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AN ALGEBRAIC CHARACTERIZATION OF CARTESIAN PRODUCTS OF FUZZY RELATIONS

By

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Abstract

This paper provides an algebraic characterization of mathematical structures formed by cartesian products of fuzzy relations with sup-min composition. A simple proof of a representation theorem for Boolean relation algebras satisfying Tarski rule and point axiom was given by Schmidt and Ströhlein, and cartesian products of Boolean relation algebras were investigated by Jónsson and Tarski. Unlike Boolean relation algebras, fuzzy relation algebras are not Boolean but equipped with semi-scalar multiplication. First we present a set of axioms for fuzzy relation algebras and add axioms for cartesian products of fuzzy relation algebras. Second we improve the definition of point relations. Then a representation theorem for such relation algebras is deduced.

Keywords : fuzzy relations, cartesian products, relation algebras, representation theorem.

1. Introduction

In 1941 Tarski proposed a problem, that is, "Is every relation algebra isomorphic to an algebra of all Boolean (ordinary) relations on a set?". The positive answers of the question, called representation theorem for relation algebras, have been investigated. Schmidt and Ströhlein (1985,1993) gave a simple proof of the representation theorem for Boolean relation algebras satisfying Tarski rule and a point axiom. A representation theorem for fuzzy relation algebras satisfying a point axiom was proved by Kawahara and Furusawa (1995), and categorical representation theorems of fuzzy relations were proved by Kawahara, Furusawa and Mori (1996).

The investigation on fuzzy theory has begun by Zadeh in 1965. Then fuzzy relations have played an important role in mathematics, science and engineering. A methodology for processing fuzzy information in relational structures was provided by Pedrycz (1991). And Sanchez (1976) provided methodology for solution of certain basic fuzzy relational equations. Theory of fuzzy relational equations often give a theoretical back ground to the fuzzy relational modelling.

Fuzzy relations in this paper are homogeneous ones on a set X with truth values (membership degrees) in the unit interval $[0, 1]$; that is, (membership) functions α :

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$X \times X \to [0,1]$. The set of all such fuzzy relations on X constitutes a fuzzy relation algebra with sup-min composition. Fuzzy relation algebras are not Boolean algebras. A notion of fuzzy algebras is a generalization of algebras formed by functions with truth values in the unit interval, that is, functions $a : X \to [0,1]$. Hence a fuzzy algebra is a complete distributive lattice with semi-scalar multiplication by scalars in the unit interval.

Jónsson and Tarski (1952) characterized cartesian products of relation algebras by using a notion of ideal relations. Ideal relations are universal with respect to the composition with the greatest relation ∇. A representable relation algebra has no ideal relation except for the zero relation O and the greatest relation ∇.

The aim of this paper is to provide an algebraic characterization of cartesian products of fuzzy relations by adding two axioms to a set of axioms given by Kawahara and Furusawa (1995). One of axioms is called an axiom of cartesian products, introduced by Jónsson and Tarski (1952), and the other is called a point axiom for cartesian products. Ideal relations in fuzzy relation algebras are required to be crisp. The second one is defined by improving notion of point relations and the point axiom provided by Kawahara and Furusawa (1995). Finally a representation theorem for fuzzy relation algebras satisfying the axiom of cartesian products and the point axiom for cartesian products is proved.

2. Fuzzy Algebras

In this section we first introduce a notion of fuzzy algebras as a mathematical structure formed by fuzzy sets, and describe some properties of fuzzy algebras. In short fuzzy algebras are complete distributive lattices which have semi-scalar multiplications and a semi-Boolean property. Throughout of the paper real numbers $k \in [0,1]$ will be called scalars, where $[0,1]$ is the unit interval, that is, the set of all real numbers k with $0 \le k \le 1$.

DEFINITION 2.1. A fuzzy algebra $\mathcal{A} = (\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \cdot, O, \nabla)$ is an algebraic structure over a nonempty set $\mathcal{A}$ satisfying the following:

A1. [Complete Distributive Lattice] A tuple $(\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, O, \nabla)$ is a complete distributive lattice with the least element O and the greatest element ∇. That is,
(a) $\sqsubseteq$ is a partial order on $\mathcal{A}$, (b) $\forall a \in \mathcal{A} : O \sqsubseteq a \sqsubseteq \nabla$, (c) $\sqcup_\lambda b_\lambda \sqsubseteq a \iff \forall \lambda : b_\lambda \sqsubseteq a$, (d) $a \sqsubseteq \sqcap_\lambda b_\lambda \iff \forall \lambda : a \sqsubseteq b_\lambda$, (e) $a \sqcap (\sqcup_\lambda b_\lambda) = \sqcup_\lambda (a \sqcap b_\lambda)$.

A2. [Semi-Scalar Multiplication] An operation $\cdot : [0,1] \times \mathcal{A} \to \mathcal{A}$ is a semi-scalar multiplication of $\mathcal{A}$. That is,
(a) $0a = O$ *and* $1a = a$, (b) $k(k'a) = (kk')a$, (c) $k(\sqcup_\lambda a_\lambda) = \sqcup_\lambda k a_\lambda$ *and* $k(\sqcap_\lambda a_\lambda) = \sqcap_\lambda k a_\lambda$, (d) $(\wedge_\lambda k_\lambda)a = \sqcap_\lambda k_\lambda a$, (e) If $ka \sqsubseteq kb$ and $0 < k$, then $a \sqsubseteq b$.
(The semi-scalar multiplication $k \cdot a$ of $a \in \mathcal{A}$ by a scalar $k \in [0,1]$ will be written as ka, unless confusion occurs.)

A3. [Semi-Boolean Algebra] If $a \sqcap k\nabla = ka$ for all scalars k, then there is an element b such that $a \sqcup b = \nabla$ and $a \sqcap b = O$. □

Note that complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras.

A fuzzy algebras $\mathcal{A} = (\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \cdot, O, \nabla)$ with $\nabla = O$ is trivial and not worth mention. Throughout the rest of this section all discussions will be done in a fixed fuzzy algebra $\mathcal{A}$ with $\nabla \neq O$.

PROPOSITION 2.2. *Let a, b be elements of a fuzzy algebra $\mathcal{A}$ and k, k' scalars. Then the following holds:*

(a) *If $a \sqsubseteq b$, then $ka \sqsubseteq kb$.*

(b) *If $k \leq k'$, then $ka \sqsubseteq k'a$. In particular, $ka \sqsubseteq a$ and $kO = O$.*

(c) $ka \sqcup k'a = (k \vee k')a$

(d) *If $k\nabla = \nabla$, then $k = 1$.*

(e) *If $k\nabla \sqsubseteq k'\nabla$, then $k \leq k'$.*

PROOF. (a) If $a \sqsubseteq b$, then $ka = k(a \sqcap b) = ka \sqcap kb \sqsubseteq kb$ by A2(c). (b) If $k \leq k'$, then $ka = (k \wedge k')a = ka \sqcap k'a \sqsubseteq k'a$ by A2(d). In particular, $ka \sqsubseteq 1a = a$ by A2(a) and $k \leq 1$, and also $O \sqsubseteq kO \sqsubseteq 1O = O$ by A1(b) and A2(a). (c) Assume $k \leq k'$. Then $ka \sqcup k'a = k'a = (k \vee k')a$ by (b). (d) Assume $k\nabla = \nabla$ and $0 \leq k < 1$. Then by A2(b) it is trivial that $k^n\nabla = \nabla$ for all natural numbers n. Hence it holds that $\nabla = \sqcap_{n \leq 0} k^n \nabla = (\wedge_{n \leq 0} k^n)\nabla = 0\nabla = O$ by A2(d), which contradicts the hypothesis $\nabla \neq O$. (e) Assume $k\nabla \sqsubseteq k'\nabla$ and $k' < k$. Then $0 \leq k'/k < 1$ and $k\nabla \sqsubseteq k[(k'/k)\nabla]$. Therefore $\nabla \sqsubseteq (k'/k)\nabla$ by A2(e) and so $1 = k'/k$ by (d), which contradicts the assumption $k' < k$. □

Following Kawahara and Furusawa (1995) the concept of crisp elements in fuzzy algebras is defined as follows:

DEFINITION 2.3. An element a of a fuzzy algebra $\mathcal{A}$ is crisp if $a \sqcap k\nabla = ka$ for all scalars k. □

In the above definition of crisp elements of fuzzy algebras it is trivial that $ka \sqsubseteq a \sqcap k\nabla$ by 2.2(a) and 2.2(b). Note that $\nabla \sqcap k\nabla = k\nabla$ since $k\nabla \sqsubseteq \nabla$ by A1(b). This means that the universal element ∇ is crisp. Also the zero element O is clearly crisp.

PROPOSITION 2.4. *Let a,b be elements of $\mathcal{A}$ and k a scalar. Then the following holds:*

(a) *If a and b are crisp, then so are $a \sqcup b$ and $a \sqcap b$.*

(b) *If $a \sqcup b = \nabla$ and $a \sqcap b = O$, then both of a and b are crisp.*

(c) *If b is crisp and $ka \sqsubseteq b$ for $k > 0$, then $a \sqsubseteq b$.*

(d) *If a and b are crisp and $a \sqsubseteq b$, then $a \sqcap kb = ka$ for each scalar k.*

PROOF. (a) Let a and b be crisp. Then $(a \sqcup b) \sqcap k\nabla = (a \sqcap k\nabla) \sqcup (b \sqcap k\nabla) = ka \sqcup kb = k(a \sqcup b)$ by A1(e) and A2(c). Also $(a \sqcap b) \sqcap k\nabla = (a \sqcap k\nabla) \sqcap (b \sqcap k\nabla) = ka \sqcap kb = k(a \sqcap b)$ by A2(c). (b) Assume $a \sqcup b = \nabla$ and $a \sqcap b = O$. First note that $ka \sqsubseteq a$ and $a \sqcap kb = O$ by 2.2(b). Then $a \sqcap k\nabla = a \sqcap k(a \sqcup b) = (a \sqcap ka) \sqcup (a \sqcap kb) = ka$ by A2(c) and A1(e). (c) Note that $ka \sqsubseteq k\nabla$ by 2.2(a). As b is crisp we have $ka \sqsubseteq b \sqcap k\nabla = kb$. Hence $a \sqsubseteq b$ by A2(e). (d) Let a and b be crisp with $a \sqsubseteq b$. Then $a \sqcap kb = a \sqcap (b \sqcap k\nabla) = (a \sqcap k\nabla) \sqcap b = ka$. □

From the last proposition 2.4(a) and 2.4(b) it is immediate that the set of all crisp elements in a fuzzy algebra $\mathcal{A}$ forms a Boolean algebra.

3. Fuzzy Relation Algebras

Section 3 provides a set of the axioms R1-R5 for fuzzy relation algebras and some basic properties of fuzzy relation algebras are described. A fuzzy relation algebra $\mathcal{R}$, which will be defined below, is an algebraic structure over a nonempty set set $\mathcal{R}$. In other words fuzzy relation algebras are fuzzy algebras with composition and involution. Elements of $\mathcal{R}$ are denoted by Greek letters such as $\alpha, \beta, \cdots$ and so on. The composite of a relation α followed by a relation β will be noted by $\alpha\beta$, unless confusion occurs.

DEFINITION 3.1. A fuzzy relation algebra $\mathcal{R} = (\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O, \nabla, \mathrm{id})$ is an algebraic structure over a nonempty set $\mathcal{R}$ satisfying the following:
R1. **[Fuzzy Algebra]** A tuple $(\mathcal{R}, \sqsubseteq, \sqcup, \sqcap, \cdot, O, \nabla)$ is a fuzzy algebra.
R2. **[Involutive Monoid]** A tuple $(\mathcal{F}, ;, ^\sharp, \mathrm{id}, O)$ is an involutive monoid with a unit element id and a zero element O. That is,
(a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, (b) $\alpha\mathrm{id} = \mathrm{id}\alpha = \alpha$, (c) $\alpha O = O\alpha = O$, (d) $(\alpha^\sharp)^\sharp = \alpha$,
(e) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (f) If $\alpha \sqsubseteq \beta$, then $\alpha^\sharp \sqsubseteq \beta^\sharp$.
R3. **[Distributive Law]** $\alpha(\sqcup_\lambda \beta_\lambda) = \sqcup_\lambda \alpha\beta_\lambda$.
R4. **[Dedekind Formula]** $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$.
R5. **[Compatibility with Semi-Scalar Multiplication]**
(a) $k(\alpha\beta) = (k\alpha)(k\beta)$, (b) $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla)$, (c) $(k\alpha)^\sharp = k\alpha^\sharp$. □

All elements in a fuzzy relation algebra $\mathcal{R}$ are called relations for short.

PROPOSITION 3.2. *Let α, β, β' be relations and k, k' scalars. Then the following holds:*

(a) *If $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta'$ and $\beta\alpha \sqsubseteq \beta'\alpha$.*

(b) $O^\sharp = O$, $\nabla^\sharp = \nabla$ *and* $\mathrm{id}^\sharp = \mathrm{id}$.

(c) $(\alpha \sqcup \beta)^\sharp = \alpha^\sharp \sqcup \beta^\sharp$ *and* $(\alpha \sqcap \beta)^\sharp = \alpha^\sharp \sqcap \beta^\sharp$.

(d) $(k\alpha)\beta \sqsubseteq k(\alpha\nabla)$ *and* $\alpha(k\beta) \sqsubseteq k(\nabla\beta)$.

PROOF. (a) If $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha\beta \sqcup \alpha\beta' = \alpha(\beta \sqcup \beta') = \alpha\beta'$ by R3. (b) $O^\sharp \sqsubseteq O^{\sharp\sharp} = O$ since $O \sqsubseteq O^\sharp$, and $\nabla = \nabla^{\sharp\sharp} \sqsubseteq \nabla^\sharp$ since $\nabla^\sharp \sqsubseteq \nabla$, and $\mathrm{id}^\sharp = \mathrm{id}^\sharp\mathrm{id} = \mathrm{id}^\sharp\mathrm{id}^{\sharp\sharp} = (\mathrm{id}^\sharp\mathrm{id})^\sharp = \mathrm{id}^{\sharp\sharp} = \mathrm{id}$. (c) First note that $\alpha^\sharp \sqcup \beta^\sharp \sqsubseteq (\alpha \sqcup \beta)^\sharp$. Hence $\alpha \sqcup \beta = \alpha^{\sharp\sharp} \sqcup \beta^{\sharp\sharp} \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^\sharp$

and $(\alpha \sqcup \beta)^\sharp \sqsubseteq (\alpha^\sharp \sqcup \beta^\sharp)^{\sharp\sharp} = \alpha^\sharp \sqcup \beta^\sharp$. (d) $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla) \sqsubseteq (k\alpha)(k\nabla) = k(\alpha\nabla)$ by R5(b) and R5(a). $\Box$

Note that $\alpha(\sqcap_\lambda \beta_\lambda) \sqsubseteq \sqcap_\lambda(\alpha\beta_\lambda)$ and $\nabla\nabla = \nabla$ holds immediately by the last proposition 3.2(a).

The next proposition mentions basic properties of crisp relations.

PROPOSITION 3.3. *Let α and β be relations and k a scalar. Then the following holds:*

(a) *If β is crisp, then $(k\alpha)\beta = k(\alpha\beta)$. (If α is crisp, then $\alpha(k\beta) = k(\alpha\beta)$).*

(b) *If α and β are crisp, then so are $\alpha^\sharp$ and $\alpha\beta$.*

(c) *If $\nabla = \alpha \sqcup \beta$, $\alpha \sqcap \beta = O$ and $\nabla\alpha = \alpha$, then $\nabla\beta = \beta$.*

PROOF. (a) Assume that β is crisp. Then $(k\alpha)\beta = (k\alpha)(\beta \sqcap k\nabla) = (k\alpha)(k\beta) = k(\alpha\beta)$ by R5(b) and R5(a). (b) Assume that α and β are crisp. Then $\alpha^\sharp \sqcap k\nabla = (\alpha \sqcap k\nabla)^\sharp = (k\alpha)^\sharp = k\alpha^\sharp$ by R2, R5(c), 3.2(c) and 3.2(b). And $\alpha\beta \sqcap k\nabla \sqsubseteq \alpha[\beta \sqcap \alpha^\sharp(k\nabla)] \sqsubseteq \alpha[\beta \sqcap k(\nabla\nabla)] \sqsubseteq \alpha(\beta \sqcap k\nabla) = \alpha(k\beta) = k(\alpha\beta)$ by R4, 3.2(d), 3.2(b) and (a). (c) Note that $\beta = \mathrm{id}\beta \sqsubseteq \nabla\beta$ by R2(b) and A1(b), and $\nabla\beta \sqcap \alpha \sqsubseteq \nabla(\beta \sqcap \nabla^\sharp\alpha) \sqsubseteq \nabla(\beta \sqcap \nabla\alpha) = \nabla(\beta \sqcap \alpha) = \nabla O = O$ by R4, A1(b) and R2(c). Hence $\nabla\beta = \nabla\beta \sqcap \nabla = \nabla\beta \sqcap (\alpha \sqcup \beta) = (\nabla\beta \sqcap \alpha) \sqcup (\nabla\beta \sqcap \beta) = \beta$. $\Box$

From proposition 2.4(a), 2.4(b) and 3.4(c) the set of all crisp relations in fuzzy relation algebra $\mathcal{R}$ forms a (Boolean) relation algebra in the sense of Schmidt and Ströhlein (1985).

4. Ideal Relations

Jónsson and Tarski (1952) investigated ideal relations in Boolean relation algebras. In this section we define ideal relations in fuzzy relation algebras, and consider some properties of ideal relations. Though in Boolean relation algebras ideal relations are just universal with respect to the composition with the greatest relation ∇, ideal relations are also required to be crisp in fuzzy relation algebras. In the next section ideal relations play an important role.

DEFINITION 4.1. A fuzzy relation ξ is an ideal if ξ is crisp and $\nabla\xi\nabla = \xi$. $\Box$

Note that $\nabla\nabla\nabla = \nabla$ by A1 and 3.2(a). So the universal relation ∇ is ideal. Also the zero relation O is ideal by R2.

PROPOSITION 4.2. *Let ξ, η, be ideal, α, β relations and k a scalar. Then the following holds:*

(a) *$\xi \sqcup \eta$, $\xi \sqcap \eta$ and $\xi^\sharp$ are ideal.*

(b) $\xi = \xi\nabla = \nabla\xi$.

(c) $\xi = \xi^\sharp$.

(d) $\xi\eta = \xi \sqcap \eta$.

(e) $\alpha\beta \sqcap \xi = (\alpha \sqcap \xi)(\beta \sqcap \xi)$.

PROOF. (a) $\nabla(\xi \sqcup \eta)\nabla = \nabla\xi\nabla \sqcup \nabla\eta\nabla = \xi \sqcup \eta$ by R3. $\nabla(\xi \sqcap \eta)\nabla \sqsubseteq \nabla\xi\nabla \sqcap \nabla\eta\nabla = \xi \sqcap \eta = \xi \sqcap \nabla\eta\nabla \sqsubseteq \nabla(\nabla^\sharp\xi\nabla^\sharp \sqcap \eta)\nabla = \nabla(\xi \sqcap \eta)\nabla$ by R4 and 3.2(b). $\nabla\xi^\sharp\nabla = (\nabla\xi\nabla)^\sharp = \xi^\sharp$ by 3.2(b) and R2. (b) Because $\xi = \nabla\xi\nabla$, $\nabla\xi = \nabla\nabla\xi\nabla = \xi$. (c) $\xi^\sharp = \xi^\sharp\nabla = \xi^\sharp\nabla \sqcap \nabla \sqsubseteq \xi^\sharp(\nabla \sqcap \xi\nabla) \sqsubseteq \nabla\xi\nabla = \xi$ by (b) and R4. Similarly it is shown that $\xi \sqsubseteq \xi^\sharp$. (d) $\xi\eta = \xi\eta \sqcap \nabla \sqsubseteq (\xi \sqcap \nabla\eta^\sharp)\eta \sqsubseteq (\xi \sqcap \eta)\nabla = \xi \sqcap \eta$ by R4, (b), (c) and (a). Conversely $\xi \sqcap \eta = \xi \sqcap \nabla\eta \sqsubseteq (\xi\eta^\sharp \sqcap \nabla)\eta \sqsubseteq \xi\eta$ by (b), R4 and (c). (e) $\alpha\beta \sqcap \xi \sqsubseteq (\alpha \sqcap \xi\beta^\sharp)\beta = (\alpha \sqcap \xi\beta^\sharp)\beta \sqcap \xi \sqsubseteq (\alpha \sqcap \xi\beta^\sharp)[\beta \sqcap (\alpha \sqcap \xi\beta^\sharp)^\sharp\xi] \sqsubseteq (\alpha \sqcap \xi\beta^\sharp)(\beta \sqcap \alpha^\sharp\xi) \sqsubseteq (\alpha \sqcap \xi\nabla)(\beta \sqcap \nabla\xi) = (\alpha \sqcap \xi)(\beta \sqcap \xi) \sqsubseteq \alpha\beta \sqcap \xi\xi \sqsubseteq \alpha\beta \sqcap \xi$ by R4 and (b). □

Now we consider the function $\phi_\xi : \mathcal{R} \to \mathcal{R}$ such that $\phi_\xi(\alpha) = \alpha \sqcap \xi$ for a fixed ideal relation ξ and any relation α. The following proposition show that $\phi_\xi : \mathcal{R} \to \mathcal{R}$ preserves all operations of fuzzy relation algebras except for nullary operations.

PROPOSITION 4.3. *Let ξ be an ideal relation, α, β relations and k a scalar. Then the following holds:*

(a) *If $\alpha \sqsubseteq \beta$, then $\phi_\xi(\alpha) \sqsubseteq \phi_\xi(\beta)$.*

(b) $\phi_\xi(\alpha \sqcup \beta) = \phi_\xi(\alpha) \sqcup \phi_\xi(\beta)$ *and* $\phi_\xi(\alpha \sqcap \beta) = \phi_\xi(\alpha) \sqcap \phi_\xi(\beta)$.

(c) $\phi_\xi(\alpha^\sharp) = \phi_\xi(\alpha)^\sharp$.

(d) $\phi_\xi(k\alpha) = k\phi_\xi(\alpha)$.

(e) $\phi_\xi(\alpha\beta) = \phi_\xi(\alpha)\phi_\xi(\beta)$.

PROOF. (a) Assume that $\alpha \sqsubseteq \beta$, then $\phi_\xi(\alpha) = \alpha \sqcap \xi \sqsubseteq \beta \sqcap \xi = \phi_\xi(\beta)$. (b) $\phi_\xi(\alpha \sqcup \beta) = (\alpha \sqcup \beta) \sqcap \xi = (\alpha \sqcap \xi) \sqcup (\beta \sqcap \xi) = \phi_\xi(\alpha) \sqcup \phi_\xi(\beta)$. Similarly it is shown that $\phi_\xi(\alpha \sqcap \beta) = \phi_\xi(\alpha) \sqcap \phi_\xi(\beta)$. (c) $\phi_\xi(\alpha^\sharp) = \alpha^\sharp \sqcap \xi = (\alpha \sqcap \xi)^\sharp = \phi_\xi(\alpha)^\sharp$ by 4.2(d). (d) $\phi_\xi(k\alpha) = k\alpha \sqcap \xi = k(\alpha \sqcap \nabla) \sqcap \xi = k\alpha \sqcap (k\nabla \sqcap \xi) = k\alpha \sqcap k\xi = k(\alpha \sqcap \xi) = k\phi_\xi(\alpha)$ since ξ is crisp. (e) $\phi_\xi(\alpha\beta) = \alpha\beta \sqcap \xi = (\alpha \sqcap \xi)(\beta \sqcap \xi) = \phi_\xi(\alpha)\phi_\xi(\beta)$ by 4.2(e). □

If ξ is not crisp, proposition 4.3(d) doesn't hold. Because there is a scalar k such that $k\nabla \sqcap \xi \neq k\xi$ by the definition of crisp relations. That is a difference between fuzzy relation algebras and Boolean relation algebras in the sense of Jónsson and Tarski (1952).

For a fixed relation ξ we denote a set of all relations α such that $\alpha \sqsubseteq \xi$ $\mathcal{R}(\xi)$. If ξ is an ideal relation, $\mathcal{R}(\xi)$ is the image of the function ϕ_ξ which appeared in the before proposition 4.3. Note that $\mathrm{id} \sqcap \xi$ is a unit element in $\mathcal{R}(\xi)$ since $\alpha(\mathrm{id} \sqcap \xi) \sqsubseteq \alpha\mathrm{id} = \alpha$ and

$$\alpha = \alpha\mathrm{id} = \alpha\mathrm{id} \sqcap \nabla \sqsubseteq \alpha(\mathrm{id} \sqcap \alpha^\sharp\xi) \sqsubseteq \alpha(\mathrm{id} \sqcap \xi\xi) \sqsubseteq \alpha(\mathrm{id} \sqcap \xi\nabla) = \alpha(\mathrm{id} \sqcap \xi)$$

for each $\alpha \in \mathcal{R}(\xi)$. Then a tuple $(\mathcal{R}(\xi), \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O, \mathrm{id} \sqcap \xi, \xi)$ is a fuzzy relation algebra and ϕ_ξ is a homomorphism from $\mathcal{R}$ onto $\mathcal{R}(\xi)$.

5. Cartesian Products of Fuzzy Relation Algebras

Jónnson and Tarski (1952) showed that a cartesian product of Boolean relation algebras is again a Boolean relation algebra. And they also characterized cartesian products of Boolean relation algebras. It is clear that a cartesian product of fuzzy relation algebras is so if we consider a relation $\alpha = \prod_{i \in I} \alpha_i$ and semi-scalar multiplication $k\alpha = k \prod_{i \in I} \alpha_i = \prod_{i \in I} k\alpha_i$, where a symbol $\prod$ is used as a cartesian product notation, and I is a finite set. (Throughout the rest of this paper we use a symbol $\prod$ as a cartesian product notation.) In this section we characterize cartesian products of fuzzy relation algebras.

DEFINITION 5.1. Let I be an arbitrary finite set. Then a fuzzy relation algebra $\mathcal{R}$ satisfies the axiom of cartesian products if:
R6. [Axiom of Cartesian Products] There exist finite number of ideal relations $\nabla_i \neq O$ $(i \in I)$ such that:
(a) $\sqcup_{i \in I} \nabla_i = \nabla$. (b) If $i, j \in I$ and $i \neq j$, then $\nabla_i \sqcap \nabla_j = O$. □

Let $\mathcal{R}$ be a fuzzy relation algebra satisfying axiom of cartesian products and ϕ_i a function from $\mathcal{R}$ to $\mathcal{R}(\nabla_i)$ such that $\phi_i(\alpha) = \alpha \sqcap \nabla_i$ for each $i \in I$ and each relation $\alpha \in \mathcal{R}$. Then the function $\phi_i : \mathcal{R} \to \mathcal{R}(\nabla_i)$ is homomorphism and a tuple $(\mathcal{R}(\nabla_i), \sqsubseteq, \sqcup, \sqcap, ;, ^\sharp, \cdot, O_i, \mathrm{id}_i, \nabla_i)$ is a fuzzy relation algebra, where $O_i = O \sqcap \nabla_i$ and $\mathrm{id}_i = \mathrm{id} \sqcap \nabla_i$, since ∇_i is an ideal relation.

PROPOSITION 5.2. *Let α, β be relations in a fuzzy relation algebra $\mathcal{R}$ satisfying the axiom of cartesian products and $i, j \in I$. If $\alpha \sqsubseteq \nabla_i$, $\beta \sqsubseteq \nabla_j$ and $i \neq j$, then $\alpha\beta = O$*

PROOF. Let $i, j \in I$ and $i \neq j$. Then $\nabla_i \nabla_j = \nabla_i \sqcap \nabla_j = O$ by 4.2(d) and R6(b). Thus $\alpha\beta = O$ since $\alpha \sqsubseteq \nabla_i$ and $\beta \sqsubseteq \nabla_j$. □

We show that every relation in a fuzzy relation algebra $\mathcal{R}$ satisfying the axiom of cartesian products can be represented as a supremum of relations in $\mathcal{R}(\nabla_i)$ for $i \in I$.

THEOREM 5.3. *Let $\mathcal{R}$ be a fuzzy relation algebra satisfying the axiom of cartesian products. Then every relation $\alpha \in \mathcal{R}$ has a unique representation $\alpha = \sqcup_{i \in I} \phi_i(\alpha)$.*

PROOF. $\alpha = \alpha \sqcap \nabla = \alpha \sqcap (\sqcup_{i \in I} \nabla_i) \doteq \sqcup_{i \in I}(\alpha \sqcap \nabla_i) = \sqcup_{i \in I} \phi_i(\alpha)$ by R6(a). Next we show the uniqueness of the representation. Assume that $\alpha = \sqcup_{i \in I} \alpha_i$, where $\alpha_i \in \mathcal{R}(\nabla_i)$ for each $i \in I$. Then $\alpha_j = \alpha_j \mathrm{id}_j = (\sqcup_{i \in I} \alpha_i)\mathrm{id}_j = [\sqcup_{i \in I}(\alpha \sqcap \nabla_i)]\mathrm{id}_j = \sqcup_{i \in I}[(\alpha \sqcap \nabla_i)\mathrm{id}_j] = \alpha \sqcap \nabla_j = \phi_j(\alpha)$ for each $j \in I$ by proposition 5.2. □

Now we consider a function $\psi : \mathcal{R} \to \prod_{i \in I} \mathcal{R}(\nabla_i)$ such that $\psi(\alpha) = \prod_{i \in I} \phi_i(\alpha)$.

PROPOSITION 5.4. *Let $\mathcal{R}$ be a fuzzy relation algebra satisfying the axiom of cartesian products. Then the function $\psi : \mathcal{R} \to \prod_{i \in I} \mathcal{R}(\nabla_i)$ is a bijection.*

PROOF. First note that $\alpha = \sqcup_{i\in I}\phi_i(\alpha)$ and $\beta = \sqcup_{i\in I}\phi_i(\beta)$ by proposition 5.3. If $\psi(\alpha) \sqsubseteq \psi(\beta)$, then $\alpha \sqsubseteq \beta$ since $\phi_i(\alpha) \sqsubseteq \phi_i(\beta)$ for each $i \in I$, which show that ψ is injective. For a cartesian product $\prod_{i\in I}\alpha_i \in \prod_{i\in I}\mathcal{R}(\nabla_i)$ of relations $\alpha_i \in \mathcal{R}(\nabla_i)$ if we set $\alpha = \sqcup_{i\in I}\alpha_i$, then $\alpha_i = \phi_i(\alpha)$ by proposition 5.3. Therefore $\psi(\alpha) = \prod_{i\in I}\alpha_i$, which show that ψ is surjective. □

The following proposition show that the function $\psi : \mathcal{R} \to \prod_{i\in I}\mathcal{R}(\nabla_i)$ preserves all operations of fuzzy relation algebras.

COROLLARY 5.5. *Let α, β be relations in a fuzzy relation algebra $\mathcal{R}$ satisfying the axiom of cartesian products and k a scalar. Then the following holds:*

(a) $\psi(O) = \prod_{i\in I} O_i$, $\psi(\nabla) = \prod_{i\in I}\nabla_i$ *and* $\psi(\mathrm{id}) = \prod_{i\in I}\mathrm{id}_i$.

(b) *If* $\alpha \sqsubseteq \beta$, *then* $\psi(\alpha) \sqsubseteq \psi(\beta)$

(c) $\psi(\alpha \sqcup \beta) = \psi(\alpha) \sqcup \psi(\beta)$ *and* $\psi(\alpha \sqcap \beta) = \psi(\alpha) \sqcap \psi(\beta)$.

(d) $\psi(\alpha^\sharp) = \psi(\alpha)^\sharp$.

(e) $\psi(k\alpha) = k\psi(\alpha)$.

(f) $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$.

PROOF. It is trivial by the definition of ψ and proposition 4.3. □

Consequently we have proved that a fuzzy relation algebra $\mathcal{R}$ satisfying the axiom of cartesian products is isomorphic to a cartesian product $\prod_{i\in I}\mathcal{R}(\nabla_i)$ of fuzzy relation algebras $\mathcal{R}(\nabla_i)$.

6. Representation Theorem

In Kawahara and Furusawa (1995) point relations in a fuzzy relation algebra defined as a crisp relation x such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\mathrm{id} \sqsubseteq xx^\sharp$ and $\nabla x = x$. But this definition is not suitable in the case of cartesian products of fuzzy relation algebras. In this section we improve the definition of point relations. Point relations will be denoted by lower case Roman letters such as $x, y, \cdots$. Throughout of this section we assume that a fuzzy relation algebra $\mathcal{R}$ satisfies the axiom R6 of cartesian products.

DEFINITION 6.1. A point relation x is a crisp relation such that $x^\sharp x \sqsubseteq \mathrm{id}$, $\nabla x = x$ and $x\nabla = \nabla_i$ for some $i \in I$. □

Remark that there is a unique $i \in I$ such that $x\nabla = \nabla_i$ by R6.

Let x be a point relation, then $x \sqsubseteq x\nabla = \nabla_i$ for some $i \in I$. Hence the relation x is an element of $\mathcal{R}(\nabla_i)$. Since $\mathcal{R}(\nabla_i)$ is a fuzzy relation algebra with the unit element id_i, it holds that $x\mathrm{id}_i = \mathrm{id}_i x = x$. The next proposition show that point relations in $\mathcal{R}$ are in fact point relations in $\mathcal{R}(\nabla_i)$ in the sense of Kawahara and Furusawa (1995).

PROPOSITION 6.2. *If x is a point relation such that $x\nabla = \nabla_i$, then the following holds:*

(a) $x \sqcap k\nabla_i = kx$ *for each scalar* k.

(b) $x^\sharp x \sqsubseteq \mathrm{id}_i$.

(c) $\mathrm{id}_i \sqsubseteq xx^\sharp$.

(d) $\nabla_i x = x$.

PROOF. (a) Since x and ∇_i are crisp with $x \sqsubseteq x\nabla = \nabla_i$, it is deduced from proposition 2.4(d). (b) $x^\sharp x \sqsubseteq (x\nabla)^\sharp(x\nabla) = \nabla_i^\sharp \nabla_i \sqsubseteq \nabla_i$. Thus $x^\sharp x \sqsubseteq \mathrm{id} \sqcap \nabla_i = \mathrm{id}_i$ by the definition. (c) $\mathrm{id}_i = \mathrm{id} \sqcap \nabla_i = \mathrm{id} \sqcap x\nabla \sqsubseteq x(x^\sharp \mathrm{id} \sqcap \nabla) = xx^\sharp$. (d) $\nabla_i x \sqsubseteq \nabla x = x = \mathrm{id}_i x \sqsubseteq \nabla_i x$. □

Let X_i be a set of all point relations x such that $x\nabla = \nabla_i$. Then it is immediate that X_i is a set of all point relations in $\mathcal{R}(\nabla_i)$ in the sense of Kawahara and Furusawa (1995) by the before proposition.

By making use of the last definition of point relations in fuzzy relation algebras satisfying the axiom R6 of cartesian products we add the following axiom:

DEFINITION 6.3. A fuzzy relation algebra $\mathcal{R}$ satisfies the point axiom for cartesian products:
R7. [Point Axiom for Cartesian Products] For each nonzero relation α there is a scalar $k > 0$ and point relations x, y such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ and $x^\sharp y \neq O$. □

Let x, y be point relations in the sense of Kawahara and Furusawa (1995). Assume that $x^\sharp y = O$, then $y = \mathrm{id}y \sqsubseteq xx^\sharp y = O$, which contradicts $\mathrm{id} \sqsubseteq yy^\sharp$. Therefore in it holds that $x^\sharp y \neq O$. But there are point relations $x \in X_i, y \in X_j$ with $i \neq j$ in a fuzzy relation algebras $\mathcal{R}$ satisfying R6. Then it holds that $x^\sharp y = O$ by proposition 5.2 since $x \sqsubseteq \nabla_i$ and $y \sqsubseteq \nabla_j$. So we need the additional condition $x^\sharp y \neq O$ to the point axiom in Kawahara and Furusawa (1995).

If $\mathcal{R}$ is a fuzzy relation algebra satisfying the axiom of cartesian products and the point axiom for cartesian products, $\mathcal{R}(\nabla_i)$ satisfies the point axiom in the sense of Kawahara and Furusawa (1995) for each $i \in I$.

PROPOSITION 6.4. *Let $\mathcal{R}$ be a fuzzy relation algebra satisfying the point axiom for cartesian products. Then there is a scalar $k > 0$ and point relations $x, y \in X_i$ such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ for each nonzero relation $\alpha \in \mathcal{R}(\nabla_i)$ and each $i \in I$.*

PROOF. Let α be an arbitrary nonzero relation in $\mathcal{R}(\nabla_i)$. Then there is a scalar $k > 0$ and point relations x, y such that $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ and $x^\sharp y \neq O$ by the point axiom R7. It is necessary point relations x, y are in a set X_j in order that $x^\sharp y \neq O$ holds. Assume that $j \neq i$, then $\alpha \sqcap x^\sharp y \sqsubseteq \nabla_i \sqcap \nabla_j = O$. This is a contradiction to $\alpha \sqcap x^\sharp y = k(x^\sharp y)$, since $x^\sharp y \neq O$ and $k > 0$. Therefore $x, y \in X_i$. □

In Kawahara and Furusawa (1995) we proved that every relation α in a fuzzy relation algebra satisfying the point axiom can be represented as a supremum of pairs of point relations with semi-scalar weights, and also such a fuzzy relation algebra is isomorphic to an algebra of fuzzy relations on a set. Following Kawahara and Furusawa (1995) we define functions $\chi_i : \mathcal{R}(\nabla_i) \to \mathbf{Rel}(X_i)$ as $\chi_i(\alpha)(x,y) = k$ with $\alpha \sqcap x^\sharp y = k(x^\sharp y)$ for each $\alpha \in \mathcal{R}(\nabla_i)$, $x, y \in X_i$ and $i \in I$, where $\mathbf{Rel}(X_i)$ is a set of all fuzzy relations on X_i. By proposition 6.4 if $\mathcal{R}$ satisfies the axiom of cartesian products and the point axiom for cartesian products, $\mathcal{R}(\nabla_i)$ is a fuzzy relation algebra satisfying the point axiom in the sense of Kawahara and Furusawa (1995) for each $i \in I$. Thus every relation $\alpha \in \mathcal{R}(\nabla_i)$ has a unique representation

$$\alpha = \sqcup_{x,y \in X_i} \chi_i(\alpha)(x,y)(x^\sharp y),$$

and χ_i is an isomorphism from $\mathcal{R}(\nabla_i)$ to $\mathbf{Rel}(X_i)$.

By the above discussion the following two propositions hold.

COROLLARY 6.5 REPRESENTATION THEOREM. *Let $\mathcal{R}$ be a fuzzy relation algebra satisfying the point axiom for cartesian products. Then every relation $\alpha \in \mathcal{R}$ has a unique representation*

$$\alpha = \sqcup_{i \in I} \sqcup_{x,y \in X_i} \chi_i(\phi_i(\alpha))(x,y)(x^\sharp y).$$

PROOF. For each $i \in I$ and each relation $\alpha \in \mathcal{R}$, a relation $\phi_i(\alpha)$ has a unique representation $\phi_i(\alpha) = \sqcup_{x,y \in X_i} \chi_i(\phi_i(\alpha))(x,y)(x^\sharp y)$ since $\phi_i(\alpha) \in \mathcal{R}(\nabla_i)$, and also a relation α has a unique representation $\alpha = \sqcup_{i \in I} \phi_i(\alpha)$ by theorem 5.3. Thus relation α has such a unique representation. □

COROLLARY 6.6 ISOMORPHISM THEOREM. *Every fuzzy relation algebra $\mathcal{R}$ satisfying the point axiom for cartesian products is isomorphic to the cartesian product $\prod_{i \in I} \mathbf{Rel}(X_i)$ of fuzzy relations $\mathbf{Rel}(X_i)$.*

PROOF. Since the function ψ is an isomorphism from $\mathcal{R}$ to $\prod_{i \in I} \mathcal{R}(\nabla_i)$ and the function χ_i is an isomorphism from $\mathcal{R}(\nabla_i)$ to $\mathbf{Rel}(X_i)$ for each $i \in I$. Therefore the composite function $(\prod_{i \in I} \chi_i) \circ \psi$ is an isomorphism from $\mathcal{R}$ to $\prod_{i \in I} \mathbf{Rel}(X_i)$. □

It is clear that a fuzzy relation algebra $\mathcal{R}$ satisfying the point axiom for cartesian products is isomorphic to (ordinary) fuzzy relations $\mathbf{Rel}(X)$ on a set of all point relations in $\mathcal{R}$ if a set I is a one point set. So we can say that the characterization in this paper is natural extension of Kawahara and Furusawa (1995).

References

Jónsson, B. and Tarski, A. (1952), Boolean algebras with operators. Part II, Amer. J. Math. **74** 127-162.

Tarski, A. (1941), On the calculus of relations, J. Symbolic Logic **6** 73–89.

Kawahara, Y. and Furusawa, H. (1995), An Algebraic Formalization of Fuzzy Relations, RIFIS Technical Report TR-CS-98, Kyushu University.

Kawahara, Y., Furusawa, H. and Mori, M. (1996), Categorical Representation Theorems of Fuzzy Relations, Proceedings of 4th International Workshop on Rouph Sets, Fuzzy Sets, and Machine Discovery (RSFD 96), 190–197.

Schmidt, G. and Ströhlein, T. (1985), Relation algebras : Concept of points and representability, Discrete Mathematics **54** 83–92.

Schmidt, G. and Ströhlein, T. (1993), Relations and graphs – Discrete Mathematics for Computer Science – (Springer-Verlag, Berlin).

Pedrycz, W. (1991), Processing in relational structures : Fuzzy relational equations, Fuzzy Sets and Systems **40** 77–106.

Pedrycz, W. (1996), Fuzzy Modelling *Paradigms and practice* (Kluwer Academic Publishers).

Sanchez, E. (1976), Resolution of composite fuzzy relation equations, Information and Control **30** 38–48.

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