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Lee, Hyen Yeal
Department of Computer Science, Pusan University

Kawahara, Yasuo
Research Institute of Fundamental Information Science, Kyushu University

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TRANSITION DIAGRAMS OF FINITE CELLULAR AUTOMATA

By

Hyen Yeal LEE* and **Yasuo KAWAHARA†**

Abstract

This paper provides some simple recursive formulas generating transition diagrams of finite cellular automata with triplet local transition functions.

1. Introduction

The theory of cellular automata has been studied by many researchers in various methods, for example, numerical, statistical and asymptotic methods. On the other hand dynamical behaviors of finite (or discrete) cellular automata are extremely complicated and interested. Many scientists and mathematicians have extensively developed theory of finite cellular automata, and their major interests focused fixed points, period lengths, transient lengths and so on, which are parts of global dynamical behaviors of cellular automata. Generally it is absolutely difficult to decide just transition diagrams of cellular automata. But in fact some special cellular automata have simpler transition diagrams. The aim of this paper is to challenge the problem to decide transition diagram in an algebraic method. It seems that the most important point for the problem is how to represent transition diagrams by simple formulas. To this end we introduce symbolic notations called tree and cycle expressions without and with roots.

First we review the definition of dynamical systems as formal systems. A dynamical system is a pair (X, δ) of a set X and a transition function $\delta : X \to X$. A dynamorphism $\varphi : (X, \delta) \to (Y, \sigma)$ from a dynamical system (X, δ) into another dynamical system (Y, σ) is a set function $\varphi : X \to Y$ such that $\sigma\varphi = \varphi\delta$, that is, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow[\varphi]{} & Y. \end{array}$$

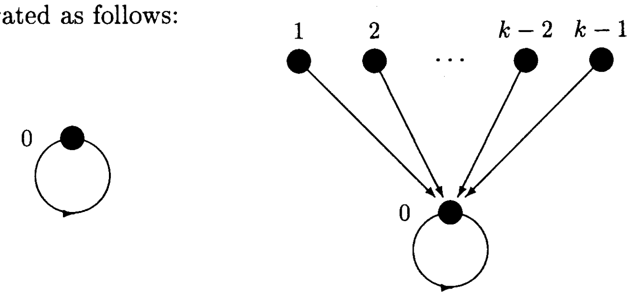
A dynamorphism $\varphi : (X, \delta) \to (Y, \sigma)$ is called an isomorphism of dynamical systems if there exists an inverse dynamorphism $\psi : (Y, \sigma) \to (X, \delta)$ such that $\psi\varphi = \mathrm{id}_X$ and

* Department of Computer Science, Pusan University, Pusan 605, Korea.

† Research Institute of Fundamental Information Science, Kyushu University 33, Fukuoka 812-81, Japan.

$\varphi\psi = \mathrm{id}_Y$, where id_X and id_Y are the identity functions on X and Y, respectively. It is clear that isomorphic dynamical systems have isomorphic transition diagrams and vice versa.

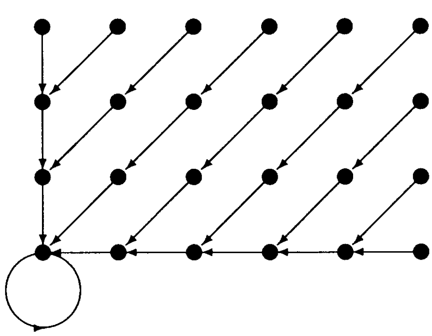
In the sequel we treat only with finite dynamical systems. For a positive integer k we define a dynamical system $[k]$ to be a pair of a set $\{0, 1, \cdots, k-2, k-1\}$ and a constant function $\delta(x) = 0$ for all $x = 0, 1, \cdots, k-2, k-1$. The transition diagram of $[k]$ is illustrated as follows:



It is clear that a product formula $[k] \times [k'] = [kk']$ holds for any positive integers k, k', where $\times$ denotes the cartesian product of dynamical systems and $=$ the equality up to isomorphisms.

For a positive integer k we define a dynamical system $[1]^k$ to be a pair of a set $\{0, 1, \cdots, k-2, k-1\}$ and a predecessor function δ such that $\delta(0) = 0$ and $\delta(x) = x - 1$ for all $x = 1, 2, \cdots, k-2, k-1$. The transition diagram of $[1]^k$ is illustrated as follows:

Note that $[1]^1 = [1]$ and $[1]^2 = [2]$. The cartesian product $[1]^6 \times [1]^4$ is illustrated as follows:



Now we will recall (one dimensional) finite cellular automata with triplet local transition functions. Let m be a natural number and Q the set $\{0,1\}$ of symbols 0 and 1. The set of all strings of symbols in Q with a length of m is denoted by B^m. For example, $B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$. A triplet local transition function f is a function $f : B^3 \to B$. Set $r_k = f(xyz)$ $(k = 4x + 2y + z,\ 0 \le k \le 7)$ for all strings $xyz \in B^3$. Following to Wolfram we define the rule number R of a triplet local transition function f by $R = \sum_{k=0}^{7} 2^k r_k$. Note that $0 \le R \le 255$. A triplet local transition function f with rule number R is illustrated as follows:

Rule R	111	110	101	100	011	010	001	000
	r_7	r_6	r_5	r_4	r_3	r_2	r_1	r_0

A triplet local transition function g such that $g(xyz) = f(zyx)$ for all x, y and z, is called symmetric to f. The rule number of g is called symmetric to the rule number of f. The triplet local transition function h such that $h(xyz) = \overline{f(\overline{x}\overline{y}\overline{z})}$ for all x, y and z, is called complemented to f. (Note that $\overline{0} = 1$ and $\overline{1} = 0$.) The rule number of g is called complemented to the rule number of f. It is easy to see that the symmetric complemented function is identical with the complemented symmetric function. For example, the symmetric, the complemented, the symmetric complemented rule numbers to 12 are 68, 107 and 121, respectively.

A cellular automaton $CA - R_{a-b}(m)$ with boundary condition $a - b$ $(a, b = 0$ or $1)$ is a dynamical system (B^m, δ_{a-b}) such that

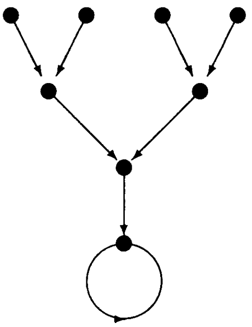
$$\delta_{a-b}(x_1 x_2 \cdots x_{m-1} x_m) = f(a x_1 x_2) f(x_1 x_2 x_3) \cdots f(x_{m-2} x_{m-1} x_m) f(x_{m-1} x_m b),$$

where f is a triplet local transition function with rule number R.

Note that if a local transition function f satisfies $f(acz) = c$ for $z = 0, 1$ then a function $j : CA - R_{c-b}(m-1) \to CA - R_{a-b}(m)$ with $(x_1 \cdots x_{m-1}) = c x_1 \cdots x_{m-1}$ is a dynamorphism, that is, a transition diagram of $CA - R_{a-b}(m)$ contains that of $CA - R_{c-b}(m-1)$ as a subgraph. Moreover, if $f(a0z) = 0$ and $f(a1z) = 1$ for $z = 0, 1$ then a transition diagram of $CA - R_{a-b}(m)$ is a disjoint union of two transition diagrams of $CA - R_{0-b}(m-1)$ and $CA - R_{1-b}(m-1)$, namely

$$CA - R_{a-b}(m) = CA - R_{0-b}(m-1) + CA - R_{1-b}(m-1).$$

The aim of this paper is to give some recursive formulas for transition diagrams of cellular automata $CA - R_{a-b}(m)$. The following facts is obvious. Transition diagrams of $CA - 0_{a-b}(m)$ and $CA - 255_{a-b}(m)$ are isomorphic to $[2^m]$. Those of $CA - 204_{a-b}(m)$ consists of 2^m fixed points, that is, $CA - 204_{a-b}(m) = 2^m[1]$, and those of $CA - 51_{a-b}(m)$ consists 2^{m-1} cycles of period length two. Also those of $CA - 15_{a-b}(m)$ and $CA - 170_{a-b}(m)$ are isomorphic to binary trees with a depth of m. For example, $CA - 170_{a-b}(3)$ is illustrated by the following tree:



In this paper we write $A(m) = CA-R_{0-0}(m)$, $B(m) = CA-R_{0-1}(m)$, $C(m) = CA\text{–}R_{1-0}(m)$, $D(m) = CA\text{–}R_{1-1}(m)$, $X(m) = CA\text{–}R_{0-b}(m)$ and $Y(m) = CA\text{–}R_{1-b}(m)$ $(a = 0, 1)$, unless no confusion occurs. Moreover we set $A(0) = B(0) = C(0) = D(0) = [1]$ for convenience.

2. Tree and Cycle Expressions

In this section we introduce tree and cycle expressions in order to algebraically represent tree structures of finite dynamical systems.

DEFINITION 2.1. Tree expressions and cycle expressions to represent the graphical structure of finite dynamical systems are recursively defined as follows:

(a) If $E_1, \cdots, E_k$ $(k \geq 0)$ are expressions and n is a positive integer such that $k \leq n$, then $[n + E_1 + \cdots + E_k]$ is an expression.

(b) If $E_1, \cdots, E_k$ $(k \geq 0)$ are expressions and n is a positive integer such that $k < n$, then an expression $[n + E_1 + \cdots + E_k]$ is a tree expression. (Note that tree expressions are expressions.)

(c) All tree expressions are cycle expressions.

(d) If $T_1, \cdots, T_k$ $(k \geq 2)$ are tree expressions, then $< T_1, \cdots, T_k >$ is a cycle expression.

(e) If $S_1, \cdots, S_k$ are cycle expressions, then $S_1 + \cdots + S_k$ is a cycle expression.

Define formal expressions F_m by $F_1 = 0$ and $F_{m+1} = 1 + [F_m]$ for $m \geq 1$, and define tree expressions $[1]^m$ by $[1]^m = [1 + F_m]$. (The notation $[0] = 0$ is a conventional symbol representing the empty tree.) For example, $[1]^1 = [1]$, $[1]^2 = [2]$, $[1]^3 = [2 + [1]]$ and $[1]^4 = [2 + [1 + [1]]]$.

Define formal expressions H_m by $H_0 = 0$ and $H_{m+1} = 1 + [2H_m]$ for $m \geq 0$, and define tree expressions T_m by $T_m = [1 + H_m]$. For example, $T_0 = [1]$, $T_1 = [2]$, $T_2 = [2 + [2]]$ and $T_3 = [2 + [2 + 2[2]]]$.
Then it is easy to see that the transition diagrams of $CA\text{–}15_{a-b}(m)$ and $CA\text{–}170_{a-b}(m)$

are represented by a tree expression T_m.

Now we define two functions H and L to extract major informations for transient lengths and limit cycles of cycle expressions:

(a) $H([1]) = 0$ and $H([n]) = 1$ if $n \geq 2$,

(b) $H([n + E_1 + \cdots + E_k]) = 1 + \max\{H(E_1), \cdots, H(E_k)\}$

(c) $H(< T_1, \cdots, T_k >) = \max\{H(T_1), \cdots, H(T_k)\}$

(d) $H(S_1 + \cdots + S_k) = \max\{H(S_1), \cdots, H(S_k)\}$

(e) $L(T) =< 1 >$,

(f) $L(< T_1, \cdots, T_k >) =< k >$,

(g) $L(S_1 + \cdots + S_k) = L(S_1) + \cdots + L(S_k)$.

The function H assigns to a cycle expression S the greatest length of paths reaching to the first root of S. Usually $H(S)$ is called the transient length (or the height) of a cycle expression S. The function L extracts just information for limit cycles of cycle expressions neglecting tree structures. It is known that $L(ca - 5_{0-0}(m)) = \gamma_1(m) < 1 > + \gamma_2(m) < 2 >$, where $\gamma_1(m) = \gamma_1(m-2) + \gamma_1(m-3)$ and $\gamma_2(m) = \gamma_2(m-1) + \gamma_2(m-2) + \gamma_2(m-3) - \gamma_2(m-4) + \gamma_2(m-5) - \gamma_2(m-6) + \gamma_1(m-5)$.

A notion of cartesian products of dynamical systems is an elementary operation to induce higher dimensional ones. The following theorem gives a very simple formula for formation of cartesian products of cycle expressions:

THEOREM 2.2. *[Product Formula]*

(a)

$$[m] \times [n + E_1 + \cdots + E_k] = [mn + [m] \times E_1 + \cdots + [m] \times E_k],$$

(b)

$$[m+D_1+\cdots+D_h]\times[n+E_1+\cdots+E_k] = [mn+\sum_{i=1}^{k}[m]\times E_i+\sum_{j=1}^{h} D_j\times[n]+\sum_{i=1,j=1}^{k,h} D_j\times E_i],$$

(c)

$$T\times < T_1, \cdots, T_p >=< T \times T_1, \cdots, T \times T_p > .$$

The next operations results a new expression after simultaneously attaching an expression to all leaves of another expression:

(a) $[n + E_1 + \cdots + E_k] * E = [n + (n-k)E + E_1 * E + \cdots + E_k * E]$,

(b) $[n + E_1 + \cdots + E_k]E = [n + (n-k-1)E + E_1 * E + \cdots + E_k * E]$.

For example, $[1]*[1] = [1+[1]]$, $[1+[1]]*[1] = [1+[1]*[1]] = [1+[1+[1]]]$, $[2]*[2] = [2+2[2]]$ and $[2+2[2]]*[2] = [2+2[2+2[2]]]$.

Recall a canonical dynamical system (N^*, ρ) over the set N^* of all finite strings consisting of positive integers. The transition function ρ is defined by $\rho(i_1 i_2 \cdots i_s) = i_1 \cdots i_{s-1}$ if $s > 0$ and $\rho(\varepsilon) = \varepsilon$, where ε is a null string. The following function D naturally assigns to a tree expression a subsystem of the canonical system (N^*, ρ).

(a) $V([n + E_1 + \cdots + E_k]) = \{1, \cdots, n\} \cup 1 \cdot V(E_1) \cup \cdots \cup k \cdot V(E_k)$,

(b) $D([n + E_1 + \cdots + E_k]) = \{\varepsilon\} \cup V([n - 1 + E_1 + \cdots + E_k])$.

The following example illustrates how D behaves:

$$D([2 + [2 + [2] + [2]]]) = \{\varepsilon, 1, 11, 12, 111, 112, 121, 122\}.$$

It is easy to see that the function D can be extended a function on cycle expressions.

To formulate recursive formulas of transition diagrams for finite cellular automata we need formal expressions with root. Because infromations on roots of cycles are generally indispensable for such recursive formulas.

DEFINITION 2.3. Tree expressions and cycle expressions with root in a set X are recursively defined as follows:

(a) If $T_1, \cdots, T_k$ are expressions with root, n is a nonnegative integer and x is an element of X, then $[n + T_1 + \cdots + T_k]_x$ is a tree expression with root.

(b) If $T_1, \cdots, T_k$ are normal expressions with root, n is a nonnegative integer such that $0 \le k \le n$ and x is an element of X, then $[n + T_1 + \cdots + T_k]_x$ is a normal expression with root. (Note that normal expressions are expressions.)

(c) If $T_1, \cdots, T_k$ are normal expressions with root, n is a positive integer such that $0 \le k < n$ and x is an element of X, then an expression $[n + T_1 + \cdots + T_k]_x$ is a tree expression with root.

(d) If $T_1, \cdots, T_k$ are tree expressions with root, then $< T_1, \cdots, T_k >$ is a cycle expression with root.

(e) If $S_1, \cdots, S_k$ are cycle expressions with root, then $S_1 + \cdots + S_k$ is a cycle expression with root.

For expressions with root over the set of all strings consisting of two symbols 0 and 1 we assume the following addition and product rules to manipulate recursive formulas for transition diagrams of finite cellular automata.

Addition Rule
$[n+T_1+\cdots+T_k]_x+[m+U_1+\cdots+U_h]_x=[n+m+T_1+\cdots+T_k+U_1+\cdots+U_h]_x.$

Product Rule
$[m]_y\times[n+T_1+\cdots+T_k]_x=[mn+[m]_y\times T_1+\cdots+[m]_y\times T_k]_{yx}$, where yx denotes the concatenation of y followed by x.

We freely use these manipulation axioms in recursive formulas obtained in the later sections.

3. CA–12

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-12_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 12 is illustrated by the following figure:

Rule 12	111	110	101	100	011	010	001	000
	0	0	0	0	1	1	0	0

The symmetric, the complemented, the symmetric complemented rule numbers to 12 are 68, 107 and 121, respectively.

LEMMA 3.1. *Assume $y=\delta_{a-b}(x)$ for two configurations $x=x_1x_2\cdots x_m$ and $y=y_1y_2\cdots y_m$ of $CA-12_{a-b}(m)$. Then $x_kx_{k+1}=01$ if and only if $y_ky_{k+1}=01$ for $k=1,2,\cdots,m-1$.*

Proof. Assume $x_kx_{k+1}=01$. Then $y_k=f(x_{k-1}01)=0$ by $f(x01)=0$ and $y_{k+1}=f(01x_{k+2})=1$ by $f(01z)=1$. Conversely assume $y_ky_{k+1}=01$. Then $x_kx_{k+1}x_{k+2}=011$ or 010 by $y_{k+1}=f(x_kx_{k+1}x_{k+2})=1$. □

THEOREM 3.2. *For cellular automata $CA-12$ the following formulas hold:*

(a) $B(m)=A(m)$ *and* $D(m)=C(m)$,

(b) $A(m)=A(m-1)+C(m-1)$,

(c) $C(m)=[m+1]+\sum_{k=1}^{m-1}[k]\times C(m-k-1)$.

Proof. (a) Since $f(xy0)=f(xy1)$ it is trivial that $CA-12_{a-0}(m)=CA-12_{a-1}(m)$ for $a=0,1$. (b) Define two functions $i_m: A(m-1)\to A(m)$ and $j_m: C(m-1)\to A(m)$ by $i_m(x_1x_2\cdots x_{m-1})=0x_1x_2\cdots x_{m-1}$ and $j_m(x_1x_2\cdots x_{m-1})=1x_1x_2\cdots x_{m-1}$. As $f(00z)=0$ and $f(01z)=1$, i_m and j_m are dynamorphisms of dynamical systems. Finally the images of i_m and j_m give a partition of the configuration set Q^m of $A(m)$. (c) Define subsets $H_0,H_1,\cdots,H_{m-1}$ of the configuration set Q^m of $C(m)$ as follows. H_0 is the set of all configurations containing no subsequence 01, and H_k $(k=1,2,\cdots,m-1)$ is the set of all configurations $x=x_1x_2\cdots x_m$ such that $x_kx_{k+1}=01$ and $x_1x_2\cdots x_{k-1}$ contains no subsequence 01. Then using Lemma 3.1 it is easy to see that $C(m)=$

$H_0 + H_1 + \cdots + H_{m-1}$ (disjoint union of dynamical systems). On the other hand $H_0 = \{1^p 0^{m-p} | 0 \le p \le m\}$ and $\delta_{1-0}(x) = 0^m$ for all $x \in H_0$. Hence $H_0 = [m+1]$ and $H_k = [k] \times C(m-k-1)$. □

Using the last theorem we can recursively compute transition diagrams of $CA-12_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1].$

$A(1) = A(0) + C(0) = 2[1]$, $C(1) = [2]$,

$A(2) = A(1) + C(1) = [2] + 2[1]$,
$C(2) = [3] + [1] \times C(0) = [3] + [1]$,

$A(3) = A(2) + C(2) = [3] + [2] + 3[1]$,
$C(3) = [4] + [1] \times C(1) + [2] \times C(0) = [4] + [2] + [2] = [4] + 2[2]$,

$$\begin{aligned} A(4) &= A(3) + C(3) = [4] + [3] + 3[2] + 3[1], \\ C(4) &= [5] + [1] \times C(2) + [2] \times C(1) + [3] \times C(0) \\ &= [5] + [1] \times ([3] + [1]) + [2] \times [2] + [3] \times [1] \\ &= [5] + [4] + 2[3] + [1]. \end{aligned}$$

4. CA–200

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA\text{–}200_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 200 is illustrated by the following figure:

Rule 200	111	110	101	100	011	010	001	000
	1	1	0	0	1	0	0	0

The symmetric and the complemented rule numbers to 200 are 200 and 236, respectively.

LEMMA 4.1. *Assume* $y = \delta_{a-b}(x)$ *for two configurations* $x = x_1 x_2 \cdots x_m$ *and* $y = y_1 y_2 \cdots y_m$ *of* $CA - 200_{a-b}(m)$. *Then* $x_k x_{k+1} = 11$ *if and only if* $y_k y_{k+1} = 11$ *for* $k = 1, 2, \cdots, m-1$.

Proof. Assume $x_k x_{k+1} = 11$. Then $y_k = f(x_{k-1}11) = 1$ by $f(x11) = 1$ and $y_{k+1} = f(11x_{k+2}) = 1$ by $f(11z) = 1$. Conversely assume $y_k y_{k+1} = 11$. Then $x_k x_{k+1} x_{k+2} = 111$, 110 or 011 by $y_{k+1} = f(x_k x_{k+1} x_{k+2}) = 1$. But if $x_k x_{k+1} x_{k+2} = 011$, then $y_k = f(x_{k-1}01) = 0$. Hence $x_k x_{k+1} = 11$. □

THEOREM 4.2. *For cellular automata* $CA - 200$ *the following formulas hold:*

(a) $B(m) = C(m)$ (Symmetry),

(b) $C(m) = A(m-1) + C(m-1)$ *and* $D(m) = B(m-1) + D(m-1)$,

(c) $A(m) = [a_{m+1}] + \sum_{k=0}^{m-2} [a_k] \times C(m-k-2)$,

where $a_0 = a_1 = 1$, $a_k = a_{k-1} + a_{k-2}$ $(k \geq 2)$.

Proof. (a) It is trivial from the symmetry $f(xyz) = f(zyx)$.
(b) We will show that $Y(m) = X(m-1) + Y(m-1)$, where $X(m) = CA{-}200_{0-a}(m)$ and $Y(m) = CA-200_{1-a}(m)$. Define two functions $i_m : X(m-1) \to Y(m)$ and $j_m : Y(m-1) \to Y(m)$ by $i_m(x_1x_2\cdots x_{m-1}) = 0x_1x_2\cdots x_{m-1}$ and $j_m(x_1x_2\cdots x_{m-1}) = 1x_1x_2\cdots x_{m-1}$. As $f(10z) = 0$ and $f(11z) = 1$, i_m and j_m are dynamorphisms of dynamical systems. Finally note that the images of i_m and j_m give a partition of the configuration set Q^m of $Y(m)$. (c) Define subsets $H, H_0, H_1, \cdots, H_{m-2}$ of the set Q^m of all configurations of $A(m)$ as follows. H is the set of all configurations containing no subsequence 11, and H_k $(k = 0, 1, \cdots, m-2)$ is the set of all configurations $x = x_1x_2\cdots x_m$ such that $x_kx_{k+1}x_{k+2} = 011$ and $x_1x_2\cdots x_{k-1}$ contains no subsequence 11. Then using Lemma 4.1 it is easy to see that $A(m) = H + H_0 + H_1 + \cdots + H_{m-2}$ (disjoint union of dynamical systems). On the other hand $|H| = a_{m+1}$ $(= \sum_{j=0}^{\infty} \binom{m-j+1}{j})$ and $\delta_{1-0}(x) = 0^m$ for all $x \in H$. Hence $H = [a_{m+1}]$ and $H_k = [a_k] \times C(m-k-2)$. □

Using the last theorem we can recursively compute transition diagrams of $CA-200_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]$.

$C(1) = A(0) + C(0) = 2[1]$, $D(1) = C(0) + D(0) = 2[1]$, $A(1) = [a_2] = [2]$.

$C(2) = A(1) + C(1) = [2] + 2[1]$,
$D(2) = C(1) + D(1) = 4[1]$,
$A(2) = [a_3] + [a_0] \times C(0) = [3] + [1] \times [1] = [3] + [1]$.

$C(3) = A(2) + C(2) = [3] + [2] + 3[1]$,
$D(3) = C(2) + D(2) = [2] + 6[1]$,
$A(3) = [a_4] + [a_0] \times C(1) + [a_1] \times C(0) = [5] + [1] \times 2[1] + [1] \times [1] = [5] + 3[1]$,

$C(4) = A(3) + C(3) = [5] + [3] + [2] + 6[1]$,
$D(4) = C(3) + D(3) = [3] + 2[2] + 9[1]$,
$$\begin{aligned} A(4) &= [a_5] + [a_0] \times C(2) + [a_1] \times C(1) + [a_2] \times C(0) \\ &= [8] + [1] \times ([2] + 2[1]) + [1] \times 2[1] + [2] \times [1] \\ &= [8] + 2[2] + 4[1]. \end{aligned}$$

5. CA–140

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA\text{--}140_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 140 is illustrated by the following figure:

Rule 140	111	110	101	100	011	010	001	000
	1	0	0	0	1	1	0	0

The symmetric, the complemented and the symmetric complemented rule numbers to 140 are 196, 206 and 220, respectively.

LEMMA 5.1. *Assume $y = \delta_{a-b}(x)$ for two configurations $x = x_1x_2\cdots x_m$ and $y = y_1y_2\cdots y_m$ of $CA-140_{a-b}(m)$. Then $x_kx_{k+1} = 01$ if and only if $y_ky_{k+1} = 01$ for $k = 1, 2, \cdots, m-1$.*

Proof. Assume $x_kx_{k+1} = 01$. Then $y_k = f(x_{k-1}01) = 0$ by $f(x01) = 0$ and $y_{k+1} = f(01x_{k+2}) = 1$ by $f(01z) = 1$. Conversely assume $y_ky_{k+1} = 01$. Then $x_kx_{k+1}x_{k+2} = 111$, 011 or 010 by $y_{k+1} = f(x_kx_{k+1}x_{k+2}) = 1$. But if $x_kx_{k+1}x_{k+2} = 111$, then $y_k = f(x_{k-1}11) = 1$. Hence $x_kx_{k+1} = 01$. □

THEOREM 5.2. *For cellular automata $CA-140$ the following formulas hold:*

(a) $A(m) = A(m-1) + C(m-1)$ *and* $B(m) = B(m-1) + D(m-1)$,

(b) $B(m) = A(m-1) + B(m-1)$ *and* $D(m) = C(m-1) + D(m-1)$,

(c) $C(m) = [1]^{m+1} + \sum_{k=1}^{m-1}[1]^k \times C(m-k-1)$.

Proof. (a) Set $X(m) = CA-140_{0-a}(m)$ and $Y(m) = CA-140_{1-a}(m)$. Then two functions $i_m : X(m-1) \to X(m)$ and $j_m : Y(m-1) \to X(m)$ with $i_m(x_1x_2\cdots x_{m-1}) = 0x_1x_2\cdots x_{m-1}$ and $j_m(x_1x_2\cdots x_{m-1}) = 1x_1x_2\cdots x_{m-1}$ are dynamorphisms of dynamical systems from $f(00z) = 0$ and $f(01z) = 1$. Then we can easily show that $X(m) = X(m-1) + Y(m-1)$.
(b) Remark that two functions $u_m : CA-140_{a-0}(m-1) \to CA-140_{a-1}(m)$ and $v_m : CA-140_{a-1}(m-1) \to CA-140_{a-1}(m)$ with $u_m(x_1x_2\cdots x_{m-1}) = x_1x_2\cdots x_{m-1}0$ and $v_m(x_1x_2\cdots x_{m-1}) = x_1x_2\cdots x_{m-1}1$ are dynamorphisms of dynamical systems from $f(x01) = 0$ and $f(x11) = 1$. Then we can easily show that $CA-140_{a-1}(m) = CA-140_{a-0}(m-1) + CA-140_{a-1}(m-1)$.
(c) Define subsets $H_0, H_1, \cdots, H_{m-1}$ of the set Q^m of all configurations of $C(m)$ as follows. H_0 is the set of all configurations containing no subsequence 01, and H_k $(k = 1, 2, \cdots, m-1)$ is the set of all configurations $x = x_1x_2\cdots x_m$ such that $x_kx_{k+1} = 01$ and $x_1x_2\cdots x_{k-1}$ contains no subsequence 01. Then using Lemma 5.1 it is easy to see that $C(m) = H_0 + H_1 + \cdots + H_{m-1}$ (disjoint union of dynamical systems). On the other hand $H_0 = \{1^p0^{m-p} | 0 \leq p \leq m\}$ and $\delta_{1-0}(1^p0^{m-p}) = 1^{p-1}0^{m-p+1}$. Hence $H_0 = [1]^{m+1}$ and $H_k = [1]^k \times C(m-k-1)$ for $k \geq 1$. □

Remark. $A(m) = D(m)$ by 5.2(a) and (b).

Using the last theorem we can recursively compute transition diagrams of $CA-140_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]$.

$A(1) = B(1) = D(1) = [1] + [1] = 2[1]$, $C(1) = [1]^2$.

$$\begin{aligned}
A(2) &= A(1) + C(1) = [1]^2 + 2[1],\\
B(2) &= A(1) + B(1) = 4[1],\\
D(2) &= C(1) + D(1) = [1]^2 + 2[1],\\
C(2) &= [1]^3 + [1] \times C(0) = [1]^3 + [1].
\end{aligned}$$

$$\begin{aligned}
A(3) &= A(2) + C(2) = [1]^3 + [1]^2 + 3[1],\\
B(3) &= A(2) + B(2) = [1]^2 + 6[1],\\
D(3) &= C(2) + D(2) = [1]^3 + [1]^2 + 3[1],\\
C(3) &= [1]^4 + [1] \times C(1) + [1]^2 \times C(0) = [1]^4 + 2[1]^2.
\end{aligned}$$

$$\begin{aligned}
A(4) &= A(3) + C(3) = [1]^4 + [1]^3 + 3[1]^2 + 3[1],\\
B(4) &= A(3) + B(3) = [1]^3 + 2[1]^2 + 9[1],\\
D(4) &= C(3) + D(3) = [1]^4 + [1]^3 + 3[1]^2 + 3[1],\\
C(4) &= [1]^5 + [1] \times C(2) + [1]^2 \times C(1) + [1]^3 \times C(0)\\
&= [1]^5 + [1]^3 + [1]^2 \times [1]^2 + [1]^3 + [1].
\end{aligned}$$

6. CA–4

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-4_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 4 is illustrated by the following figure:

Rule 4	111	110	101	100	011	010	001	000
	0	0	0	0	0	1	0	0

The symmetric and the complemented rule numbers to 4 are 4 and 223, respectively.

The recursive formulas for transition diagrams of cellular automata appearing hereafter require tree and cycle expressions with roots.

THEOREM 6.1. *For cellular automata $CA-4$ the following formulas hold:*

(a) $B(m) = C(m)$ (Symmetry),

(b) $X(m) = [1]_0 \times X(m-1) + [1]_{10} \times X(m-2) + [1]_{00} \times Y(m-2)$.

(c) $Y(m) = [1]_0 \times X(m-1) + [1]_0 \times Y(m-1)$,

Proof. (a) It is trivial from the symmetry $f(xyz) = f(zyx)$.
(b) Remark that two functions $u_m : X(m-1) \to X(m)$ with $u_m(x_1x_2\cdots x_{m-1}) = 0x_1x_2\cdots x_{m-1}$ and $v_m : X(m-2) \to X(m)$ with $v_m(x_1x_2\cdots x_{m-2}) = 10x_1x_2\cdots x_{m-2}$ are dynamorphisms by $f(00z) = 0$, $f(010) = 1$ and $f(10z) = 0$. Next note that if $\delta_{1-a}(x) = y$ in $Y(m-2)$ then $\delta_{0-a}(0y) = 0y$ in $X(m-1)$ and $\delta_{0-a}(11x) = 00y$ in $X(m)$. This proves $[1]_{00} \times [p]_x + [1]_0 \times [q]_{0x} = [p+q]_{00x}$.
(c) Remark that a function $j_m : X(m-1) \to Y(m)$ with $j_m(x_1x_2\cdots x_{m-1}) = 0x_1x_2\cdots x_{m-1}$ is a dynamorphism by $f(10z) = 0$. Next note that if $\delta_{1-a}(y) = x$ in $Y(m-1)$ then $\delta_{0-a}(x) = x$ in $X(m-1)$ and $\delta_{0-a}(1y) = 0x$ in $X(m)$. This proves $[1]_0 \times [p]_x + [1]_0 \times [q]_x = [p+q]_{0x}$. □

Using the last theorem we can recursively compute transition diagrams of $CA-4_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$, $A(1) = [1]_0 + [1]_1$.

$C(1) = [1]_0 \times [1]_\varepsilon + [1]_0 \times [1]_\varepsilon = [1]_0 + [1]_0 = [2]_0$, $B(1) = [2]_0$,
$D(1) = [1]_0 \times [1]_\varepsilon + [1]_0 \times [1]_\varepsilon = [1]_0 + [1]_0 = [2]_0$.

$C(2) = [1]_0 \times [2]_0 + [1]_0 \times ([1]_0 + [1]_1) = [2]_{00} + ([1]_{00} + [1]_{01}) = [3]_{00} + [1]_{01}$,
$D(2) = [1]_0 \times [2]_0 + [1]_0 \times [2]_0 = [2]_{00} + [2]_{00} = [4]_{00}$,
$A(2) = [1]_{00} \times [1]_\varepsilon + [1]_0 \times ([1]_0 + [1]_1) + [1]_{10} \times [1]_\varepsilon = [1]_{00} + ([1]_{00} + [1]_{01}) + [1]_{10}$
$\quad = [2]_{00} + [1]_{01} + [1]_{10}$,
$B(2) = [1]_{00} \times [1]_\varepsilon + [1]_0 \times [2]_0 + [1]_{10} \times [1]_\varepsilon = [1]_{00} + [2]_{00} + [1]_{10} = [3]_{00} + [1]_{10}$.

$C(3) = [1]_0 \times ([3]_{00} + [1]_{01}) + [1]_0 \times ([2]_{00} + [1]_{01} + [1]_{10})$
$\quad = ([3]_{000} + [1]_{001}) + ([2]_{000} + [1]_{001} + [1]_{010})$
$\quad = [5]_{000} + [2]_{001} + [1]_{010}$,
$D(3) = [1]_0 \times [4]_{00} + [1]_0 \times ([3]_{00} + [1]_{10}) = [4]_{000} + ([3]_{000} + [1]_{010}) = [7]_{000} + [1]_{010}$,
$A(3) = [1]_{00} \times [2]_0 + [1]_0 \times ([2]_{00} + [1]_{01} + [1]_{10}) + [1]_{10} \times ([1]_0 + [1]_1)$
$\quad = [2]_{000} + ([2]_{000} + [1]_{001} + [1]_{010}) + ([1]_{100} + [1]_{101})$
$\quad = [4]_{000} + [1]_{001} + [1]_{010} + [1]_{100} + [1]_{101}$,
$B(3) = [1]_{00} \times [2] + [1]_0 \times ([3]_{00} + [1]_{10}) + [1]_{10} \times [2]_0 = [2]_{000} + ([3]_{000} + [1]_{010}) + [2]_{100}$
$\quad = [5]_{000} + [1]_{010} + [2]_{100}$.

7. CA–76

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-76_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 76 is illustrated by the following figure:

Rule 76	111	110	101	100	011	010	001	000
	0	1	0	0	1	1	0	0

The symmetric and the complemented rule numbers to 76 are 76 and 205, respectively.

THEOREM 7.1. *For cellular automata $CA-76$ the following formulas hold:*

(a) $C(m) = B(m)$ (Symmetry),

(b) $X(m) = [1]_0 \times X(m-1) + [1]_1 \times Y(m-1)$,

(c) $D(m) = [1]_{0^m} + [1]_0 \times B(m-1) + [1]_{10} \times B(m-2) + \sum_{k=3}^{m} [1]_{0^{k-2}10} \times B(m-k)$.

Proof. (a) It is trivial from the symmetry $f(xyz) = f(zyx)$.
(b) Remark that two functions $i_m : X(m-1) \to X(m)$ and $j_m : Y(m-1) \to X(m)$ with $i_m(x_1x_2\cdots x_{m-1}) = 0x_1x_2\cdots x_{m-1}$ and $j_m(x_1x_2\cdots x_{m-1}) = 1x_1x_2\cdots x_{m-1}$ are dynamorphisms of dynamical systems from $f(00z) = 0$ and $f(01z) = 1$. Then we can easily show that $X(m) = [1]_0 \times X(m-1) + [1]_1 \times Y(m-1)$.
(c) Remark that two functions $u_m : B(m-2) \to D(m)$ with $u_m(x_1x_2\cdots x_{m-2}) = 10x_1x_2\cdots x_{m-2}$ and $v_m : B(m-1) \to D(m)$ with $v_m(x_1x_2\cdots x_{m-1}) = 0x_1x_2\cdots x_{m-1}$ are dynamorphisms by $f(110) = 1$ and $f(10z) = 0$. Next note that if $\delta_{0-1}(y) = x$ in $B(k)$ $(k = 0, 1, \cdots, m-3)$ then $\delta_{0-1}(0^{m-k-3}10x) = 0^{m-k-3}10x$ in $B(m-1)$ and $\delta_{1-1}(1^{m-k-1}0y) = 0^{m-k-2}10x$ in $D(m)$. This proves ${}_{1^{m-k-1}0}[p]_x + {}_0[q]_{0^{m-k-3}10x} = [p+q]_{0^{m-k-2}10x}$. Finally it is easy to see that $\delta_{1-1}(1^m) = 0^m$. □

Using the last theorem we can recursively compute transition diagrams of $CA-76_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$, $D(1) = [2]_0$, $D(2) = [2]_{00} + [1]_{01} + [1]_{10}$.

$A(1) = [1]_0 \times A(0) + [1]_1 \times C(0) = [1]_0 + [1]_1$,
$B(1) = [1]_0 \times B(0) + [1]_1 \times D(0) = [1]_0 + [1]_1$,
$C(1) = B(1) = [1]_0 + [1]_1$.

$A(2) = [1]_0 \times A(1) + [1]_1 \times C(1) = [1]_{00} + [1]_{01} + [1]_{10} + [1]_{11}$,
$B(2) = [1]_0 \times B(1) + [1]_1 \times D(1) = [1]_{00} + [1]_{01} + [2]_{10}$,
$C(2) = B(2) = [1]_{00} + [2]_{01} + [1]_{10}$.

$A(3) = [1]_0 \times A(2) + [1]_1 \times C(2) = [1]_{000} + [1]_{001} + [1]_{010} + [1]_{011} + [1]_{100} + [2]_{101} + [1]_{110}$,
$B(3) = [1]_0 \times B(2) + [1]_1 \times D(2) = [1]_{000} + [1]_{001} + [2]_{010} + [2]_{100} + [1]_{101} + [1]_{110}$,
$C(3) = B(3) = [1]_{000} + [2]_{001} + [2]_{010} + [1]_{011} + [1]_{100} + [1]_{101}$,
$D(3) = [1]_{000} + [1]_{010} \times B(0) + [1]_0 \times B(2) + [1]_{10} \times B(1)$
$= [1]_{000} + [1]_{010} + ([1]_{000} + [1]_{001} + [2]_{010}) + ([1]_{100} + [1]_{101})$
$= [2]_{000} + [1]_{001} + [3]_{010} + [1]_{100} + [1]_{101}$.

8. CA–29

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA{-}29_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 29 is illustrated by the following figure:

Rule 29	111	110	101	100	011	010	001	000
	0	0	0	1	1	1	0	1

The symmetric and the complemented rule numbers to 29 are 71 and 71, respectively.

THEOREM 8.1. *For cellular automata $CA-29$ the following formulas hold:*

(a) $C(m) = B(m)$ (Symmetry),

(b) $X(m) = [1]_{1^{m-1}\overline{a}} + [1]_1 \times Y(m-1) + [1]_{01} \times Y(m-2) + \sum_{k=3}^{m}[1]_{1^{k-2}01} \times Y(m-k)$,

(c) $Y(m) = [1]_{1^{m-1}\overline{a}} + [1]_0 \otimes Y(m-1) + [1]_{01} \times Y(m-2) + \sum_{k=3}^{m}[1]_{1^{k-2}01} \times Y(m-k)$,

Proof. (a) $< 0^k1 * \cdots * >$
If $\delta_{1-a}(x) = y$ in $Y(m-1)$, then $\delta_{0-a}(1x) = 1y$ in $X(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-k)$ for $k \geq 2$, then $\delta_{0-a}(0^{k-1}1x) = 1^{k-2}01y$ in $X(m)$. $\delta_{0-a}(0^m) = 1^{m-1}\overline{a}$ and $\delta_{0-a}(1^{m-1}\overline{a}) = 10^{m-1}$ in $X(m)$.
(b) $< 0^k1 * \cdots * >$
If $\delta_{1-a}(x) = y$ in $Y(m-1)$, then $\delta_{1-a}(1x) = 0y$ in $Y(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-k)$ for $k \geq 2$, then $\delta_{1-a}(0^{k-1}1x) = 1^{k-2}01y$ in $Y(m)$. $\delta_{1-a}(0^m) = 1^{m-1}\overline{a}$ and $\delta_{0-a}(1^{m-1}\overline{a}) = 0^m$ in $Y(m)$. □

Using the last theorem we can recursively compute transition diagrams of $CA-29_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$.

$A(1) = [2]_1$, $C(1) =< [1]_0, [1]_1 >$, $B(1) = [1]_0 + [1]_1$, $D(1) = [2]_0$.

$A(2) = [1]_{01}+ < [1]_{10}, [2]_{11} >$, $C(2) =< [1]_{00}, [1]_{11} > +[2]_{01}$,
$B(2) = [3]_{10} + [1]_{01}$, $D(2) =< [2]_{00}, [1]_{10} > +[1]_{01}$.

$A(3) =< [1]_{010}, [1]_{011} > + < [1]_{100}, [2]_{111} > +[3]_{101}$,
$C(3) =< [1]_{000}, [1]_{111} > + < [2]_{001}, [1]_{101} > + < [1]_{010}, [2]_{011} >$,
$B(3) = [2]_{010}+ < [2]_{100}, [2]_{110} > +[2]_{101}$,
$D(3) =< [2]_{000}, [1]_{110} > + < [1]_{001}, [1]_{101} > +[3]_{010}$.

9. CA–8

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-8_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 8 is illustrated by the following figure:

Rule 8	111	110	101	100	011	010	001	000
	0	0	0	0	1	0	0	0

The symmetric, the complemented, the symmetric complemented rule numbers to 8 are 64, 239 and 253, respectively.

THEOREM 9.1. *For cellular automata $CA-8$ the following formulas hold:*

(a) $X(m) = [1]_0 \times X(m-1) + [1]_{00} \times X(m-2) + [1]_{10} \otimes Y(m-2)$,

(b) $Y(m) = [1]_0 \times X(m-1) + [1]_0 \times Y(m-1)$,

where $[p]_x \times_u [q]_y = [pq]_{xuy}$, $[1]_{10} \otimes [p + [q_1]_{x_1} + \cdots + [q_k]_{x_k}]_y = [[p]_{10y} + [q_1]_{10x_1} + \cdots + [q_k]_{10x_k}]_{00y}$.

Proof. (a) $< 0*\cdots*, 10*\cdots*, 11*\cdots* >$ If $\delta_{0-a}(x) = y$ in $X(m-1)$, then $\delta_{0-a}(0x) = 0y$ in $X(m)$. If $\delta_{0-a}(x) = y$ in $X(m-2)$, then $\delta_{0-a}(10x) = 00y$ in $X(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-2)$, then $\delta_{0-a}(11x) = 10y$ in $X(m)$ and so the transient length of $11x$ is equal to 2.
(b) $< 0*\cdots*, 1*\cdots* >$ If $\delta_{0-a}(x) = y$ in $X(m-1)$, then $\delta_{1-a}(0x) = 0y$ in $Y(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-1)$, then $\delta_{1-a}(1x) = 0y$ in $Y(m)$ and the transient length of $1x$ is equal to that of x. □

Using the last theorem we can recursively compute transition diagrams of $CA-8_{a-b}(m)$ as follows:

$A(0) = C(0) = B(0) = D(0) = [1]_\varepsilon$, $A(1) = [2]_0$, $B(1) = [1]_0 + [1]_1$.

$C(1) = [1]_0 \times A(0) + [1]_0 \times C(0) = [1]_0 + [1]_0 = [2]_0$,
$D(1) = [1]_0 \times B(0) + [1]_0 \times D(0) = [1]_0 + [1]_0 = [2]_0$.

$C(2) = [1]_0 \times [2]_0 + [1]_0 \times [2]_0 = [2]_{00} + [2]_{00} = [4]_{00}$,
$A(2) = [1]_0 \times [2]_0 + [1]_{00} \times [1]_\varepsilon + [1]_{10} \otimes [1]_\varepsilon = [2]_{00} + [1]_{00} + [[1]_{10}]_{00} = [3 + [1]_{10}]_{00}$,
$D(2) = [1]_0 \times ([1]_0 + [1]_1) + [1]_0 \times [2]_0 = [1]_{00} + [1]_{01} + [2]_{00} = [3]_{00} + [1]_{01}$,
$B(2) = [1]_0 \times ([1]_0 + [1]_1) + [1]_{00} \times [1]_\varepsilon + [1]_{10} \otimes [1]_\varepsilon = [1]_{00} + [1]_{01} + [1]_{00} + [[1]_{10}]_{00}$
$= [2 + [1]_{10}]_{00} + [1]_{01}$.

$C(3) = [1]_0 \times [3 + [1]_{10}]_{00} + [1]_0 \times [4]_{00} = [3 + [1]_{010}]_{000} + [4]_{000} = [7 + [1]_{010}]_{000}$,
$A(3) = [1]_0 \times [3 + [1]_{10}]_{00} + [1]_{00} \times [2]_0 + [1]_{10} \otimes [2]_0 = [3 + [1]_{010}]_{000} + [2]_{000} + [[2]_{100}]_{000}$
$= [5 + [1]_{010} + [2]_{100}]_{000}$,

$$
\begin{aligned}
D(3) &= [1]_0 \times ([2+[1]_{10}]_{00} + [1]_{01}) + [1]_0 \times ([3]_{00} + [1]_{01}) \\
&= [2+[1]_{010}]_{000} + [1]_{001} + [3]_{000} + [1]_{001} \\
&= [5+[1]_{010}]_{000} + [2]_{001}, \\
B(3) &= [1]_0 \times ([2+[1]_{10}]_{00} + [1]_{01}) + [1]_{00} \times ([1]_0 + [1]_1) + [1]_{10} \otimes [2]_0 \\
&= [2+[1]_{010}]_{000} + [1]_{001} + [1]_{000} + [1]_{001} + [[2]_{100}]_{000} \\
&= [3+[1]_{010} + [2]_{100}]_{000} + [2]_{001}.
\end{aligned}
$$

10. CA–72

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA{-}72_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 72 is illustrated by the following figure:

Rule 72	111	110	101	100	011	010	001	000
	0	1	0	0	1	0	0	0

The symmetric and the complemented rule numbers to 72 are 72 and 237, respectively.

THEOREM 10.1. *For cellular automata $CA{-}72$ the following formulas hold:*

(a) $C(m) = B(m)$ (Symmetry),

(b) $X(m) = [[1]_{10^{m-1}\bar{a}}]_{0^m} + [1]_0 \times X(m-1) + [1]_{00} \times X(m-2) + [1]_{110} \times X(m-3) + \sum_{k=4}^{m} [1]_{10^{k-3}10} \otimes X(m-k)$,

(c) $Y(m) = [[1]_{10^{m-1}\bar{a}}]_{0^m} + [1]_0 \times X(m-1) + [1]_{10} \times X(m-2) + \sum_{k=3}^{m} [1]_{0^{k-2}10} \otimes X(m-k)$,

where $[1]_{10^{k-3}10} \otimes [p + [q_1]_{x_1} + \cdots + [q_k]_{x_k}]_y = [[p]_{10^{k-3}10y} + [q_1]_{10^{k-3}10x_1} + \cdots + [q_k]_{10^{k-3}10x_k}]_{0^k}$

Proof. (b) $< 1^k0 * \cdots * >$
If $\delta_{0-a}(x) = y$ in $X(m-1)$, then $\delta_{0-a}(0x) = 0y$ in $X(m)$. If $\delta_{0-a}(x) = y$ in $X(m-2)$, then $\delta_{0-a}(10x) = 00y$ in $X(m)$. If $\delta_{0-a}(x) = y$ in $X(m-k)$ for $k \geq 3$, then $\delta_{0-a}(1^{k-1}0x) = 10^{k-3}10y$ in $X(m)$. $\delta_{0-a}(1^m) = 10^{m-2}\bar{a}$ and $\delta_{0-a}(10^{m-2}\bar{a}) = 0^m$ in $X(m)$.
(c) $< 1^k0 * \cdots * >$
If $\delta_{0-a}(x) = y$ in $X(m-1)$, then $\delta_{1-a}(0x) = 0y$ in $Y(m)$. If $\delta_{0-a}(x) = y$ in $X(m-k)$ for $k \geq 2$, then $\delta_{1-a}(1^{k-1}0x) = 0^{k-2}10y$ in $Y(m)$. $\delta_{0-a}(1^m) = 0^{m-1}\bar{a}$ and $\delta_{0-a}(0^{m-1}\bar{a}) = 0^m$ in $Y(m)$. □

Using the last theorem we can recursively compute transition diagrams of $CA{-}72_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$.

$A(1) = [2]_0,\ B(1) = [1]_0 + [1]_1,$
$C(1) = [1]_0 + [1]_1,$
$D(1) = [2]_0.$

$A(2) = [3]_{00} + [1]_{11},$
$B(2) = [2 + [1]_{10}]_{00} + [1]_{01},$
$C(2) = [2 + [1]_{01}]_{00} + [1]_{10},$
$D(2) = [2]_{00} + [1]_{01} + [1]_{10}.$

$A(3) = [5 + [1]_{101}]_{0^3} + [1]_{011} + [1]_{110},$
$B(3) = [3 + [1]_{010} + [1]_{100}]_{0^3} + [2]_{001} + [1]_{110},$
$C(3) = [3 + [1]_{001} + [1]_{010}]_{0^3} + [1]_{011},$
$D(3) = [3 + [2]_{010}]_{0^3} + [1]_{001} + [1]_{100} + [1]_{101}.$

11. CA–108

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA{-}108_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 108 is illustrated by the following figure:

Rule 108	111	110	101	100	011	010	001	000
	0	1	1	0	1	1	0	0

The symmetric and the complemented rule numbers to 108 are 108 and 201, respectively.

THEOREM 11.1. *For cellular automata $CA-108$ the following formulas hold:*

(a) $C(m) = B(m)$ (Symmetry),

(b) $X(m) = [1]_0 \times X(m-1) + [1]_1 \times Y(m-1),$

(c) $D(m) = [1]_{0^m} + [1]_{0^{m-2}11} + [1]_{00} \times B(m-2) + [1]_{11} \times D(m-2) + [1]_{100} \times B(m-3) + \sum_{k=4}^{m} [1]_{0^{k-3}100} \times B(m-k) + \sum_{k=3}^{m} [1]_{0^{k-3}111} \times D(m-k),$

Proof. (b) $< 0 * \cdots *, 1 * \cdots * >$
If $\delta_{0-a}(x) = y$ in $X(m-1)$, then $\delta_{0-a}(0x) = 0y$ in $X(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-1)$, then $\delta_{1-a}(1x) = 1y$ in $X(m)$.
(c) $< 1^k 00 * \cdots *, 1^k 01 * \cdots * >$
If $\delta_{0-a}(x) = y$ in $X(m-2)$, then $\delta_{1-a}(00x) = 00y$ in $Y(m)$. If $\delta_{0-a}(x) = y$ in $X(m-k)$ for $k \geq 3$, then $\delta_{1-a}(1^{k-2}00x) = 0^{k-3}100y$ in $Y(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-2)$, then $\delta_{1-a}(01x) = 11y$ in $Y(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-k)$ for $k \geq 3$, then $\delta_{1-a}(1^{k-2}01x) = 0^{k-3}111y$ in $Y(m)$. □

Using the last theorem we can recursively compute transition diagrams of $CA-108_{a-b}(m)$ as follows:

$A(0) = C(0) = B(0) = D(0) = [1]_\varepsilon.$
$A(1) = [1]_0 + [1]_1,\ C(1) = [1]_0 + [1]_1,$
$B(1) = [1]_0 + [1]_1,$
$D(1) = [1]_0 + [1]_1.$

$A(2) = [1]_{00} + [1]_{01} + [1]_{10} + [1]_{11}$
$C(2) = [1]_{00} + ([1]_{01} + [1]_{11}) + [1]_{10}$
$B(2) = [1]_{00} + [1]_{01} + ([1]_{10} + [1]_{11})$
$D(2) = [2 + [2]_{11}]_{00}$

$$\begin{aligned}
A(3) &= [1]_0 \times A(2) + [1]_1 \times C(2)\\
&= [1]_0 \times ([1]_{00} + [1]_{01} + [1]_{10} + [1]_{11}) + [1]_1 \times ([1]_{00} + ([1]_{01} + [1]_{11}) + [1]_{10})\\
&= [1]_{000} + [1]_{001} + [1]_{010} + [1]_{011} + [1]_{100} + ([1]_{101} + [1]_{111}) + [1]_{110},\\
C(3) &= C(2) \times [1]_0 + D(2) \times [1]_1\\
&= ([1]_{00} + ([1]_{01} + [1]_{11}) + [1]_{10}) \times [1]_0 + ([2 + [2]_{11}]_{00}) \times [1]_1\\
&= [1]_{000} + [2 + [2]_{111}]_{001} + ([1]_{010} + [1]_{110}) + [1]_{100},\\
B(3) &= [1]_0 \times B(2) + [1]_1 \times D(2)\\
&= [1]_0 \times ([1]_{00} + [1]_{01} + ([1]_{10} + [1]_{11})) + [1]_1 \times [2 + [2]_{11}]_{00}\\
&= [1]_{000} + [1]_{001} + [1]_{010} + [1]_{011} + [2 + [2]_{111}]_{100},\\
D(3) &= [2 + [2]_{111}]_{000} + [1]_{001} + ([1]_{011} + [1]_{110}) + [1]_{100}.
\end{aligned}$$

12. CA–1

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-1_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 1 is illustrated by the following figure:

Rule 1	111	110	101	100	011	010	001	000
	0	0	0	0	0	0	0	1

The symmetric and the complemented rule numbers to 1 are 1 and 127, respectively.

THEOREM 12.1. *For cellular automata $CA-1$ the following formulas hold:*

(a) $B(m) = C(m)$ (Symmetry),

(b) $X(m) = [1]_{1^{m-1}\bar{a}} + [1]_0 \times Y(m-1) + [1]_{00} \times Y(m-2) + \sum_{k=3}^{m}[1]_{1^{k-2}00} \times Y(m-k)$ *for* $m \geq 3$,

(c) $Y(m) = [1]_0 \times X(m-1) + [1]_0 \otimes Y(m-1)$ *for* $m \geq 2$,

where
(i) $[1]_0 \otimes [p + [q_i]_{x_i}]_{01y} = [[p]_{001y}]_{000z} + [[q_1]_{0x_1} + \cdots + [q_k]_{0x_k}]]_{011y}$
if $01y \to 00z$ *in* $Y(m-1)$,
(ii) $[1]_0 \otimes [p + [q_i]_{01x_i} + [q_j]_{001x_j}]_{000y} = [p + [q_i]_{001x_i}]_{0000y} + [[q_j]_{0001x_j}]_{0100y}$

if $000y \to 011z$ *in* $Y(m-1)$ *and* $x_j \neq z$ *for all* j,
(iii) $[1]_0 \otimes [p + [q_i]_{01x_i} + [q_j]_{001x_j}]_{000y} = [p + [q_i]_{001x_i}]_{0000y} + [[q_j]_{0001x_j}]_{0100y} + [q_{j_0}]_{0001x_{j_0}}$
if $000y \to 011z$ *in* $Y(m-1)$ *and* $x_{j_0} = z$.

Proof. (b) $< 0^{k-1}1 * \cdots * >$
$\delta_{0-a}(1x) = 0y$, $\delta_{0-a}(01x) = 00y$, $\delta_{0-a}(0^{k-1}1x) = 1^{k-2}00y\ (k \geq 3)$, $\delta_{0-a}(0^m) = 1^{m-1}\bar{a}$.
(c) $< 1^{k-1}0 * \cdots * >$
$\delta_{1-a}(0x) = 0y$, $\delta_{1-a}(10x) = 00y$, $\delta_{1-a}(1^{k-1}0x) = 0^k y\ (k \geq 2)$.
$\delta_{1-a}(1^m) = 0^m$.
(Note that if $y = 00\sharp$ then $y = 000\sharp$ and $x_i = 01x_i'$ or $x_i = 001x_i'$.) □

The following detailed formulas for the operation $\otimes$ also are valid:

$[1]_0 \otimes < [p + [q_i]_{01x_i} + [q_j]_{001x_j}]_{000y}, [r + [s_k]_{u_k}]_{01z} > = < [p + [q_i]_{001x_i} + [r]_{001z}]_{0000y}, [[s_k]_{0u_k}]_{011z} >$
$+ < [0]_{000z}, [[q_j]_{0001x_j}]_{0100y} >$
if $000y \to 011z$ in $Y(m-1)$ and $1x_j \neq z$ for all j.

$[1]_0 \otimes < [p + [q_i]_{01x_i} + [q_j]_{001x_j}]_{000y}, [r + [s_k]_{u_k}]_{01z} > = < [p + [q_i]_{001x_i} + [r]_{001z}]_{0000y}, [[s_k]_{0u_k}]_{011z} >$
$+ < [q_{j_0}]_{000z}, [[q_j]_{0001x_j}]_{0100y} >$
if $000y \to 011z$ in $Y(m-1)$ and $1x_{j_0} = z$.

Using the last theorem we can recursively compute transition diagrams of $CA - 1_{a-b}(m)$ as follows:

$A(0) = [1]_\varepsilon$, $A(1) = ([1]_0 + [1]_1)$, $A(2) = ([3]_{00} + [1]_{11})$, $C(0) = [1]_\varepsilon$, $D(0) = [1]_\varepsilon$.

$C(1) = [1]_0 \times [1]_\varepsilon + [1]_0 \otimes [1]_\varepsilon = [2]_0$,
$D(1) = [1]_0 \times [1]_\varepsilon + [1]_0 \otimes [1]_\varepsilon = [2]_0$.

$C(2) = ([1]_{00} + [1]_{01}) + [1]_0 \otimes [2]_0 = [3]_{00} + [1]_{01}$
$D(2) = [1]_0 \times [2]_0 + [1]_0 \otimes [2]_0 = [4]_{00}$

$$
\begin{aligned}
A(3) &= [1]_{1^2\bar{a}} + [1]_0 \times C(2) + [1]_{00} \times C(1) + \textstyle\sum_{k=3}^{3} [1]_{1^{k-2}00} \times C(3-k) \\
&= [1]_{1^3} + [1]_0 \times ([3]_{00} + [1]_{01}) + [1]_{00} \times ([2]_0) + [1]_{100} \times [1]_\varepsilon \\
&= [1]_{111} + [3]_{0^3} + [1]_{001} + [2]_{0^3} + [1]_{100} \\
&= [1]_{1^3} + [5]_{0^3} + [1]_{001} + [1]_{100}, \\
C(3) &= [1]_0 \times ([3]_{00} + [1]_{11}) + [1]_0 \otimes ([3]_{00} + [1]_{01}) = [3]_{000} + [1]_{011} + [3]_{000} + [[1]_{001}]_{000} \\
&= [6 + [1]_{001}]_{000} + [1]_{011}, \\
D(3) &= [1]_0 \times ([3]_{00} + [1]_{10}) + [1]_0 \otimes [4]_{00} = [7]_{000} + [1]_{010}.
\end{aligned}
$$

13. CA–5

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-5_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 5 is illustrated by the following figure:

*Rule*5	111	110	101	100	011	010	001	000
	0	0	0	0	0	1	0	1

The symmetric and the complemented rule numbers to 5 are 5 and 95, respectively.

THEOREM 13.1. *For cellular automata $CA-5$ the following formulas hold:*

(a) $B(m) = C(m)$ (Symmetry),

(b) $X(m) = [1]_{1^{m-1}\bar{a}} + [1]_{1^{m-2}0\bar{a}} + [1]_{10} \times X(m-2) + [1]_{00} \times Y(m-2) + \sum_{k=3}^{m}\{[1]_{1^{k-3}010} \otimes X(m-k) + [1]_{1^{k-3}000} \otimes Y(m-k)\}$,

(c) $Y(m) = [1]_0 \times X(m-1) + [1]_0 \otimes Y(m-1)$,

where

(i) $[1]_0 \otimes [p + [q_i]_{010x_i} + [q_j]_{000x_j}]_{010y} = [p + [q_i]_{0010x_i}]_{0010y} + [[q_j]_{0000x_j}]_{0110y}$,

(ii) $[1]_0 \otimes [p + [q_i]_{x_i}]_{011y} = [[p]_{0011y}]_{00000z} + [[q_i]_{0x_i}]_{0111y}$ *if* $011y \to 0000z$ *in* $Y(m-1)$,

(iii) $[1]_0 \otimes [p + [q_i]_{01x_i} + [q_j]_{00x_j}]_{000y} = [p + [q_i]_{001x_i}]_{0000y} + [[q_j]_{000x_j}]_{0100y}$.

Proof. (a) $< 0^{k-2}10 * \cdots *, 0^{k-2}11 * \cdots * >$
If $\delta_{0-a}(10x) = 10y$, then $\delta_{0-a}(11x) = 00y$. If $\delta_{0-a}(0^{k-2}10x) = 1^{k-3}010y$ $(k \geq 3)$, then $\delta_{0-a}(0^{k-2}11x) = 1^{k-3}000y$ $(k \geq 3)$. Finally note that $\delta_{0-a}(0^m) = 1^{m-1}\bar{a}$ and $\delta_{0-a}(0^{m-1}) = 1^{m-2}0\bar{a}$.
(b) $< 1^{k-1}0 * \cdots * >$
If $\delta_{1-a}(0x) = 0y$, then $\delta_{1-a}(10x) = 00y$ and $\delta_{1-a}(1^{k-1}0x) = 0^k y$ $(k \geq 2)$.
Finally note that $\delta_{1-a}(1^m) = 0^m$.
(Note that if $y = 00\sharp$ then $y = 000\sharp$ and $x_i = 01x_i'$ or $x_i = 001x_i'$.) □

The following detailed formulas for the operation $\otimes$ also are valid:

$[1]_0 \otimes [p + [q_i]_{010x_i} + [q_j]_{000x_j}]_{010y} = [p + [q_i]_{0010x_i}]_{0010y} + < [q_{j_0}]_{000y}, [[q_j]_{0000x_j}]_{0110y} >$
if $x_{j_0} = y$ and $x_j \neq y$.

$[1]_0 \otimes < [p + [q_i]_{010x_i} + [q_j]_{000x_j}]_{010y}, [r + [s_h]_{010u_h} + [s_k]_{000u_k}]_{010z} > = < [p + [q_i]_{0010x_i}]_{0010y}, [r + [s_h]_{0010u_h}]_{0010z} > + < [q_{j_0}]_{000y}, [[q_j]_{0000x_j}]_{0110y} > + < [s_{k_0}]_{000z}, [[s_k]_{0000u_k}]_{0110z} >$
if $x_{j_0} = y$, $x_j \neq y$, $u_{k_0} = z$ and $u_k \neq z$.

$[1]_0 \otimes < [p + [q_i]_{01x_i} + [q_j]_{00x_j}]_{000y}, [r + [s_k]_{u_k}]_{011z} > = < [p + [q_i]_{001x_i} + [r]_{0011z}]_{0000y}, [[s_k]_{0u_k}]_{0111z} >$
$+ < [[q_j]_{000x_j}]_{0100y}, [0]_{0101z} >$

Using the last theorem we can recursively compute transition diagrams of $CA-5_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$, $A(1) = [2]_1$, $B(1) = C(1) = D(1) = [2]_0$,
$A(2) = [1]_{00} + [1]_{01} + [1]_{10} + [1]_{11}$, $B(2) = [2 + [2]_{00}]_{10}$, $C(2) = [2]_{00} + [2]_{01}$,
$D(2) = [4]_{00}$.

$$\begin{aligned}
A(3) &= [1]_{1^3} + [1]_{101} + [1]_{10} \times [2]_1 + [1]_{00} \times [2]_0 + [1]_{010} \otimes [1]_\varepsilon + [1]_{0^3} \otimes [1]_\varepsilon \\
&= [1]_{1^3} + [1]_{101} + [2]_{101} + [2]_{0^3} + [1]_{010} + [1]_{0^3} \\
&= [3]_{0^3} + [1]_{1^3} + [3]_{101} + [1]_{010}, \\
B(3) &= [1]_{1^20} + [1]_{100} + [1]_{10} \times [2]_0 + [1]_{00} \times [2]_0 + [1]_{010} \otimes [1]_\varepsilon + [1]_{000} \otimes [1]_\varepsilon \\
&= [1]_{110} + [1]_{100} + [2]_{100} + [2]_{0^3} + [1]_{010} + [1]_{0^3} \\
&= [3]_{000} + [1]_{010} + [3]_{100} + [1]_{110}, \\
C(3) &= [1]_{0^3} + [1]_0 \times ([1]_{00} + [1]_{01} + [1]_{10} + [1]_{11}) + [1]_{0^2} \times [2]_1 + [1]_{0^3} \times [1]_\varepsilon \\
&= [3]_{000} + [3]_{001} + [1]_{010} + [1]_{011}, \\
D(3) &= [1]_{0^3} + [1]_0 \times ([2 + [2]_{00}]_{10}) + [1]_{0^2} \otimes [2]_0 + [1]_{0^3} \otimes [1]_\varepsilon \\
&= [1]_{0^3} + [2 + [2]_{0^3}]_{010} + [2]_{0^3} + [1]_{0^3}.
\end{aligned}$$

14. CA–19

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA-19_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 19 is illustrated by the following figure:

Rule 19	111	110	101	100	011	010	001	000
	0	0	0	1	0	0	1	1

The symmetric and the complemented rule numbers to 19 are 19 and 55, respectively.

THEOREM 14.1. *For cellular automata $CA-19$ the following formulas hold:*

(a) $C(m) = B(m)$ (Symmetry),

(b) $X(m) = [1]_1 \times X(m-1) + [1]_{00} \times Y(m-2) + [1]_{011} \times X(m-3) + [1]_{000} \times Y(m-3)$,

(c) $Y(m) = [1]_0 \times Y(m-1) + [1]_{11} \times X(m-2) + [1]_{00} \times Y(m-2)$,

Proof. (a) $< 0 * \cdots *, 11 * \cdots *, 100 * \cdots *, 101 * \cdots * >$
Observe that $\delta_{0-a}(0x) = 1y$, $\delta_{0-a}(11x) = 00y$, $\delta_{0-a}(100x) = 011y$ and $\delta_{0-a}(101x) = 000y$.
(b) $< 1 * \cdots *, 00 * \cdots *, 01 * \cdots * >$
Observe that $\delta_{1-a}(1x) = 0y$, $\delta_{1-a}(00x) = 11y$ and $\delta_{1-a}(01x) = 00y$ □

Using the last theorem we can recursively compute transition diagrams of $CA-19_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$, $A(1) = B(1) = C(1) = [1]_0 + [1]_1$, $D(1) = [2]_0$,
$A(2) = [1]_{00} + [1]_{11} + [1]_{01} + [1]_{10}$, $B(2) = [1 + [2]_{10}]_{00} + [1]_{11}$, $C(2) = [1 + [2]_{01}]_{00} + [1]_{11}$,
$D(2) = [3]_{00} + [1]_{11}$.

$$\begin{aligned}
A(3) &= [1]_1 \times ([1]_{00} + [1]_{11} + [1]_{01} + [1]_{10}) + [1]_{011} \times [1]_\varepsilon + [1]_{000} \times [1]_\varepsilon + [1]_{00} \times ([1]_0 + [1]_1) \\
&= [1 + [2]_{101}]_{000} + [1]_{111} + [1]_{001} + [1]_{110} + [1]_{011} + [1]_{100}, \\
B(3) &= [1]_1 \times ([1 + [2]_{10}]_{00} + [1]_{11}) + [1]_{011} \times [1]_\varepsilon + [1]_{000} \times [1]_\varepsilon + [1]_{00} \times [2]_0 \\
&= [2 + [2]_{110}]_{000} + [1]_{111} + [2]_{100} + [1]_{011}, \\
C(3) &= [1]_{11} \times [1]_\varepsilon + [1]_{00} \times ([1]_0 + [1]_1) + [1]_0 \times ([1 + [2]_{01}]_{00} + [1]_{11}) \\
&= [2 + [2]_{011}]_{000} + [1]_{111} + [2]_{001} + [1]_{110}, \\
D(3) &= [1]_{11} \times ([1]_0 + [1]_1) + [1]_{00} \times [2]_0 + [1]_0 \times ([3]_{00} + [1]_{11}) \\
&= [3 + [2]_{110} + [2]_{011}]_{000} + [1]_{111}.
\end{aligned}$$

15. CA–36

In this section we state recursive formulas generating transition diagrams of finite cellular automata $CA{-}36_{a-b}(m)$. First we recall that a triplet local transition function f with rule number 36 is illustrated by the following figure:

Rule 36	111	110	101	100	011	010	001	000
	0	0	1	0	0	1	0	0

The symmetric and the complemented rule numbers to 36 are 36 and 219, respectively.

THEOREM 15.1. *For cellular automata $CA{-}36$ the following formulas hold:*

(a) $C(m) = B(m)$ (Symmetry),

(b) $X(2m) = [[1]_{1^{2m-1}0}]_{0^{2m}} + [1]_0 \times X(2m-1) + [1]_{100} \times X(2m-3) + \sum_{k=2}^{m-1} [1]_{1^{2k-1}00} \otimes X(2m-2k-1) + \sum_{k=1}^{m} [1]_{1^{2k-2}00} \otimes Y(2m-2k)$,

(c) $X(2m+1) = [[1]_{1^{2m+1}}]_{0^{2m+1}} + [1]_0 \times X(2m) + [1]_{100} \times X(2m-2) + \sum_{k=2}^{m} [1]_{1^{2k-1}00} \otimes X(2m-2k) + \sum_{k=1}^{m} [1]_{1^{2k-2}00} \otimes Y(2m-2k+1)$,

(d) $Y(2m) = [[1]_{1^{2m}}]_{0^{2m}} + [1]_0 \times Y(2m-1) + \sum_{k=1}^{m-1} [1]_{1^{2k-1}00} \otimes Y(2m-2k-1) + \sum_{k=1}^{m} [1]_{1^{2k-2}00} \otimes X(2m-2k)$,

(e) $Y(2m+1) = [[1]_{1^{2m}0}]_{0^{2m+1}} + [1]_0 \times Y(2m) + \sum_{k=1}^{m} [1]_{1^{2k-1}00} \otimes Y(2m-2k) + \sum_{k=1}^{m} [1]_{1^{2k-2}00} \otimes X(2m-2k+1)$,

where $X(m) = CA{-}36_{0-a}(m)$ and $Y(m) = CA{-}36_{1-a}(m)$ for $a = 0, 1$.

Proof. (b) $< (10)^k 0 * \cdots *, (10)^k 11 * \cdots * >$
If $\delta_{0-a}(x) = y$ in $X(m-1)$, then $\delta_{0-a}(0x) = 0y$ in $X(m)$. If $\delta_{0-a}(x) = y$ in $X(m-2k-1)$ for $k \geq 1$, then $\delta_{0-a}((10)^k 0x) = 1^{2k-1}00y$ in $X(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-2k)$ for $k \geq 1$, then $\delta_{0-a}((10)^{k-1} 11x) = 1^{2k-2}00y$ in $X(m)$.

(c) $< (01)^k 1 * \cdots *, (01)^k 00 * \cdots * >$
If $\delta_{1-a}(x) = y$ in $Y(m-1)$, then $\delta_{1-a}(1x) = 0y$ in $Y(m)$. If $\delta_{1-a}(x) = y$ in $Y(m-2k-1)$ for $k \geq 1$, then $\delta_{1-a}((01)^k 1x) = 1^{2k-1}00y$ in $Y(m)$. If $\delta_{0-a}(x) = y$ in $X(m-2k)$ for $k \geq 1$, then $\delta_{0-a}((01)^{k-1}00x) = 1^{2k-2}00y$ in $Y(m)$. □

Using the last theorem we can recursively compute transition diagrams of $CA-36_{a-b}(m)$ as follows:

$A(0) = B(0) = C(0) = D(0) = [1]_\varepsilon$,
$A(1) = [1]_0 + [1]_1$, $B(1) = [2]_0$, $C(1) = [2]_0$, $D(1) = [1]_0 + [1]_1$,
$A(2) = [2]_{00} + [1]_{01} + [1]_{10}$, $B(2) = [3 + [1]_{11}]_{00}$,
$C(2) = [3 + [1]_{11}]_{00}$, $D(2) = [1]_{01} + [1]_{10} + [2]_{00}$.

$$
\begin{aligned}
A(3) &= [1]_0 \times ([2]_{00} + [1]_{01} + [1]_{10}) + [1]_{100} \times ([1]_\varepsilon) + [1]_{00} \otimes C(1) + [1]_{1^3} \\
&= [4 + [1]_{1^3}]_{0^3} + [1]_{001} + [1]_{010} + [1]_{100}, \\
B(3) &= [1]_0 \times ([3 + [1]_{11}]_{00}) + [1]_{100} \times ([1]_\varepsilon) + [1]_{00} \otimes ([1]_0 + [1]_1) + [1]_{110} \\
&= [4 + [1]_{011} + [1 + [1]_{110}]_{001}]_{001} + [1]_{100}, \\
C(3) &= [1]_0 \otimes ([3 + [1]_{11}]_{00}) + [1]_{100} + [1]_{00} \times ([1]_0 + [1]_1) + [1]_{110} \\
&= [4 + [1]_{110} + [1 + [1]_{011}]_{100}]_{000} + [1]_{001}, \\
D(3) &= [1]_0 \otimes ([1]_{01} + [1]_{10} + [2]_{00}) + [1]_{100} \times [1]_\varepsilon + [1]_{00} \times [2]_0 + [1]_{1^3} \\
&= [4 + [1]_{001} + [1]_{100} + [1 + [1]_{010}]_{1^3}]_{0^3}.
\end{aligned}
$$

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