ *defined by (8) is*

$$\hat{\nu}_n = 8(-2e_1 - 3e_2) + \frac{2}{n}\sum_{i=1}^{n}\lambda_1^*(X_i) + \frac{2}{n(n-1)}\sum_{C_{n,2}}\lambda_2^*(X_i, X_j) + \frac{8\delta^*}{n} + o_p(n^{-1}).$$

3. Asymptotic results for jackknife skewness estimators

3.1. Asymptotic representations and biases

In this section, using the asymptotic representations of $n\hat{\sigma}_n^2$, $\hat{\mu}_n$ and $\hat{\nu}_n$, we will discuss the representations of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ and obtain the Edgeworth expansions of them.

Similarly as Lemma 3 in Maesono (1994), we have the following lemma.

LEMMA 3.1. *If* $E|h(X_1, X_2)|^9 < \infty$ *and* $\xi_1^2 > 0$, *we have*

$$\begin{aligned}(n\sigma_n^2)^{\frac{3}{2}}(n\hat{\sigma}_n^2)^{-\frac{3}{2}} &= 1 - \frac{3}{4n\xi_1^2}\sum_{i=1}^{n}f_1(X_i) - \frac{3}{4n(n-1)\xi_1^2}\sum_{C_{n,2}}[f_2(X_i, X_j) \\ &\quad - \frac{5}{4\xi_1^2}f_1(X_i)f_1(X_j)] + \frac{1}{n}\{\frac{15E[f_1^2(X_1)]}{32\xi_1^4} - \frac{3\xi_2^2}{4\xi_1^2}\} + o_p(n^{-1}).\end{aligned} \tag{15}$$

PROOF. Since

$$(n\sigma_n^2)^{\frac{3}{2}}(n\hat{\sigma}_n^2)^{-\frac{3}{2}} = (1 + \frac{n\hat{\sigma}_n^2 - n\sigma_n^2}{n\sigma_n^2})^{-\frac{3}{2}},$$

we can use a Taylor expansion

$$(1+x)^{-\frac{3}{2}} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}(1+\vartheta)^{-\frac{9}{2}}x^3$$

where $0 \le |\vartheta| \le |x|$. From the equation (9), we obtain that $(n\sigma_n^2)^{-1} = (4\xi_1^2)^{-1} + O(n^{-1})$ and $(n\sigma_n^2)^{-2} = (4\xi_1^2)^{-2} + O(n^{-1})$. Similarly as the proof of Lemma 3 in Maesono (1994), it follows from Lemma 2.2 and Lemma 5.2 that under the moment condition

$$(\frac{n\hat{\sigma}_n^2 - n\sigma_n^2}{n\sigma_n^2})^2 = \frac{1}{2n^2\xi_1^4}\sum_{C_{n,2}}f_1(X_i)f_1(X_j) + \frac{Ef_1^2(X_1)}{4n\xi_1^4} + o_p(n^{-1})$$

and

$$\frac{35}{16}(1+\vartheta)^{-\frac{9}{2}}(\frac{n\hat{\sigma}_n^2 - n\sigma_n^2}{n\sigma_n^2})^3 = o_p(n^{-1}).$$

Thus we have the equation (15).

From the equation (9) and Theorem 1 in Maesono (1995b), we can show that

$$(n\sigma_n^2)^{-\frac{3}{2}} = \frac{1}{8\xi_1^3} - \frac{3\xi_2^2}{32n\xi_1^5} + O(n^{-2})$$

and

$$\mu_n = 8(e_1 + 3e_2) + \frac{8(3e_3 + e_4)}{n} + O(n^{-2}).$$

Thus we have

$$\kappa_3 = \frac{e_1 + 3e_2}{\xi_1^3} + \frac{1}{n}\{\frac{3e_3 + e_4}{\xi_1^3} - \frac{3\xi_2^2(e_1 + 3e_2)}{4\xi_1^5}\} + O(n^{-2}). \tag{16}$$

Using Lemma 3.1, we can obtain the asymptotic representation of $\hat{\kappa}_3$.

THEOREM 3.2. *If $E|h(X_1, X_2)|^{10+\varepsilon} < \infty$ for some $\varepsilon > 0$ and $\xi_1^2 > 0$, an asymptotic representation of the jackknife skewness estimator $\hat{\kappa}_3$ in (4) is given by*

$$\hat{\kappa}_3 = \kappa_3 + \frac{2}{n\xi_1^3}\sum_{i=1}^{n} \zeta_1(X_i) + \frac{2}{n(n-1)\xi_1^3}\sum_{C_{n,2}} \zeta_2(X_i, X_j) + \frac{d}{n\xi_1^3} + o_p(n^{-1}) \tag{17}$$

where

$$\zeta_1(x) = \frac{1}{8}\lambda_1(x) - \frac{3(e_1 + 3e_2)}{8\xi_1^2} f_1(x),$$

$$\zeta_2(x,y) = \frac{1}{8}\lambda_2(x,y) - \frac{3}{16\xi_1^2}\{f_1(x)\lambda_1(y) + f_1(y)\lambda_1(x)\}$$

$$-\frac{3(e_1 + 3e_2)}{8\xi_1^2} f_2(x,y) + \frac{5(e_1 + 3e_2)}{8\xi_1^4} f_1(x) f_1(y)$$

and

$$d = \delta + (e_1 + 3e_2)\{\frac{15E[f_1^2(X_1)]}{32\xi_1^4} - \frac{3\xi_2^2}{4\xi_1^2}\} - \frac{3}{16\xi_1^2} E[f_1(X_1)\lambda_1(X_1)].$$

PROOF. See Appendix.

Similarly as Theorem 3.2, using Lemma 2.5 and Lemma 3.1, the asymptotic representation of $\hat{\kappa}_3^*$ in (7) is obtained as follows.

THEOREM 3.3. *If $E|h(X_1, X_2)|^{10+\varepsilon} < \infty$ for some $\varepsilon > 0$ and $\xi_1^2 > 0$, an approximation of the jackknife skewness estimator $\hat{\kappa}_3^*$ defined by (7) is*

$$\hat{\kappa}_3^* = \kappa_3^* + \frac{2}{n\xi_1^3}\sum_{i=1}^{n} \zeta_1^*(X_i) + \frac{2}{n(n-1)\xi_1^3}\sum_{C_{n,2}} \zeta_2^*(X_i, X_j) + \frac{d^*}{n\xi_1^3} + o_p(n^{-1})$$

where

$$\zeta_1^*(x) = \frac{1}{8}\lambda_1^*(x) + \frac{3(2e_1 + 3e_2)}{8\xi_1^2} f_1(x),$$

$$\zeta_2^*(x,y) = \frac{1}{8}\lambda_2^*(x,y) - \frac{3}{32\xi_1^2}\{f_1(x)\lambda_1^*(y) + f_1(y)\lambda_1^*(x)\}$$

$$-\frac{3(2e_1 + 3e_2)}{8\xi_1^2} f_2(x,y) + \frac{5(2e_1 + 3e_2)}{8\xi_1^4} f_1(x) f_1(y)$$

and

$$\begin{aligned}
d^* = &\{\frac{519}{64}e_1 + \frac{909}{128}e_2 - 9e_3 - e_4\} \\
&+\frac{1}{\xi_1^2}\{\xi_2^2(6e_1 + \frac{21}{2}e_2) + \frac{3}{2}e_8 + \frac{3}{4}e_9 + \frac{21}{4}e_{10} + \frac{39}{4}e_{11} + \frac{27}{2}e_{12} + \frac{21}{4}e_{13}\} \\
&-\frac{1}{\xi_1^4}\{e_1(\frac{39}{64}e_5 + \frac{231}{16}e_6 + \frac{279}{16}e_7) + e_2(\frac{237}{128}e_5 + \frac{1005}{32}e_6 + \frac{1197}{32}e_7)\} \\
&+\frac{5}{2\xi_1^6}(e_1 + 2e_2)^3.
\end{aligned}$$

REMARK. d/ξ_1^3 and d^*/ξ_1^3 are asymptotic biases of $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$. d^* is more complicated than d. In the case of variance estimation, we will study the biases d/ξ_1^3 and d^*/ξ_1^3 in Section 4.

3.2. Edgeworth expansions

Since the jackknife skewness estimators $\hat{\kappa}_3$ and $\hat{\kappa}_3^*$ are asymptotic U-statistics, using the Edgeworth expansion for U-statistics, we can obtain the Edgeworth expansion with remainder term $o(n^{-1/2})$.

THEOREM 3.4. *If* $E|h(X_1, X_2)|^{10+\varepsilon} < \infty$ *for some* $\varepsilon > 0$ *and* $\limsup_{|t|\to\infty} |E[\exp\{it\zeta_1(X_1)\}]| < 1$, *an Edgeworth expansion* $Q_n(x)$ *for the jackknife skewness estimator* $\hat{\kappa}_3$ *defined by (4) satisfies*

$$\sup_x |P\{\frac{\sqrt{n}\xi_1^3(\hat{\kappa}_3 - \kappa_3)}{2\tilde{\xi}_1} \le x\} - Q_n(x)| = o(n^{-1/2})$$

where

$$\begin{aligned}
Q_n(x) &= \Phi(x) - \frac{\phi(x)}{6\sqrt{n}}(\tilde{\kappa}_3 x^2 + \frac{3d}{\tilde{\xi}_1} - \tilde{\kappa}_3), \\
\tilde{\kappa}_3 &= (\tilde{\xi}_1)^{-3}\{E[\zeta_1^3(X_1)] + 3E[\zeta_1(X_1)\zeta_1(X_2)\zeta_2(X_1, X_2)]\}
\end{aligned}$$

and

$$(\tilde{\xi}_1)^2 = E[\zeta_1^2(X_1)].$$

PROOF. Let us define

$$\tilde{U}_n = \frac{2}{n}\sum_{i=1}^{n}\zeta_1(X_i) + \frac{2}{n(n-1)}\sum_{C_{n,2}}\zeta_2(X_i, X_j).$$

Then from the equation (17), we have

$$\frac{\sqrt{n}\xi_1^3(\hat{\kappa}_3 - \kappa_3)}{2\tilde{\xi}_1} = \frac{\sqrt{n}\tilde{U}_n}{2\tilde{\xi}_1} + \frac{d}{2\tilde{\xi}_1\sqrt{n}} + \sqrt{n}o_p(n^{-1}).$$

Since $\tilde{U}_n$ is a U-statistic with degree 2, using the Edgeworth expansion for U-statistics (see Bickel et al. (1986)), we have

$$P\{\frac{\sqrt{n}\tilde{U}_n}{2\tilde{\xi}_1} \leq x\} = \Phi(x) - \frac{\tilde{\kappa}_3}{6\sqrt{n}}\phi(x)(x^2-1) + o(n^{-1/2}).$$

It is easy to see that

$$P\{\frac{\sqrt{n}\tilde{U}_n}{2\tilde{\xi}_1} + \frac{d}{2\tilde{\xi}_1\sqrt{n}} \leq x\} = P\{\frac{\sqrt{n}\tilde{U}_n}{2\tilde{\xi}_1} \leq x - \frac{d}{2\tilde{\xi}_1\sqrt{n}}\} = H_n(x) + o(n^{-1/2}).$$

Since

$$P\{|\sqrt{n}o_p(n^{-1})| \geq n^{-1/2}(\log n)^{-1}\} = o(n^{-1/2}),$$

from Lemma 2.2, we have the desired result.

It is easy to obtain an Edgeworth expansion for $\hat{\kappa}_3^*$.

THEOREM 3.5. *If* $E|h(X_1,X_2)|^{10+\varepsilon} < \infty$ *for some* $\varepsilon > 0$ *and* $\limsup_{|t|\to\infty} |E[\exp\{it\zeta_1(X_1)\}]| < 1$, *an Edgeworth expansion* $Q_n^*(x)$ *for the jackknife skewness estimator* $\hat{\kappa}_3^*$ *defined by (7) is*

$$Q_n^*(x) = \Phi(x) - \frac{\phi(x)}{6\sqrt{n}}(\tilde{\kappa}_3^* x^2 + \frac{3d^*}{\tilde{\xi}_1^*} - \tilde{\kappa}_3^*)$$

where

$$\tilde{\kappa}_3^* = (\tilde{\xi}_1^*)^{-3}\{E[\zeta_1^*(X_1)]^3 + 3E[\zeta_1^*(X_1)\zeta_1^*(X_2)\zeta_2^*(X_1,X_2)]\}$$

and

$$(\tilde{\xi}_1^*)^2 = E[\zeta_1^*(X_1)]^2.$$

And $Q_n^*(x)$ *satisfies*

$$\sup_x |P\{\frac{\sqrt{n}\xi_1^3(\hat{\kappa}_3^* - \kappa_3^*)}{2\tilde{\xi}_1^*} \leq x\} - Q_n^*(x)| = o(n^{-1/2}).$$

4. Example

Let $\sigma^2 = Var(X_1)$. Then U-statistic with kernel $h(x,y) = (x-y)^2/2$ is an unbiased estimator of σ^2. We will discuss the biases of the jackknife estimators of the skewness of U-statistic

$$\frac{2}{n(n-1)}\sum_{C_{n,2}}\frac{1}{2}(X_i - X_j)^2.$$

It is easy to see that

$$\theta = \sigma^2, \quad g_1(x) = \frac{1}{2}(x^2 - \sigma^2) \quad \text{and} \quad g_2(x,y) = -xy.$$

For the sake of simplicity, we will consider the case that the distribution $F(x)$ is symmetric about the origin. Let us define

$$m_k = E[X_1^k].$$

Then because of symmetry of F, if k is odd number, $m_k = 0$. Using this fact, we can obtain that

$$\begin{aligned}
\xi_1^2 &= \frac{1}{4}(m_4 - \sigma^4), \quad \xi_2^2 = \sigma^4, \quad e_1 = \frac{1}{8}(m_6 - 3\sigma^2 m_4 + 2\sigma^6), \\
e_2 &= 0, \quad e_3 = \frac{1}{2}(\sigma^2 m_4 - \sigma^6), \quad e_4 = -\sigma^6, \\
e_5 &= \frac{1}{16}(m_8 - 4\sigma^2 m_6 + 6\sigma^4 m_4 - 3\sigma^8), \\
e_8 &= \frac{1}{32}(m_{10} - 5\sigma^2 m_8 + 10\sigma^4 m_6 - 10\sigma^6 m_4 + 4\sigma^{10}), \\
e_6 &= e_7 = e_9 = e_{10} = e_{11} = e_{12} = e_{13} = 0, \\
f_1(x) &= 2\{g_1^2(x) - \xi_1^2\}, \quad \lambda_1(x) = 4\{g_1^3(x) - e_1\} - 12\xi_2^2 g_1(x), \\
E[f_1^2(X_1)] &= \frac{1}{4}(m_8 - 4\sigma^2 m_6 + 8\sigma^4 m_4 - m_4^2 - 4\sigma^8)
\end{aligned}$$

and

$$\begin{aligned}
&E[f_1(X_1)\lambda_1(X_1)] \\
= \ &\frac{1}{4}(m_{10} - 5\sigma^2 m_8 + 14\sigma^4 m_6 - 30\sigma^6 m_4 + 12\sigma^2 m_4^2 - 4m_4 m_6 + 12\sigma^{10}).
\end{aligned}$$

Here we will study the following three underlying distributions.

Normal distribution: If the distribution is normal, that is $X_i \sim N(0, \sigma^2)$,

$$m_4 = 3\sigma^4, \quad m_6 = 15\sigma^6, \quad m_8 = 105\sigma^8 \ \textit{and} \ m_{10} = 945\sigma^{10}.$$

Logistic distribution: We consider the logistic distribution which has the density function

$$\frac{\pi e^{-\frac{\pi x}{\sqrt{3}\sigma}}}{\sqrt{3}\sigma(1 + e^{-\frac{\pi x}{\sqrt{3}\sigma}})}.$$

In this case we have that

$$\begin{aligned}
&Var(X_1) = \sigma^2, \quad m_4 = \frac{21}{5}\sigma^4, \quad m_6 = \frac{279}{7}\sigma^6, \\
&m_8 = \frac{3429}{5}\sigma^8 \ \textit{and} \ m_{10} = \frac{206955}{11}\sigma^{10}.
\end{aligned}$$

Laplace distribution: Finally we consider the Laplace distribution which has the density function

$$\frac{1}{\sqrt{2}\sigma} e^{-\frac{\sqrt{2}}{\sigma}|x|}.$$

Also we have that

$$Var(X_1) = \sigma^2, \quad m_4 = 6\sigma^4, \quad m_6 = 90\sigma^6,$$
$$m_8 = 2520\sigma^8 \text{ and } m_{10} = 113400\sigma^{10}.$$

Table lists the values of κ_3 and κ_3^* until the order n^{-1} and the biases d/ξ_1^3 and d^*/ξ_1^3 of order n^{-1}.

Table

	κ_3	d/ξ_1^3	κ_3^*	d^*/ξ_1^3
Normal	$2.83 + n^{-1}1.41$	-54.45	$-5.66 - n^{-1}146.37$	209.22
Logistic	$5.11 + n^{-1}0.52$	-785.04	$-10.22 - n^{-1}923.78$	1517.35
Laplace	$6.62 + n^{-1}0.68$	-1626.00	$-13.24 - n^{-1}1922.95$	3168.54

The asymptotic biases of the above cases are all downward, which mean the biases make the absolute value of the parameter small. It seems that if the distribution has heavy tail, the biases are large.

5. Appendix

At first we review H-decomposition or $ANOVA$-decomposition which is a basic tool of the studies of the analysis of variance, the jackknife inference, etc. Let $\nu(x_1, \cdots, x_r)$ be a function which is symmetric in its arguments and $E[\nu(X_1, \cdots, X_r)] = 0$. Let us define

$$\begin{aligned} \rho_1(x_1) &= E[\nu(x_1, X_2, \cdots, X_r)], \\ \rho_2(x_1, x_2) &= E[\nu(x_1, x_2, \cdots, X_r)] - \rho_1(x_1) - \rho_1(x_2), \cdots, \end{aligned}$$

and

$$\rho_r(x_1, x_2, \cdots, x_r) = \nu(x_1, x_2, \cdots, x_r) - \sum_{j=1}^{r-1} \sum_{C_{r,j}} \rho_j(x_{i_1}, x_{i_2}, \cdots, x_{i_j}).$$

Then we can show that

$$E[\rho_k(X_1, \cdots, X_k) | X_1, \cdots, X_{k-1}] = 0 \ \ a.s. \tag{18}$$

and

$$\sum_{C_{n,r}} \nu(X_{i_1}, \cdots, X_{i_r}) = \sum_{k=1}^{r} \binom{n-k}{r-k} \Lambda_k$$

where

$$\Lambda_k = \sum_{C_{n,k}} \rho_k(X_{i_1}, \cdots, X_{i_k}).$$

Using moment evaluations of martingales (Dharmadhikari, Fabian and Jogdeo (1968)), we have the upper bounds of the absolute moments of Λ_k as follows.

LEMMA 5.1. *For $q \geq 2$, if $E|\nu(X_1, \cdots, X_r)|^q < \infty$, there exists a positive constant c, which may depend on ν and F but not on n, such that*

$$E|\Lambda_k|^q \leq cn^{\frac{qk}{2}}. \tag{19}$$

PROOF. Using the equation (18), we can show that $\sum_{C_{n,k}} \rho_k(X_{i_1}, \cdots, X_{i_k})$ has a martingale property and $\sum_{C_{n-1,k-1}} \rho_k(X_{i_1}, \cdots, X_{i_{k-1}}, X_n)$ has a martingale property and so on. Then applying the result of Dharmadhikari et al. (1968) repeatedly, we can obtain the inequality (19).

From (10), (11) and (19), we can easily obtain the following lemma which is useful for obtaining the asymptotic representation.

LEMMA 5.2. *(i) If $E|\nu(X_1, \cdots, X_r)|^{2+\varepsilon} < \infty$ for $\varepsilon > 0$, we have*

$$n^{-r-1} \sum_{C_{n,r}} \nu(X_1, \cdots, X_r) = o_p(n^{-1}). \tag{20}$$

(ii) If $E|\nu(X_1, \cdots, X_r)|^{2+\varepsilon} < \infty$ for $\varepsilon > 0$ and $\rho_1(X_1) = \rho_2(X_1, X_2) = 0$ a.s., we have

$$n^{-r} \sum_{C_{n,r}} \nu(X_1, \cdots, X_r) = o_p(n^{-1}). \tag{21}$$

Proof of Lemma 2.4.

Let us define

$$\tilde{U}_n^* = U_n^* - \frac{1}{n^2\xi_1} \sum_{i=1}^n a_1(X_i).$$

From direct computation, we can show that

$$\begin{aligned} n^2 E(U_n^*)^3 &= n^2 E(\tilde{U}_n^*)^3 + \frac{9}{2n\xi_1^3} E[g_1^2(X_1)a_1(X_1)] \\ &+ \frac{9}{n\xi_1^3} E[a_1(X_1)g_1(X_2)g_2(X_1, X_2)] + O(n^{-2}). \end{aligned}$$

Since $\tilde{U}_n^*$ is a U-statistic with kernel degree 3, it follows from Maesono (1995b) that

$$\begin{aligned} & n^2 E(\tilde{U}_n^*)^3 = e_1 + 9E[g_1(X_1)g_1(X_2)a_2(X_1, X_2)] \\ + & \frac{27}{n}\{E[g_1(X_1)a_2^2(X_1, X_2)] + E[a_2(X_1, X_2)a_2(X_1, X_3)a_2(X_2, X_3)] \\ & + E[g_1(X_1)a_2(X_2, X_3)a_3(X_1, X_2, X_3)]\}. \end{aligned}$$

From long but direct computation, we can get the skewness κ_3^*.

Proof of Theorem 3.2.

From the definition we have

$$\begin{aligned}\hat{\kappa}_3 &= \kappa_3(n\sigma_n^2)^{\frac{3}{2}}(n\hat{\sigma}_n^2)^{-\frac{3}{2}} + \frac{\hat{\mu}_n - \mu_n}{(n\sigma_n^2)^{\frac{3}{2}}}(n\sigma_n^2)^{\frac{3}{2}}(n\hat{\sigma}_n^2)^{-\frac{3}{2}}\\ &= p_1 + p_2 \quad (say).\end{aligned}$$

Using the equations (13), (15) and (16), we can show that

$$\begin{aligned}p_1 &= \kappa_3 - \frac{e_1+3e_2}{\xi_1^3}\{\frac{3}{4n\xi_1^2}\sum_{i=1}^n f_1(X_i) - \frac{3}{4n(n-1)\xi_1^2}\sum_{C_{n,2}}[f_2(X_i,X_j)\\ &\quad -\frac{5}{4\xi_1^2}f_1(X_i)f_1(X_j)]\} + \frac{e_1+3e_2}{n\xi_1^3}\{\frac{15E[f_1^2(X_1)]}{32\xi_1^4} - \frac{3\xi_2^2}{4\xi_1^2}\} + o_p(n^{-1}).\end{aligned}$$

Let us define $b(x,y) = f_2(x,y) - 5f_1(x)f_1(y)/(4\xi_1^2)$. Since $E[b(X_1,X_2)|X_1] = 0$ *a.s.*, we can obtain the same order upper bound of the inequality (13) for $\sum b(X_i,X_j)$. From the equation (12), we can show that under the moment condition

$$E|\sum_{C_{n,2}}\lambda_2(X_i,X_j)\sum_{C_{n,2}}b(X_i,X_j)|^{1+\varepsilon} = O(n^{2+2\varepsilon}).$$

Therefore it follows from the equations (10) and (11) that

$$O(n^{-4})\sum_{C_{n,2}}\lambda_2(X_i,X_j)\sum_{C_{n,2}}b(X_i,X_j) = o_p(n^{-1}).$$

Further we have

$$\begin{aligned}&O(n^{-3})\sum_{i=1}^n\lambda_1(X_i)\sum_{C_{n,2}}b(X_i,X_j)\\ =\ & O(n^{-3})\sum_{C_{n,2}}\{\lambda_1(X_i)+\lambda_1(X_j)\}b(X_i,X_j)\\ +\ & O(n^{-3})\sum_{C_{n,3}}\{\lambda_1(X_i)b(X_j,X_k)+\lambda_1(X_j)b(X_i,X_k)+\lambda_1(X_k)b(X_i,X_j)\}.\end{aligned}$$

Since $E[\{\lambda_1(X_i)+\lambda_1(X_j)\}b(X_i,X_j)] = 0$, from the equation (20) we have

$$O(n^{-3})\sum_{C_{n,2}}\{\lambda_1(X_i)+\lambda_1(X_j)\}b(X_i,X_j) = o_p(n^{-1}).$$

Since

$$E[\lambda_1(X_i)b(X_j,X_k)+\lambda_1(X_j)b(X_i,X_k)+\lambda_1(X_k)b(X_i,X_j)|X_i,X_j] = 0 \quad a.s.,$$

it follows from the equation (21) that

$$\begin{aligned}&O(n^{-3})\sum_{C_{n,3}}\{\lambda_1(X_i)b(X_j,X_k)+\lambda_1(X_j)b(X_i,X_k)+\lambda_1(X_k)b(X_i,X_j)\}\\ =\ & o_p(n^{-1}).\end{aligned}$$

Similarly we can show that

$$O(n^{-3})\sum_{i=1}^{n} f_1(X_i)\sum_{C_{n,2}} \lambda_2(X_i, X_j) = o_p(n^{-1}).$$

Since

$$\begin{aligned}
&-\frac{3}{2n^2\xi_1^2}\sum_{i=1}^{n} f_1(X_i)\sum_{i=1}^{n}\lambda_1(X_i) \\
= \quad & -\frac{3}{2n^2\xi_1^2}\sum_{i=1}^{n} f_1(X_i)\lambda_1(X_i) - \frac{3}{2n^2\xi_1^2}\sum_{C_{n,2}}\{f_1(X_i)\lambda_1(X_j) + f_1(X_j)\lambda_1(X_i)\}
\end{aligned}$$

and under the moment condition

$$E|O(n^{-2})\sum_{i=1}^{n}\{f_1(X_i)\lambda_1(X_i) - E[f_1(X_1)\lambda_1(X_1)]\}|^{2+\epsilon} = O(n^{-3-\frac{3\epsilon}{2}}),$$

from the equations (10) and (11) we have

$$\begin{aligned}
&-\frac{3}{2n^2\xi_1^2}\sum_{i=1}^{n} f_1(X_i)\sum_{i=1}^{n}\lambda_1(X_i) \\
= \quad & -\frac{3}{2n(n-1)\xi_1^2}\sum_{C_{n,2}}\{f_1(X_i)\lambda_1(X_j) + f_1(X_j)\lambda_1(X_i)\} \\
& -\frac{3}{2n\xi_1^2}E[f_1(X_1)\lambda_1(X_1)] + o_p(n^{-1}).
\end{aligned}$$

Using the equation (20), we can ignore the rest terms which multiply by $const.n^{-1}$. Finally since $(n\sigma_n^2)^{-3/2} = 1/(8\xi_1^3) + O(n^{-1})$, we have

$$\begin{aligned}
p_2 = &\frac{\delta}{n\xi_1^3} + \frac{1}{4n\xi_1^3}\sum_{i=1}^{n}\lambda_1(X_i) \\
&+\frac{1}{4n(n-1)\xi_1^3}\sum_{C_{n,2}}\{\lambda_2(X_i, X_j) - \frac{3}{4\xi_1^2}[f_1(X_i)\lambda_1(X_j) + f_1(X_j)\lambda_1(X_i)]\} \\
&-\frac{3}{16n\xi_1^5}E[f_1(X_1)\lambda_1(X_1)] + o_p(n^{-1}).
\end{aligned}$$

This completes the proof of theorem.

References

Arvesen, J.N. (1969). *Jackknifing U-statistics*, Ann. Math. Statist., 40, 2076-2100.

Beran, R. (1984). *Jackknife approximations to bootstrap estimates*, Ann. Statist., 12, 101-118.

Bickel, P.J., Goetze, F. and van Zwet W.R. (1986). *The Edgeworth expansion for U-statistics of degree two* , Ann. Statist., 14, 1463-1484.

Callaert, H., Janssen, P. and Veraverbeke, N. (1980). *An Edgeworth expansion for U-statistics*, Ann. Statist., 8, 299-312.

Dharmadhikari, S.W., Fabian, V. and Jogdeo, K.(1968). *Bounds on the moments of martingales,* Ann. Math. Statist., 39, 1719-1723.

Efron, B. and Stein, C. (1981). *The jackknife estimate of variance*, Ann. Statist., 9, 586-596.

Hall, P. (1992). *The Bootstrap and Edgeworth Expansion*, Springer, New York.

Hinkley, D.V. (1978). *Improving the jackknife with special reference to correlation estimation*, Biometrika, 65, 13-21.

Hinkley, D.V. and Wei, B.C. (1984). *Improvements of jackknife confidence limit methods*, Biometrika, 71, 331-339.

Hoeffding, W. (1948). *A class of statistics with asymptotically normal distribution*, Ann. Math. Statist., 19, 293-325.

Lee A.J. (1990). *U-statistics: Theory and Practice*, Marcel Dekker, New York.

Maesono, Y. (1994). *Edgeworth expansions of a studentized U-statistic and a jackknife estimator of variance* , The Australian National University Statistics Research Report No.SRR 038-94.

Maesono, Y. (1995a). *On the normal approximations of studentized U-statistic*, J. Japan Statist. Soc., 25, 19-33.

Maesono, Y. (1995b). *On biases of jackknife estimators of third central moments*, The Australian National University Statistics Research Report No.SRR 028-95.

Maesono, Y. (1995c). *Asymptotic representations and expansions of jackknife estimators of third moments*, The Australian National University Statistics Research Report No.SRR 030-95

Petrov, V.V. (1975). *Sums of Independent Random Variables*, Springer Berlin.

Schemper, M. (1987). *Nonparametric estimation of variance, skewness and kurtosis of the distribution of a statistic by jackknife and bootstrap techniques*, Statist. Neerlandica, 41, 59-64.

Tu, D. and Gross, A.J. (1994). *Bias reduction for jackknife skewness estimators*, Commun. Statist.-Theory Meth., 23, 2323-2341.

Tu, D. and Zhang, L. (1992). *On the estimation of skewness of a statistic by using the jackknife and bootstrap*, Statist. Papers, 33, 39-56.

van Zwet, W.R. (1984). *A Berry-Esséen bound for symmetric statistics*, Z. Wharsch. und Verw. Gebiete, 66, 425-440.

Received June 28, 1996
Revised March 31, 1997