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Iwamoto, Seiichi
Department of Economic Engineering, Faculty of Economics, Kyushu University

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ON EXPECTED VALUES OF MARKOV STATISTICS

By

Seiichi IWAMOTO*

Abstract

In this paper we give both forward and backward iterative algorithms for computing expected value of some associative statistics associated with a stationary Markov chain on finite state space. Both algorithms are based upon an invariant imbedding technique in dynamic programming.

1. Introduction

In this paper we consider how to compute the expected value of some associative statistics from a stationary Markov chain on a finite state space. We are concerned with two kinds of related associative statistics. One is simple statistics. The other is compound one. The simple statistics include sample sum, sample mean, sample maximum, sample minimum, and others from iid populations. As three compound ones we consider range statistics, ratio statistics, and variance statistics from the Markov chain (Sniedovich (1983, 1987, 1989, 1992)).

Introducing a new real-parameter at the head of statistics we imbed the expectation problem into a family of parametric problems, one of which reduces to the original problem. We show both forward and backward iterative algorithms for computing the expected value. Both algorithms are based upon an invariant imbedding technique (Bellman and Denman(1971), Iwamoto(1996), Iwamoto and Fujita(1995), Lee(1968), Sniedovich (1983, 1989)) in stochastic dynamic programming (Bellman(1957), Blaclwell (1965), Denardo(1968, 1972), Furukawa and Iwamoto(1973a, 1973b), Hinderer(1970), Howard(1960), Iwamoto(1974, 1975a, 1975b, 1977, 1993, 1994), Kreps(1977a, 1977b), Lipfert(1985), Mitten(1964), Nemhauser(1966), Porteus(1975, 1982), Puterman(1994), Sniedovich(1986)).

In Section 2, we define associative statistics from Markov chain. Deriving the conditional probability function, we give the direct computation method for associative statistics. Not only for associative statistics but also for functions of it we give two iterative computation methods, based upon forward and backward recursive equations in dynamic programming.

* Department of Economic Engineering, Faculty of Economica, Kyushu University 27, Fukuoka 812-81, Japan
e-mail: iwamoto@en.kyushu-u.ac.jp

In Section 3, as two examples of associative statistics, we consider extremum statistics from Markov chain: maximum statistics and minimum statistics. We specify forward and backward iterative algorithms. In Section 4, as three examples of compound statistics, we consider range statistics, ratio statistics, and variance statistics from Markov chain. Imbedding the original problem into an appropriately large family of parametric ones, we derive forward and backward iterative algorithms. In Section 5, we illustrate typical examples both of associative binary relation and of nonassociative one. In Section 6, we give a numeical example, which assures the same expected value through three methods: direct computation, forward iterative computation, and backward computation. In the last section we conclude with some remarks on associative/nonassociative statistics and stationary/nonstationary and forward/backward recursive equations.

2. Associative Statistics and Related Statistics

Let $S = \{1, 2, \ldots, N\}$ be a finite state space, where $N \geq 1$ is an integer. Throughout the paper, let X_n, $n \geq 0$ be a stationary Markov chain on S having the one-step transition function $p(j|i)$:

$$p(j|i) = P(X_{n+1} = j \,|X_n = i\,) \qquad i \in S, \quad j \in S \quad n \geq 0.$$

It is such that

$$p(j|i) \geq 0 \qquad i \in S, \quad j \in S,$$

and

$$\sum_{j \in S} p(j|i) = 1 \qquad i \in S.$$

It now follows from the stationary Markov property that

$$\begin{aligned} & P(X_{m+n+1} = j \,, X_{m+n+2} = k \,, \cdots, X_{m+n+l-1} = s \,, X_{m+n+l} = t \\ & \qquad\qquad |X_{m+n} = i \,, X_{m+n-1} = i_{n-1} \,, \cdots, X_{m+1} = i_1 \,, X_m = i_0) \\ = \;& P(X_{m+n+1} = j \,, X_{m+n+2} = k \,, \cdots, X_{m+n+l-1} = s \,, X_{m+n+l} = t \,|X_{m+n} = i) \\ = \;& p(t|s) \cdots p(k|j) p(j|i). \end{aligned}$$

We also remark that

$$\begin{aligned} & P(X_1 = j \,, X_2 = k \,, \cdots, X_{n-1} = s \,, X_n = t | X_1 = j \,, X_0 = i) \\ = \;& P(X_2 = k \,, X_3 = l \,, \cdots, X_{n-1} = s \,, X_n = t | X_1 = j) \\ = \;& p(t|s) \cdots p(l|k) p(k|j). \end{aligned}$$

Let $R \subset R^1$ be any interval. Let $\circ : R \times R \to R$ be an *associative* binary relation on R :

$$(x \circ y) \circ z = x \circ (y \circ z). \tag{1}$$

Any $\tilde{x}$ satisfying

$$\tilde{x} \circ y = y \qquad \forall y \in R$$

is called a *left-identity* element for $\circ$. The common value (1) is denoted by $x \circ y \circ z$. We also use the notation $x_1 \circ x_2 \circ \cdots \circ x_n$ in the following.

2.1. Associative Statistics

Let $r : S \to R$ be a function. We consider how to compute the conditional expected value of the related *associative* statistics $R_0 \circ R_1 \circ \cdots \circ R_n$:

$$u_n(i) = E[\, R_0 \circ R_1 \circ \cdots \circ R_n \,|\, X_0 = i\,] \qquad i \in S \tag{2}$$

where

$$R_k = r(X_k).$$

First, we consider the direct computation as follows. The conditional probability distribution $p_n(x_0, x_1, x_2, \cdots, x_{n-1}, x_n | x_0)$ of $(X_0, X_1, X_2, \cdots, X_{n-1}, X_n)$ given $X_0 = i$ becomes

$$\begin{aligned}
& p_n(j, k, l, \cdots, s, t|i) \\
= \;& P(X_0 = j\,, X_1 = k\,, X_2 = l, \cdots, X_{n-1} = s,\; X_n = t | X_0 = i) \\
= \;& p(t|s) \cdots p(l|k) p(k|j) \delta_{ij}
\end{aligned}$$

where δ_{ij} is the Dirac's notation :

$$\delta_{ij} = \begin{cases} 1 & \text{for } \; i = j \\ 0 & \text{for } \; i \neq j. \end{cases}$$

Then we have

$$\begin{aligned}
u_n(i) &= \sum_{j,k,\ldots,s,t} [r(j) \circ r(k) \circ r(l) \circ \cdots \circ r(s) \circ r(t)] p_n(j, k, l, \cdots, s, t|i) \\
&= \sum_{j,k,\ldots,s,t} [r(j) \circ r(k) \circ r(l) \circ \cdots \circ r(s) \circ r(t)] p(t|s) \cdots p(l|k) p(k|j) \delta_{ij} \\
&= \sum_{k,\ldots,s,t} [r(i) \circ r(k) \circ r(l) \circ \cdots \circ r(s) \circ r(t)] p(t|s) \cdots p(l|k) p(k|i).
\end{aligned}$$

where each of $j, k, ..., s, t (k, ..., s, t)$ ranges over $S = \{1, 2, ..., N\}$.

Second, we imbed the problem (2) into the class of parametrized problems:

$$u_n(i; \lambda) = E[\, \lambda \circ R_0 \circ R_1 \circ \cdots \circ R_n \,|\, X_0 = i\,] \qquad \lambda \in R. \tag{3}$$

Then the stationarity implies that

$$u_n(i; \lambda) = E[\, \lambda \circ R_k \circ R_{k+1} \circ \cdots \circ R_{k+n} \,|\, X_k = i\,] \qquad k \geq 0.$$

We have for any left-identity value $\tilde{\lambda}$

$$u_n(i; \tilde{\lambda}) = u_n(i).$$

The Markov property together with the associativity in $\circ$ yields

$$\begin{aligned}
& E[\, \lambda \circ R_0 \circ R_1 \circ \cdots \circ R_n \,|\, X_0 = i\,] \\
= \;& E[\, E[\, (\lambda \circ R_0) \circ R_1 \circ \cdots \circ R_n \,|\, X_1\,, X_0 = i] \,|\, X_0 = i\,] \\
= \;& E[\, E[\, (\lambda \circ r(i)) \circ R_1 \circ \cdots \circ R_n \,|\, X_1] \,|\, X_0 = i\,].
\end{aligned}$$

Thus, we get the forward recurrence equation:

$$u_0(i;\lambda) = \lambda \circ r(i) \tag{4}$$

$$u_n(i;\lambda) = \sum_{j=1}^{N} u_{n-1}(j;\lambda \circ r(i))p(j|i) \qquad n \geq 1. \tag{5}$$

Moreover, let $\pi_0 = \{\pi_0(i)\}_{i\in S}$ be an initial distribution:

$$P(X_0 = i) = \pi_0(i) \geq 0 \qquad i \in S,$$

and

$$\sum_{i\in S} \pi_0(i) = 1.$$

Then the (unconditional) expected value of the associative statistics $R_0 \circ R_1 \circ \cdots \circ R_n$:

$$u_n = E[\,R_0 \circ R_1 \circ \cdots \circ R_n\,] \qquad n \geq 1$$

is calculated as follows:

$$u_n = \sum_{i=1}^{N} \pi_0(i)u_n(i).$$

On the other hand, we imbed the problem (2) into the class of backward problems:

$$u^{n-k+1}(i;\lambda) = E[\,\lambda \circ R_k \circ R_{k+1} \circ \cdots \circ R_n \,|X_k = i\,] \qquad 0 \leq k \leq n, \qquad \lambda \in R^1. \tag{6}$$

We have for any left-identity value $\tilde{\lambda}$

$$u^{n+1}(i;\tilde{\lambda}) = u_n(i)$$

and the backward recurrence equation:

$$u^1(i;\lambda) = \lambda \circ r(i) \tag{7}$$

$$u^{n-k+1}(i;\lambda) = \sum_{j=1}^{N} u^{n-k}(j;\lambda \circ r(i))p(j|i) \qquad 0 \leq k \leq n-1. \tag{8}$$

We remark that the stationarity implies the relation between forward and backward problems (3), (6):

$$u^{n-k+1}(i;\lambda) = u_{n-k}(i;\lambda).$$

2.2. Function of Associative Statistics

Let $h : R \to R^1$ be a function. Then we consider the conditional expected value of the **function** of $R_0 \circ R_1 \circ \cdots \circ R_n$:

$$v_n(i) = E[\,h(R_0 \circ R_1 \circ \cdots \circ R_n)\,|\,X_0 = i\,] \qquad i \in S \tag{9}$$

First, we imbed this problem into the class of parametrized problems:

$$v_n(i;\lambda) = E[\,h(\lambda \circ R_0 \circ R_1 \circ \cdots \circ R_n)\,|\,X_0 = i\,] \qquad \lambda \in R.$$

We get the forward recurrence equation:

$$\begin{aligned} v_0(i;\lambda) &= h(\lambda \circ r(i)) \\ v_n(i;\lambda) &= \sum_{j=1}^{N} v_{n-1}(j;\lambda \circ r(i))p(j|i) \qquad n \geq 1. \end{aligned}$$

On the other hand, we imbed the problem (9) into the class of backward problems:

$$v^{n-k+1}(i;\lambda) = E[\,h(\lambda \circ R_k \circ R_{k+1} \circ \cdots \circ R_n)\,|X_k = i\,] \qquad 0 \leq k \leq n, \qquad \lambda \in R.$$

We have for any left-identity $\tilde{\lambda}$

$$v^{n+1}(i;\tilde{\lambda}) = v_n(i)$$

and the backward recurrence equation:

$$\begin{aligned} v^1(i;\lambda) &= h(\lambda \circ r(i)) \\ v^{n-k+1}(i;\lambda) &= \sum_{j=1}^{N} v^{n-k}(j;\lambda \circ r(i))p(j|i) \qquad 0 \leq k \leq n-1. \end{aligned}$$

The stationarity also yields

$$v^{n-k+1}(i;\lambda) = v_{n-k}(i;\lambda).$$

3. Single Statistics

Let $f : S \to R^1$ and $g : S \to R^1$ be two functions. Then we define

$$Y_n = f(X_0) \vee f(X_1) \vee \cdots \vee f(X_n) \qquad n \geq 0$$

$$Z_n = g(X_0) \wedge g(X_1) \wedge \cdots \wedge g(X_n) \qquad n \geq 0$$

$$Y_{k,n} = f(X_k) \vee f(X_{k+1}) \vee \cdots \vee f(X_n) \qquad 0 \leq k \leq n$$

$$Z_{k,n} = g(X_k) \wedge g(X_{k+1}) \wedge \cdots \wedge g(X_n) \qquad 0 \leq k \leq n$$

and

$$m = \min\{f(x)|x \in S\} \qquad M = \max\{f(x)|x \in S\}.$$

3.1. Maximum Statistics

First, we consider the conditional expected value :

$$u_n(i) = E[Y_n \,|X_0 = i] \qquad n \geq 0. \tag{10}$$

We imbed this problem into the class of parametrized problems:

$$u_n(i;\lambda) = E[\lambda \vee Y_n \,|X_0 = i] \qquad m \leq \lambda \leq M.$$

Then for a sufficiently small value $\tilde{\lambda}$ of λ we have

$$u_n(i;\tilde{\lambda}) = u_n(i).$$

In particular,

$$u_n(i;m) = u_n(i).$$

We get the forward recurrence equation:

$$\begin{aligned} u_0(i;\lambda) &= \lambda \vee f(i) \\ u_n(i;\lambda) &= \sum_{j=1}^{N} u_{n-1}(j;\lambda \vee f(i))p(j|i) \qquad n \geq 1. \end{aligned}$$

On the other hand, we imbed the problem (10) into the class of backward problems:

$$u^{n-k+1}(i;\lambda) = E[\lambda \vee Y_{k,n} \,|X_k = i] \qquad 0 \leq k \leq n, \qquad m \leq \lambda \leq M.$$

We have

$$u^{n+1}(i;m) = u_n(i)$$

and the backward recurrence equation:

$$\begin{aligned} u^1(i;\lambda) &= \lambda \vee f(i) \\ u^{n-k+1}(i;\lambda) &= \sum_{j=1}^{N} u^{n-k}(j;\lambda \vee f(i))p(j|i) \qquad 0 \leq k \leq n-1. \end{aligned}$$

3.2. Minimum Statistics

Second, we consider the conditional expected value :

$$v_n(i) = E[Z_n \,|X_0 = i] \qquad n \geq 0. \tag{11}$$

(See also Bellman and Zadeh(1970), Esogbue and Bellman(1984), Iwamoto(1996), Iwamoto and Fujita(1965), Kacprzyk(1978) for the corresponding optimization problem). We imbed this problem into the class of parametrized problems:

$$v_n(i;\mu) = E[\mu \wedge Z_n \,|X_0 = i] \qquad m \leq \mu \leq M.$$

Then for a sufficiently large value $\tilde{\mu}$ of μ we have

$$v_n(i;\tilde{\mu}) = v_n(i).$$

In particular,

$$v_n(i;M) = v_n(i).$$

We get the recurrence equation:

$$\begin{aligned} v_0(i;\mu) &= \mu \wedge f(i) \\ v_n(i;\mu) &= \sum_{j=1}^{N} v_{n-1}(j;\mu \wedge f(i))p(j|i) \qquad n \geq 1. \end{aligned}$$

On the other hand, we imbed the problem (11) into the class of backward problems:

$$v^{n-k+1}(i;\mu) = E[\mu \wedge Z_{k,n}\,|X_k = i] \qquad 0 \leq k \leq n, \qquad m \leq \mu \leq M.$$

We have

$$v^{n+1}(i;M) = v_n(i)$$

and the backward recurrence equation:

$$\begin{aligned} v^1(i;\mu) &= \mu \wedge f(i) \\ v^{n-k+1}(i;\mu) &= \sum_{j=1}^{N} v^{n-k}(j;\mu \wedge f(i))p(j|i) \qquad 0 \leq k \leq n-1. \end{aligned}$$

4. Compound Statistics

In this section we consider three kinds of compound statistics; range, ratio and variance.

4.1. Range Statistics

Let $h : R^1 \times R^1 \to R^1$ be a function. First, we consider the conditional expected value :

$$u_n(i) = E[h(Y_n, Z_n)\,|X_0 = i] \qquad n \geq 0. \tag{12}$$

If

$$h(x, y) = x - y,$$

then $u_n(i)$ represents a range for the statistics $X_0, X_1, \cdots, X_n$:

$$u_n(i) = E[Y_n - Z_n\,|X_0 = i] \qquad n \geq 0.$$

We have

$$E[Y_n - Z_n\,|X_0 = i] = E[Y_n\,|X_0 = i] - E[Z_n\,|X_0 = i].$$

This expectation problem is combined with the former two problems: maximum statistics and minimum statistics.

However, in general

$$E[h(Y_n, Z_n)\,|X_0 = i] \neq h(E[Y_n\,|X_0 = i],\ E[Z_n\,|X_0 = i]).$$

We imbed this problem into the class of two-parametrized problems:

$$u_n(i;\lambda,\mu) = E[h(\lambda \vee Y_n,\ \mu \wedge Z_n)\,|X_0 = i] \qquad m \leq \lambda, \mu \leq M.$$

Then for a sufficiently small value $\tilde{\lambda}$ of λ and a sufficiently large value $\tilde{\mu}$ of μ we have

$$u_n(i;\tilde{\lambda},\tilde{\mu}) = u_n(i).$$

In particular,

$$u_n(i;m,M) = u_n(i).$$

We get the recurrence equation:

$$\begin{aligned} u_0(i;\lambda,\mu) &= h(\lambda \vee f(i),\ \mu \wedge f(i)) \\ u_n(i;\lambda,\mu) &= \sum_{j=1}^{N} u_{n-1}(j;\lambda \vee f(i), \mu \wedge f(i))p(j|i) \qquad n \geq 1. \end{aligned}$$

On the other hand, we imbed the problem (12) into the class of backward problems:

$$u^{n-k+1}(i;\lambda,\mu) = E[h(\lambda \vee Y_{k,n},\ \mu \wedge Z_{k,n})\,|X_k = i] \qquad 0 \leq k \leq n, \quad m \leq \lambda, \mu \leq M.$$

We have

$$u^{n+1}(i;m,M) = u_n(i)$$

and the backward recurrence equation:

$$\begin{aligned} u^1(i;\lambda,\mu) &= h(\lambda \vee f(i),\ \mu \wedge f(i)) \\ u^{n-k+1}(i;\lambda,\mu) &= \sum_{j=1}^{N} u^{n-k}(j;\lambda \vee f(i), \mu \wedge f(i))p(j|i) \qquad 0 \leq k \leq n-1. \end{aligned}$$

4.2. Ratio Statistics

Further, we define

$$S_n = f(X_0) + f(X_1) + \cdots + f(X_n) \qquad n \geq 0$$

and

$$T_n = g(X_0) + g(X_1) + \cdots + g(X_n) \qquad n \geq 0.$$

Second, we consider for any fixed $n \geq 0$ the conditional expected value :

$$v_n(i) = E[\frac{S_n}{T_n}\,|X_0 = i]. \tag{13}$$

(See Sniedovich(1989, 1992) for optimization problem.) We note that

$$E[\frac{S_n}{T_n}\,|X_0 = i] \neq \frac{E[S_n\,|X_0 = i]}{E[T_n\,|X_0 = i]}.$$

Now, we imbed this problem into the class of two-parametrized problems:

$$\begin{aligned} v_k(i;\lambda,\mu) &= E[\frac{\lambda + S_k}{\mu + T_k}\,|X_0 = i] \qquad 0 \le k \le n \\ &\qquad\qquad \lambda \in [\underline{\lambda}_k, \overline{\lambda}_k], \quad \mu \in [\underline{\mu}_k, \overline{\mu}_k] \end{aligned}$$

where

$$\underline{\lambda}_k = (n-k)\underline{f} \qquad \overline{\lambda}_k = (n-k)\overline{f} \qquad 0 \le k \le n$$

$$\underline{\mu}_k = (n-k)\underline{g} \qquad \overline{\mu}_k = (n-k)\overline{g} \qquad 0 \le k \le n$$

$$\underline{f} = \min_{s \in S} f(s), \qquad \overline{f} = \max_{s \in S} f(s)$$

$$\underline{g} = \min_{s \in S} g(s), \qquad \overline{g} = \max_{s \in S} g(s).$$

Then we have

$$v_n(i;0,0) = v_n(i)$$

and the forward recurrence equation:

$$\begin{aligned} v_0(i;\lambda,\mu) &= \frac{\lambda + f(i)}{\mu + g(i)} \qquad \lambda \in [\underline{\lambda}_0, \overline{\lambda}_0], \quad \mu \in [\underline{\mu}_0, \overline{\mu}_0] \\ v_k(i;\lambda,\mu) &= \sum_{j=1}^{N} v_{k-1}(j;\lambda + f(i), \mu + g(i))p(j|i) \qquad 1 \le k \le n \\ &\qquad\qquad \lambda \in [\underline{\lambda}_k, \overline{\lambda}_k], \quad \mu \in [\underline{\mu}_k, \overline{\mu}_k]. \end{aligned}$$

On the other hand, we imbed the problem (13) into the class of backward problems:

$$\begin{aligned} v^{n-k+1}(i;\lambda,\mu) &= E[\frac{\lambda + S_{k,n}}{\mu + T_{k,n}}\,|X_k = i] \qquad 0 \le k \le n \\ &\qquad\qquad \lambda \in [k\underline{f}, k\overline{f}], \quad \mu \in [k\underline{f}, k\overline{f}] \end{aligned}$$

where

$$S_{k,n} = f(X_k) + f(X_{k+1}) + \cdots + f(X_n) \qquad 0 \le k \le n$$

and

$$T_{k,n} = g(X_k) + g(X_{k+1}) + \cdots + g(X_n) \qquad 0 \le k \le n.$$

We have

$$v^{n+1}(i;0,0) = v_n(i)$$

and the backward recurrence equation:

$$\begin{aligned} v^1(i;\lambda,\mu) &= \frac{\lambda+f(i)}{\mu+g(i)} \qquad \lambda\in[n\underline{f},n\overline{f}],\quad \mu\in[n\underline{g},n\overline{g}] \\ v^{n-k+1}(i;\lambda,\mu) &= \sum_{j=1}^{N} v^{n-k}(j;\lambda+f(i),\mu+g(i))p(j|i) \qquad 0\le k\le n-1 \\ &\qquad\qquad \lambda\in[k\underline{f},k\overline{f}],\quad \mu\in[k\underline{g},k\overline{g}]. \end{aligned}$$

4.3. Variance Statistics

Finally, we consider for any fixed $n\ge 1$ the conditional expected value :

$$E\{\,\frac{1}{n}[\sum_{k=0}^{n-1}(f_k-\mu)^2]\,|X_0=i\,\} \tag{14}$$

where

$$\mu=\frac{1}{n}\sum_{k=0}^{n-1}f_k,\qquad f_k=f(X_k).$$

(See Sniedovich(1983, 1992) for deterministic optimization problem.) Note that

$$\frac{1}{n}[\sum_{k=0}^{n-1}(f_k-\mu)^2\,]=\frac{1}{n}\sum_{k=0}^{n-1}f_k^2\,-\,(\frac{1}{n}\sum_{k=0}^{n-1}f_k\,)^2.$$

Then we calculate the expected value multiplied by n^2 instead of (14):

$$n^2\,E\{\,\frac{1}{n}[\sum_{k=0}^{n-1}(f_k-\mu)^2\,]\,|X_0=i\,\}.$$

That is, we set

$$v_n(i)=E[\,n\sum_{k=0}^{n-1}f_k^2\,-\,(\sum_{k=0}^{n-1}f_k\,)^2\,|X_0=i\,]. \tag{15}$$

Now, we imbed this problem into the class of one-parametrized problems:

$$\begin{aligned} v_k(i;\lambda) &= E[\,n\sum_{l=0}^{k-1}f_l^2\,-\,(\,\lambda+\sum_{l=0}^{k-1}f_l\,)^2\,|X_0=i\,] \\ &\qquad 1\le k\le n \qquad \lambda\in[\underline{\lambda}_k,\overline{\lambda}_k]. \end{aligned}$$

We have

$$v_n(i;0)=v_n(i)$$

and the forward recurrence equation:

$$\begin{aligned} v_1(i;\lambda) &= nf^2(i)-(\lambda+f(i))^2 \qquad \lambda\in[\underline{\lambda}_1,\overline{\lambda}_1] \\ v_k(i;\lambda) &= nf^2(i)+\sum_{j=1}^{N}v_{k-1}(j;\lambda+f(i))p(j|i) \\ &\qquad 2\le k\le n \qquad \lambda\in[\underline{\lambda}_k,\overline{\lambda}_k]. \end{aligned}$$

Then the desired expected value of (14) is given by

$$\frac{1}{n^2}v_n(i;0) = \frac{1}{n^2}v_n(i).$$

On the other hand, we imbed the problem (15) into the class of backward problems:

$$v^{n-k}(i;\lambda) \;=\; E[n\sum_{l=k}^{n-1} f_l^2 - (\lambda + \sum_{l=k}^{n-1} f_l)^2 \,|X_k = i]$$
$$0 \le k \le n-1 \qquad \lambda \in [k\underline{f}, k\overline{f}].$$

We have

$$v^n(i;0) = v_n(i)$$

and the backward recurrence equation:

$$\begin{aligned} v^1(i;\lambda) &= nf^2(i) - (\lambda + f(i))^2 \qquad \lambda \in [(n-1)\underline{f}, (n-1)\overline{f}] \\ v^{n-k}(i;\lambda) &= nf^2(i) + \sum_{j=1}^{N} v^{n-k-1}(j;\lambda + f(i))p(j|i) \\ &\qquad 0 \le k \le n-2 \quad \lambda \in [k\underline{f}, k\overline{f}]. \end{aligned}$$

5. Associative/Nonassociative Binary Relations

We illustrates both classes of typical associative binary relations and of nonassocitive ones in Markov decision process and/or stochastic dynamic programming. Since the correspondence between each binary relation below and the resulting objective functions of optimization problems discussed in the references is straightforward, we omit specifying it.

First we have the following examples of associative binary relations $\circ$ with left-identity in parenthesis.

Example 1 $\quad \circ : R^1 \times R^1 \to R^1$

$$\begin{aligned} < additive > \quad x \circ y &= x + y \quad (\tilde{x} = 0) \\ < multiplicative > \quad x \circ y &= xy \quad (\tilde{x} = 1) \\ < terminal > \quad x \circ y &= y \quad (\tilde{x} = \text{any real number}). \\ < g - additive > \quad x \circ y &= g^{-1}(g(x) + g(y)) \quad (\tilde{x} = g^{-1}(0)) \\ < g - multiplicative > \quad x \circ y &= g^{-1}(g(x)g(y)) \quad (\tilde{x} = g^{-1}(1)) \end{aligned}$$

where $g : R^1 \to R^1$ is onto continuous and strictly increasing.

Example 2 $\quad \circ : [0,1] \times [0,1] \to [0,1]$

$$\begin{aligned} < maximum > \quad x \circ y &= x \vee y \quad (\tilde{x} = 0) \\ < minimum > \quad x \circ y &= x \wedge y \quad (\tilde{x} = 1) \end{aligned}$$

Example 3 $\quad \circ : R^1 \times R^1 - \{(x,y)|xy \neq -1\} \rightarrow R^1$

$$x \circ y = \frac{x+y}{1+xy} \quad (\tilde{x} = 1).$$

Example 4 $\quad \circ : R^1 \times R^1 - \{(x,y)|x+y \neq 0\} \rightarrow R^1$

$$x \circ y = \frac{1+xy}{x+y} \quad (\tilde{x} = 1).$$

Example 5 $\quad \circ : R^1_+ \times R^1_+ \rightarrow R^1_+$

$$< p - normed > \qquad x \circ y = (x^p + y^p)^{1/p} \quad (\tilde{x} = 0)$$

where

$$p > 0.$$

On the other hand, the following are examples of *nonassociative* binary relation $\circ$.

Example 6 $\quad \circ : R^1 \times R^1 \rightarrow R^1$

$$\begin{array}{rrcl} < \textit{multiplicatively additive} > & x \circ y & = & x + xy \\ < \textit{discounted} > & x \circ y & = & x + \beta y \end{array}$$

where

$$\beta \neq 1.$$

Example 7 $\quad \circ : (0,\infty) \times (0,\infty) \rightarrow (0,\infty)$

$$\begin{array}{rrcl} < \textit{backward exponential} > & x \circ y & = & x^y \\ < \textit{forward exponential} > & x \circ y & = & y^x \end{array}$$

6. Examples

In this section we specify three-stage statistics and calculate the expected values of them on two-state Markov chain.

6.1. Three-stage statistics

In this subsection we specify explicit forms for three-stage statistics. The n-stage statistics are straightforward.

Example 1

$$\begin{array}{rl} < \textit{additive} > & E[R_0 + R_1 + R_2 + R_3|X_0 = i] \\ < \textit{multiplicative} > & E[R_0 R_1 R_2 R_3|X_0 = i] \\ < \textit{terminal} > & E[R_3|X_0 = i] \\ < g - \textit{additive} > & E[g^{-1}(g(R_0) + g(R_1) + g(R_2) + g(R_3))|X_0 = i] \\ < g - \textit{multiplicative} > & E[g^{-1}(g(R_0)g(R_1)g(R_2)g(R_3))|X_0 = i] \end{array}$$

where $g : R^1 \to R^1$ is onto continuous and strictly increasing.

Example 2

$$< maximum > \qquad E[R_0 \vee R_1 \vee R_2 \vee R_3 | X_0 = i]$$
$$< minimum > \qquad E[R_0 \wedge R_1 \wedge R_2 \wedge R_3 | X_0 = i].$$

Example 3

$$E[\frac{R_0 + R_1 + R_2 + R_3 + R_0R_1R_2 + R_0R_1R_3 + R_0R_2R_3 + R_1R_2R_3}{1 + R_0R_1 + R_0R_2 + R_0R_3 + R_1R_2 + R_1R_3 + R_2R_3 + R_0R_1R_2R_3} | X_0 = i].$$

Example 4

$$E[\frac{1 + R_0R_1 + R_0R_2 + R_0R_3 + R_1R_2 + R_1R_3 + R_2R_3 + R_0R_1R_2R_3}{R_0 + R_1 + R_2 + R_3 + R_0R_1R_2 + R_0R_1R_3 + R_0R_2R_3 + R_1R_2R_3} | X_0 = i].$$

Example 5

$$< p - normed > \quad E[(R_0^p + R_1^p + R_2^p + R_3^p)^{1/p} | X_0 = i]$$

where

$$p > 0.$$

On the other hand, the following are examples of nonassociative statistics.

Example 6

$$< multiplicatively\ additive > \ E[R_0 + R_0R_1 + R_0R_1R_2 + R_0R_1R_2R_3 | X_0 = i]$$

$$< discounted > \ E[R_0 + \beta R_1 + \beta^2 R_2 + \beta^3 R_3 | X_0 = i]$$

where

$$\beta \neq 1.$$

Example 7 (See also Golmb(1975, 1980) for the corresponding deterministic optimization problems.)

$$< backward\ exponential >$$

$$E[((R_0^{R_1})^{R_2})^{R_3} | X_0 = i]$$

$$< forward\ exponential >$$

$$E[R_3^{R_2^{R_1^{R_0}}} | X_0 = i].$$

As compound statistics, we consider the expected value of variance and ratio statistics as follows :

Example 8

$$< variance > \qquad E[\frac{1}{4}\sum_{i=0}^{3}(R_i - \overline{R})^2 \,|X_0 = i] \qquad \overline{R} = \frac{1}{4}\sum_{i=0}^{3} R_i$$

$$< ratio > \qquad E[\frac{R_0 + R_1 + R_2 + R_3}{R_0^2 + R_1^2 + R_2^2 + R_3^2} \,|X_0 = i].$$

6.2. Two-state Markov Chain

Now we consider a simple Markov chain on state space $S = \{s_1, s_2\}$ with the following numerical data :

$$r(s_1) = 2, \qquad r(s_2) = -1$$

$x_t \setminus x_{t+1}$	s_1	s_2
s_1	1/4	3/4
s_2	1/2	1/2

6.3. Simple Statistics

First we consider the expected value of multilpicative, minimum, maximum and terminal statistics.

Then the direct computation yields Table 1 as follows :

history(state, trans. prob)	path	mult.	min.	max.	ter.
s_1 1/4 s_1 1/4 s_1 1/4 s_1	1/64	16	2	2	2
s_1 1/4 s_1 1/4 s_1 3/4 s_2	3/64	-8	-1	2	-1
s_1 1/4 s_1 3/4 s_2 1/2 s_1	6/64	-8	-1	2	2
s_1 1/4 s_1 3/4 s_2 1/2 s_2	6/64	4	-1	2	-1
s_1 3/4 s_2 1/2 s_1 1/4 s_1	6/64	-8	-1	2	2
s_1 3/4 s_2 1/2 s_1 3/4 s_2	18/64	4	-1	2	-1
s_1 3/4 s_2 1/2 s_2 1/2 s_1	12/64	4	-1	2	2
s_1 3/4 s_2 1/2 s_2 1/2 s_2	12/64	4	-1	2	-1
expected value from s_1		1/4	-61/64	2	11/64
s_2 1/2 s_1 1/4 s_1 1/4 s_1	2/64	-8	-1	2	2
s_2 1/2 s_1 1/4 s_1 3/4 s_2	6/64	4	-1	2	-1
s_2 1/2 s_1 3/4 s_2 1/2 s_1	12/64	4	-1	2	2
s_2 1/2 s_1 3/4 s_2 1/2 s_2	12/64	-2	-1	2	-1
s_2 1/2 s_2 1/2 s_1 1/4 s_1	4/64	4	-1	2	2
s_2 1/2 s_2 1/2 s_1 3/4 s_2	12/64	4	-1	2	-1
s_2 1/2 s_2 1/2 s_2 1/2 s_1	8/64	-2	-1	2	2
s_2 1/2 s_2 1/2 s_2 1/2 s_2	8/64	1	-1	-1	-1
expected value from s_2		1/4	-1	13/8	7/32

(where multi. = multiplication, min. = minimum, max. = maximum, ter. = terminal)

Table 1 : expected values of simple statistics from s_1, s_2

On the other hand, the forward recurrence equation for multiplicative problem

$$\begin{aligned} u_0(i;\lambda) &= \lambda r(i) \\ u_n(i;\lambda) &= \sum_{j=1}^{2} u_{n-1}(j;\lambda r(i))p(j|i) \qquad n=1,2,3 \end{aligned}$$

yields the solution as follows :

$$\begin{aligned} &u_0(s_1;\lambda) = 2\lambda \quad u_1(s_1;\lambda) = -\frac{1}{2}\lambda \quad u_2(s_1;\lambda) = -\lambda \quad u_3(s_1;\lambda) = \frac{1}{4}\lambda \\ &u_0(s_2;\lambda) = -\lambda \quad u_1(s_2;\lambda) = -\frac{1}{2}\lambda \quad u_2(s_2;\lambda) = \frac{1}{2}\lambda \quad u_3(s_2;\lambda) = \frac{1}{4}\lambda. \end{aligned}$$

Therefore we get the desired expected values

$$u_3(s_1) = u_3(s_1;1) = \frac{1}{4}, \quad u_3(s_2) = u_3(s_2;1) = \frac{1}{4}.$$

For the minimun problem, the forward recurrence equation

$$\begin{aligned} u_0(i;\lambda) &= \lambda \wedge r(i) \\ u_n(i;\lambda) &= \sum_{j=1}^{2} u_{n-1}(j;\lambda \wedge (i))p(j|i) \qquad n=1,2,3 \end{aligned}$$

yields

$$\begin{aligned} u_0(s_1;\lambda) &= \lambda \wedge 2 \\ u_0(s_2;\lambda) &= \lambda \wedge (-1) \end{aligned}$$

$$\begin{aligned} u_1(s_1;\lambda) &= \frac{1}{4}(\lambda \wedge 2) + \frac{3}{4}(\lambda \wedge (-1)) \\ u_1(s_2;\lambda) &= \lambda \wedge (-1) \end{aligned}$$

$$\begin{aligned} u_2(s_1;\lambda) &= \frac{1}{16}(\lambda \wedge 2) + \frac{15}{16}(\lambda \wedge (-1)) \\ u_2(s_2;\lambda) &= \lambda \wedge (-1) \end{aligned}$$

$$\begin{aligned} u_3(s_1;\lambda) &= \frac{1}{64}(\lambda \wedge 2) + \frac{63}{64}(\lambda \wedge (-1)) \\ u_3(s_2;\lambda) &= \lambda \wedge (-1). \end{aligned}$$

Thus we have the desired expected values

$$u_3(s_1) = u_3(s_1;2) = -\frac{61}{64}, \quad u_3(s_2) = u_3(s_2;2) = -1.$$

For the maximum problem, the forward recurrence equation

$$\begin{aligned} u_0(i;\lambda) &= \lambda \vee r(i) \\ u_n(i;\lambda) &= \sum_{j=1}^{2} u_{n-1}(j;\lambda \vee (i))p(j|i) \qquad n = 1,2,3 \end{aligned}$$

yields

$$\begin{aligned} u_0(s_1;\lambda) &= \lambda \vee 2 \\ u_0(s_2;\lambda) &= \lambda \vee (-1) \end{aligned}$$

$$\begin{aligned} u_1(s_1;\lambda) &= \lambda \vee 2 \\ u_1(s_2;\lambda) &= \frac{1}{2}(\lambda \vee 2) + \frac{1}{2}(\lambda \vee (-1)) \end{aligned}$$

$$\begin{aligned} u_2(s_1;\lambda) &= \lambda \vee 2 \\ u_2(s_2;\lambda) &= \frac{3}{4}(\lambda \vee 2) + \frac{1}{4}(\lambda \vee (-1)) \end{aligned}$$

$$\begin{aligned} u_3(s_1;\lambda) &= \lambda \vee 2 \\ u_3(s_2;\lambda) &= \frac{7}{8}(\lambda \vee 2) + \frac{1}{8}(\lambda \vee (-1)). \end{aligned}$$

Thus we have the desired expected values

$$u_3(s_1) = u_3(s_1;-1) = 2, \quad u_3(s_2) = u_3(s_2;-1) = \frac{13}{8}.$$

For the terminal problem, the forward recurrence equation

$$\begin{aligned} u_0(i;\lambda) &= r(i) \\ u_n(i;\lambda) &= \sum_{j=1}^{2} u_{n-1}(j;r(i))p(j|i) \qquad n = 1,2,3 \end{aligned}$$

yields

$$\begin{aligned} &u_0(s_1;\lambda) = 2 \quad u_1(s_1;\lambda) = -\frac{1}{4} \quad u_2(s_1;\lambda) = \frac{5}{16} \quad u_3(s_1;\lambda) = \frac{11}{64} \\ &u_0(s_2;\lambda) = -1 \quad u_1(s_2;\lambda) = \frac{1}{2} \quad u_2(s_2;\lambda) = \frac{1}{8} \quad u_3(s_2;\lambda) = \frac{7}{32}. \end{aligned}$$

Thus we have the desired expected values

$$u_3(s_1) = u_3(s_1;\lambda) = \frac{11}{64}, \quad u_3(s_2) = u_3(s_2;\lambda) = \frac{7}{32}.$$

6.4. Variance

Second, we illustrate how to iteratively calculate the expected value of variance statistics. We remak that

$$E[\frac{1}{4}\sum_{i=0}^{3}(R_i - \overline{R})^2 \,|X_0 = i] = E[\frac{1}{4}\sum_{i=0}^{3} R_i^2 \,|X_0 = i] - E[(\frac{1}{4}\sum_{i=0}^{3} R_i\,)^2 \,|X_0 = i]$$

where

$$\overline{R} = \frac{1}{4}\sum_{i=0}^{3} R_i.$$

Thus we apply the famous formula:

$$\text{variance} = \text{mean square} - \text{square mean}.$$

Then the direct computation yields Table 2 as follows :

history(state, trans. prob)	path	sum	sq. mean	mean sq.	variance
s_1 1/4 s_1 1/4 s_1 1/4 s_1	1/64	8	4	4	0
s_1 1/4 s_1 1/4 s_1 3/4 s_2	3/64	5	25/16	13/4	27/16
s_1 1/4 s_1 3/4 s_2 1/2 s_1	6/64	5	25/16	13/4	27/16
s_1 1/4 s_1 3/4 s_2 1/2 s_2	6/64	2	1/4	10/4	9/4
s_1 3/4 s_2 1/2 s_1 1/4 s_1	6/64	5	25/16	13/4	27/16
s_1 3/4 s_2 1/2 s_1 3/4 s_2	18/64	2	1/4	10/4	9/4
s_1 3/4 s_2 1/2 s_2 1/2 s_1	12/64	2	1/4	10/4	9/4
s_1 3/4 s_2 1/2 s_2 1/2 s_2	12/64	-1	1/16	7/4	27/16
expected value from s_1		143/64	595/1024	655/256	2025/1024
s_2 1/2 s_1 1/4 s_1 1/4 s_1	2/64	5	25/16	13/4	27/16
s_2 1/2 s_1 1/4 s_1 3/4 s_2	6/64	2	1/4	10/4	9/4
s_2 1/2 s_1 3/4 s_2 1/2 s_1	12/64	2	1/4	10/4	9/4
s_2 1/2 s_1 3/4 s_2 1/2 s_2	12/64	-1	1/16	7/4	27/16
s_2 1/2 s_2 1/2 s_1 1/4 s_1	4/64	2	1/4	10/4	9/4
s_2 1/2 s_2 1/2 s_1 3/4 s_2	12/64	-1	1/16	7/4	27/16
s_2 1/2 s_2 1/2 s_2 1/2 s_1	8/64	-1	1/16	7/ 4	27/16
s_2 1/2 s_2 1/2 s_2 1/2 s_2	8/64	-4	1	1	0
expected value from s_2		-5/32	149/512	251/128	855/512

Table 2 : expected value of variance from s_1, s_2

First, we consider the problem multiplied by 4^2 as follows :

$$\begin{aligned} & 4^2E[\frac{1}{4}\sum_{i=0}^{3}(R_i - \overline{R})^2 \,|X_0 = i] \\ = \quad & E[4R_0^2 + 4R_1^2 + 4R_2^2 + 4R_3^2 - (R_0 + R_1 + R_2 + R_3)^2 \,|X_0 = i] \end{aligned}$$

Imbedding this problem into the family of parametrized ones with λ :

$$\begin{aligned} & u_n(i;\lambda) \\ = \; & E[4R_0^2 + 4R_1^2 + \cdots + 4R_{n-1}^2 - (\lambda + R_0 + R_1 + \cdots + R_{n-1})^2 \,|X_0 = i] \\ & \qquad\qquad i = 1,2 \quad n = 0,1,2,3,4 \end{aligned}$$

we have the forward recurrence equation :

$$\begin{aligned} u_0(i;\lambda) &= -\lambda^2 \\ u_n(i;\lambda) &= 4r(i)^2 + \sum_{j=1}^{2} u_{n-1}(j;\lambda + r(i))p(j\,|i) \qquad n = 1,2,3. \end{aligned}$$

Second, solving this equation, we obtain the following expressions :

$$\begin{aligned} u_0(s_1;\lambda) &= -\lambda^2 \\ u_0(s_2;\lambda) &= -\lambda^2 \end{aligned}$$

$$\begin{aligned} u_1(s_1;\lambda) &= 4\cdot 2^2 - (\lambda+2)^2 \\ u_1(s_2;\lambda) &= 4(-1)^2 - (\lambda-1)^2 \end{aligned}$$

$$\begin{aligned} u_2(s_1;\lambda) &= 5\cdot 2^2 + 3(-1)^2 - \frac{1}{4}(\lambda+4)^2 - \frac{3}{4}(\lambda+1)^2 \\ u_2(s_2;\lambda) &= 2\cdot 2^2 + 6(-1)^2 - \frac{1}{2}(\lambda+1)^2 - \frac{1}{2}(\lambda-2)^2 \end{aligned}$$

$$\begin{aligned} u_3(s_1;\lambda) &= \frac{27}{4}\cdot 2^2 + \frac{21}{4}(-1)^2 - \frac{1}{4^2}(\lambda+6)^2 - \frac{9}{4^2}(\lambda+3)^2 - \frac{3}{2\cdot 4}\lambda^2 \\ u_3(s_2;\lambda) &= \frac{7}{2}\cdot 2^2 + \frac{17}{2}(-1)^2 - \frac{1}{2\cdot 4}(\lambda+3)^2 - \frac{5}{2\cdot 4}\lambda^2 - \frac{1}{2\cdot 2}(\lambda-3)^2 \end{aligned}$$

$$\begin{aligned} u_4(s_1;\lambda) &= \frac{133}{4^2}\cdot 2^2 + \frac{3\cdot 41}{4^2}(-1)^2 - \frac{1}{4^3}(\lambda+8)^2 - \frac{15}{4^3}(\lambda+5)^2 \\ &\qquad - \frac{3^2}{4^2}(\lambda+2)^2 - \frac{3}{4^2}(\lambda-1)^2 \\ u_4(s_2;\lambda) &= \frac{41}{4\cdot 2}\cdot 2^2 + \frac{87}{8}(-1)^2 - 2\cdot 4^2(\lambda+5)^2 - \frac{11}{2\cdot 4^2}(\lambda+2)^2 \\ &\qquad - \frac{1}{2}(\lambda-1)^2 - \frac{1}{8}(\lambda-4)^2. \end{aligned}$$

Finally, substituting $\lambda = 0$, we have

$$\begin{aligned} u_4(s_1;0) &= \frac{133}{4^2}\cdot 2^2 + \frac{3\cdot 41}{4^2}(-1)^2 - \frac{1}{4^3}\cdot 8^2 - \frac{15}{4^3}\cdot 5^2 - \frac{3^2}{4^2}\cdot 2^2 - \frac{3}{4^2}\cdot(-1)^2 \\ &= \frac{3^4\cdot 5^2}{2^6} \\ u_4(s_2;0) &= \frac{41}{4\cdot 2}\cdot 2^2 + \frac{87}{8}(-1)^2 - \frac{1}{2\cdot 4^2}\cdot 5^2 - \frac{11}{2\cdot 4^2}\cdot 2^2 - \frac{1}{2}\cdot(-1)^2 - \frac{1}{8}\cdot(-4)^2 \\ &= \frac{5\cdot 9\cdot 19}{2^5}. \end{aligned}$$

Therefore we get the desired expected values

$$
\begin{aligned}
E[\frac{1}{4}\sum_{i=0}^{3}(R_i-\overline{R})^2\,|X_0=s_1] &= \frac{1}{4^2}\cdot\frac{3^4\cdot 5^2}{2^6}=\frac{3^4\cdot 5^2}{2^{10}} \\
&= \frac{2025}{1024}=1.9775390625 \\
E[\frac{1}{4}\sum_{i=0}^{3}(R_i-\overline{R})^2\,|X_0=s_2] &= \frac{1}{4^2}\cdot\frac{5\cdot 9\cdot 19}{2^5}=\frac{5\cdot 9\cdot 19}{2^9} \\
&= \frac{855}{512}=1.669921875.
\end{aligned}
$$

6.5. Ratio

Finally we consider the expected value of the following ratio statistics :

$$E[\frac{R_0+R_1+R_2+R_3}{R_0^2+R_1^2+R_2^2+R_3^2}\,|X_0=i].$$

The direct computation yields Table 3 as follows :

history(state, trans. prob)	path	sum	sum sq.	ratio
s_1 1/4 s_1 1/4 s_1 1/4 s_1	1/64	8	16	8/16
s_1 1/4 s_1 1/4 s_1 3/4 s_2	3/64	5	13	5/13
s_1 1/4 s_1 3/4 s_2 1/2 s_1	6/64	5	13	5/13
s_1 1/4 s_1 3/4 s_2 1/2 s_2	6/64	2	10	2/10
s_1 3/4 s_2 1/2 s_1 1/4 s_1	6/64	5	13	5/13
s_1 3/4 s_2 1/2 s_1 3/4 s_2	18/64	2	10	2/10
s_1 3/4 s_2 1/2 s_2 1/2 s_1	12/64	2	10	2/10
s_1 3/4 s_2 1/2 s_2 1/2 s_2	12/64	-1	7	-1/7
expected value from s_1		143/64	655/256	9969/58240
s_2 1/2 s_1 1/4 s_1 1/4 s_1	2/64	5	13	5/13
s_2 1/2 s_1 1/4 s_1 3/4 s_2	6/64	2	10	2/10
s_2 1/2 s_1 3/4 s_2 1/2 s_1	12/64	2	10	2/10
s_2 1/2 s_1 3/4 s_2 1/2 s_2	12/64	-1	7	-1/7
s_2 1/2 s_2 1/2 s_1 1/4 s_1	4/64	2	10	2/10
s_2 1/2 s_2 1/2 s_1 3/4 s_2	12/64	-1	7	-1/7
s_2 1/2 s_2 1/2 s_2 1/2 s_1	8/64	-1	7	-1/7
s_2 1/2 s_2 1/2 s_2 1/2 s_2	8/64	-4	4	-4/4
expected value from s_2		-5/32	251/32	-421/25480

Table 3 : expected value of a ratio from s_1, s_2

We consider the following ratio problem :

$$u_3(i)=E[\frac{R_0+R_1+R_2+R_3}{R_0^2+R_1^2+R_2^2+R_3^2}\,|X_0=i]$$

We imbed this problem into the family of two-parametrized ones with λ, μ :

$$u_n(i;\lambda,\mu) = E[\frac{\lambda + R_0 + \cdots + R_n}{\mu + R_0^2 + \cdots + R_n^2} |X_0 = i] \qquad i = 1,2 \quad n = 0,1,2,3$$

Then we get the forward recurrence equation :

$$\begin{aligned} u_0(i;\lambda,\mu) &= \frac{\lambda + r(i)}{\mu + r^2(i)} \\ u_n(i;\lambda,\mu) &= \sum_{j=1}^{2} u_{n-1}(j;\lambda + r(i), \mu + r^2(i))p(j|i) \qquad n = 1,2,3. \end{aligned}$$

Solving this equation, we obtain the following expressions :

$$\begin{aligned} u_0(s_1;\lambda,\mu) &= \frac{\lambda+2}{\mu+4} \\ u_0(s_2;\lambda,\mu) &= \frac{\lambda-1}{\mu+1} \end{aligned}$$

$$\begin{aligned} u_1(s_1;\lambda,\mu) &= \frac{1}{4}\cdot\frac{\lambda+4}{\mu+8} + \frac{3}{4}\cdot\frac{\lambda+1}{\mu+5} \\ u_1(s_2;\lambda,\mu) &= \frac{1}{2}\cdot\frac{\lambda+1}{\mu+5} + \frac{1}{2}\cdot\frac{\lambda-2}{\mu+2} \end{aligned}$$

$$\begin{aligned} u_2(s_1;\lambda,\mu) &= \frac{1}{4^2}\cdot\frac{\lambda+6}{\mu+12} + \frac{3^2}{4^2}\cdot\frac{\lambda+3}{\mu+9} + \frac{3}{2\cdot 4}\cdot\frac{\lambda}{\mu+6} \\ u_2(s_2;\lambda,\mu) &= \frac{1}{4\cdot 2}\cdot\frac{\lambda+3}{\mu+9} + \frac{5}{4\cdot 2}\cdot\frac{\lambda}{\mu+6} + \frac{1}{2^2}\cdot\frac{\lambda-3}{\mu+3} \end{aligned}$$

$$\begin{aligned} u_3(s_1;\lambda,\mu) &= \frac{1}{4^3}\cdot\frac{\lambda+8}{\mu+16} + \frac{3\cdot 5}{4^3}\cdot\frac{\lambda+5}{\mu+13} + \frac{3\cdot 6}{2\cdot 4^2}\cdot\frac{\lambda+2}{\mu+10} + \frac{3}{2^2\cdot 4}\cdot\frac{\lambda-1}{\mu+7} \\ u_3(s_2;\lambda,\mu) &= \frac{1}{2\cdot 4^2}\cdot\frac{\lambda+5}{\mu+13} + \frac{11}{2\cdot 4^2}\cdot\frac{\lambda+2}{\mu+10} + \frac{1}{2}\cdot\frac{\lambda-1}{\mu+7} + \frac{1}{2^3}\cdot\frac{\lambda-4}{\mu+4} \end{aligned}$$

Finally, substituting $\lambda = \mu = 0$, we have the desired expected values :

$$\begin{aligned} u_3(s_1;0,0) &= \frac{1}{4^3}\cdot\frac{8}{16} + \frac{3\cdot 5}{4^3}\cdot\frac{5}{13} + \frac{3\cdot 6}{2\cdot 4^2}\cdot\frac{1}{5} + \frac{3}{2^2\cdot 4}\cdot(-\frac{1}{7}) \\ &= \cdots = \frac{9969}{58240} = 0.17117101648 \\ u_3(s_2;0,0) &= \frac{1}{2\cdot 4^2}\cdot\frac{5}{13} + \frac{11}{2\cdot 4^2}\cdot\frac{1}{5} + \frac{1}{2}\cdot(\frac{-1}{7}) + \frac{1}{2^3}\cdot(-1) \\ &= \cdots = -\frac{421}{25480} = -0.01652276295 \end{aligned}$$

7. Concluding Remarks

We have considered the conditional expected value of the related associative statistics $R_0 \circ R_1 \circ \cdots \circ R_n$. Firstly we treat the conditional expected value of $R_1 \circ R_2 \circ \cdots \circ R_n$ instead:

$$u_n(i) = E[\, R_1 \circ R_2 \circ \cdots \circ R_n \,|\, X_0 = i\,] \qquad i \in S \tag{16}$$

Let

$$u_n(i;\lambda) = E[\, \lambda \circ R_1 \circ R_2 \circ \cdots \circ R_n \,|\, X_0 = i\,] \qquad \lambda \in R^1.$$

Then, we get the forward recurrence equation:

$$\begin{aligned} u_1(i;\lambda) &= \sum_{j=1}^{N} (\lambda \circ r(j))p(j|i) \\ u_n(i;\lambda) &= \sum_{j=1}^{N} u_{n-1}(j; \lambda \circ r(j))p(j|i) \qquad n \geq 2. \end{aligned}$$

On the other hand, we imbed the problem (16) into the class of backward problems:

$$u^{n-k}(i;\lambda) = E[\, \lambda \circ R_{k+1} \circ R_{k+2} \circ \cdots \circ R_n \,| X_k = i\,] \quad 0 \leq k \leq n-1, \quad \lambda \in R^1.$$

We have the backward recurrence equation:

$$\begin{aligned} u^1(i;\lambda) &= \sum_{j=1}^{N} (\lambda \circ r(j))p(j|i) \\ u^{n-k}(i;\lambda) &= \sum_{j=1}^{N} u^{n-k-1}(j; \lambda \circ r(j))p(j|i) \qquad 0 \leq k \leq n-2. \end{aligned}$$

Secondly we consider the expected values of the following *nonassociative* statistics :

$$u_n(i) = E[\, ((\cdots((R_0 \circ R_1) \circ R_2) \circ \cdots) \circ R_n) \,|\, X_0 = i\,] \qquad i \in S \tag{17}$$

where the nonassociativity does not assure the equality

$$(x \circ y) \circ z = x \circ (y \circ z).$$

We also imbed the problem (17) into the class of *forward* problems:

$$u_n(i;\lambda) = E[\, ((\cdots((\lambda \circ R_0) \circ R_1) \circ \cdots) \circ R_n) \,|\, X_0 = i\,] \qquad \lambda \in R^1.$$

Then we have the forward recurrence equation:

$$\begin{aligned} u_0(i;\lambda) &= \lambda \circ r(i) \\ u_n(i;\lambda) &= \sum_{j=1}^{N} u_{n-1}(j; \lambda \circ r(i))p(j|i) \qquad n \geq 1. \end{aligned}$$

This system has the same form as the system (4),(5). Solving the system, we can calculate the expected value of forward exponential statistics :

$$u_n(i) = E[R_n^{R_{n-1}^{\cdot^{\cdot^{\cdot R_1^{R_0}}}}} \,|\, X_0 = i] \qquad i \in S.$$

Moreover, we can obtain the corresponding (nonstationary) recurrence equation for class of forward problems with *nonstationary* binary relations:

$$u_n(i;\lambda) = E[((\cdots((\lambda * R_0) \diamond R_1) \circ \cdots) \bullet R_n) \,|\, X_0 = i] \qquad \lambda \in R^1,$$

where $*$, $\diamond$, $\circ$, $\cdots$, $\bullet$ are binary relations. For instance, the problem

$$u_3(i) = E[((R_0 \wedge R_1)R_2) + R_3)^{R_4} \,|\, X_0 = i]$$

belongs to the class. Because we have

$$u_3(i) = E[(((R_0 * R_1) \diamond R_2) \circ R_3) \bullet R_4 \,|\, X_0 = i]$$

where

$$a * b = a \wedge b, \quad a \diamond b = ab, \quad a \circ b = a + b, \quad a \bullet b = a^b.$$

Finally we remark that we have also the resulting backward equation (7),(8) for the nonassociative statistics (17).

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