# CHARACTERIZATION OF THE SOLUTIONS OF MULTIOBJECTIVE LINEAR PROGRAMMING WITH A GENERAL DOMINATED CONE

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# CHARACTERIZATION OF THE SOLUTIONS OF MULTIOBJECTIVE LINEAR PROGRAMMING WITH A GENERAL DOMINATED CONE

By

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**Abstract**

In this paper we give a characterization of the solutions of a multi-objective linear programming problem with a general dominated cone. In such a problem the domination structure defined by the cone plays an important role. The dominated cone we adopt as the criterion in this paper is expressed in the form of a system of linear inequalities, but is not assumed to be acute. We first give a characterization theorem of the solutions, and next show, by the use of the theorem, that when the cone is not acute our problem can be transformed to another optimization problem with respect to a certain acute cone.

## 1. Notation and definitions

We deal with the following multiple objective linear programming problem:

$$\text{(P)} \qquad \begin{array}{l} \Lambda\text{-maximize } Cx \\ \qquad \text{subject to } Ax \leq b,\ x \geq 0\ , \end{array}$$

where $C \in \mathbf{R}^{q \times n}$ and $A \in \mathbf{R}^{m \times n}$ are given matrices, $b \in \mathbf{R}^m$ is a given vector, and $\Lambda$ is a cone in $\mathbf{R}^q$ which gives the order in the space of objective functions. For $x = (x^1, x^2, \ldots, x^n)$ and $y = (y^1, y^2, \ldots, y^n)$, we shall use $x \leq y$ and $x \geq y$ to denote $x^i \leq y^i$ and $x^i \geq y^i$ for all $i = 1, 2, \ldots, n$, respectively. Then problem (P) means "Maximize $Cx$ subject to the conditions $Ax \leq b$ and $x \geq 0$ with respect to the order given by $\Lambda$.

Throughout this paper, let $X$ denote the set of all feasible points of problem (P), i.e.,

$$X := \{x \in \mathbf{R}^n \mid Ax \leq b,\ x \geq 0\},$$

and we assume that $X$ is not empty.

First we consider the traditional definition.

DEFINITION 1.1. ([1], [2] and [3]) Let $\Lambda$ be a closed polyhedral convex cone in $\mathbf{R}^q$, $C$ a $q \times n$ matrix, and let $x_1, x_2 \in \mathbf{R}^q$. Then we say that $x_1$ is dominated by $x_2$ with respect to $C$ if $Cx_1 \in Cx_2 + \Lambda$ .

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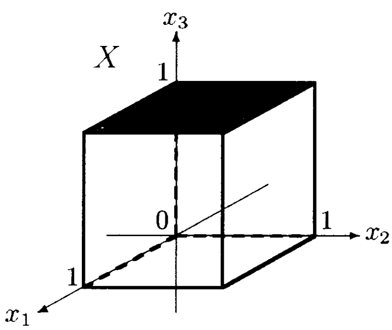
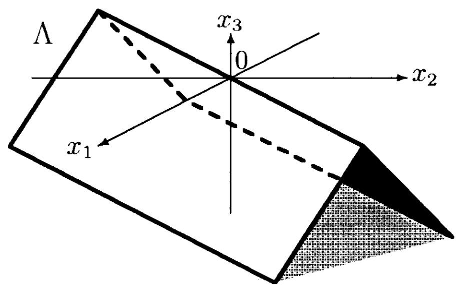
Moreover we asume that $\Lambda$ is acute and we consider problem (P). Then $x \in X$ is called a solution of problem (P), if $x$ is not dominated by any other point of $X$.

When we concretly define the order given by $\Lambda$, we get another representation of the optimal condition. To begin with, for $x_1$ and $x_2 \in \mathbf{R}^q$ we define $Cx_1 \preceq Cx_2$ if $Cx_1 \in Cx_2 + \Lambda$. Then the fact that $x$ is a solution of problem (P) is equvalent to that there does not exist $x' \in X$ such that $Cx \preceq Cx'$. That is, we want to find 'greater' points with respect to $\preceq$, and it will be successful if $\Lambda$ is acute.

However, if $\Lambda$ is not acute, we can not always obtain the solutions by using this order. The following is a typical example.

EXAMPLE 1.2. Let $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $X = \{(x_1, x_2, x_3) |\ 0 \le x_1, x_2, x_3 \le 1\}$ and $\Lambda = \{(x_1, x_2, x_3) |\ 2x_1 - x_2 + 2x_3 \le 0,\ -2x_1 + x_2 + 2x_3 \le 0\}$,




then it is natural that we consider the points on $S = \{(x_1, x_2, x_3) |\ 0 \le x_1, x_2 \le 1,\ x_3 = 1\}$ are solutions. Because arbitrary point of $S$ is 'greater' than any other point of $X$ with respect to $\preceq$. However, even though points $(0, 0, 1)$ and $(\frac{1}{2}, 1, 1)$ are different points on $S$, they are dominated by each other w.r.t. $C$.

Since Definition 1.1 can not characterize the solution of Example, more general definition must be used. Thus we introduce the following definition.

DEFINITION 1.3. Let $\Lambda$ be a cone in $\mathbf{R}^q$ and we consider problem (P). Then $x \in X$ is called a solution of problem (P), if $Cx' - Cx \in \Lambda$ for all $x' \in X$ such that $Cx - Cx' \in \Lambda$.

Note that if $\Lambda$ is acute then the optimality in Definition 1.3 is the same with the optimality in Definition 1.1.

In the following, let $\Lambda^*$ be the polar cone of $\Lambda$, i.e.,

$$\Lambda^* := \{\lambda \mid \lambda\lambda' \le 0 \text{ for } {}^\forall\lambda' \in \Lambda\},$$

we shall use aff$A$ to denote the affine hull of $A$, ri$A$ to denote the relative interior of $A$ and the inner product of two vectors $a$ and $b$ is expressed by $ab$.

## 2. Main Results

First, we prove the following lemma.

LEMMA 2.1. *Let $\Lambda$ be a closed polyhedral convex cone in $\mathbf{R}^q$. If $\mu \in \Lambda$ satisfies $\mu\lambda = 0$ for some $\lambda \in ri\Lambda^*$, then $-\mu \in \Lambda$.*

PROOF. We assume that $\mu \in \Lambda$ and $\lambda \in \mathrm{ri}\Lambda^*$ satisfy $\mu\lambda = 0$ and $-\mu \notin \Lambda$. Then there exists $\lambda' \in \Lambda^*$ such that $-\mu\lambda' > 0$. Let

$$L := \{v(\alpha) \mid \alpha \in \mathbf{R}\},$$

where $v(\alpha) := \lambda + \alpha(\lambda - \lambda')$, then $\lambda \in (\mathrm{ri}\Lambda^*) \bigcap L$. Since $\mathrm{aff}(\Lambda^* \bigcap L) = L$, the neighborhood of $\lambda$ on $\mathrm{aff}(\Lambda^* \bigcap L)$ is given by the following form:

$$B(\varepsilon_1, \varepsilon_2) := \{v(\varepsilon) \mid -\varepsilon_1 < \varepsilon < \varepsilon_2\}, \quad \varepsilon_1, \varepsilon_2 > 0.$$

On the other hand, since $\mu v(\alpha)$ is linear with respect to $\alpha$ and

$$\mu v(-1) = \mu\lambda' < 0\ ,\ \mu v(0) = \mu\lambda = 0,$$

for any $\alpha > 0$, $\mu\ v(\alpha) > 0$ , i.e., $v(\alpha) \notin \Lambda^*$. This means that

$$B(\varepsilon_1, \varepsilon_2) \not\subset \Lambda^* \bigcap L\ ,\ {}^\forall\varepsilon_1, {}^\forall\varepsilon_2 > 0$$

Hence $\lambda \notin \mathrm{ri}(\Lambda^* \bigcap L)$, which is equivalent to $\lambda \notin (\mathrm{ri}\Lambda^*) \bigcap L$. This is a contradiction. Thus Lemma 2.1 is proved. □

We can now prove the main result of this paper. By the following theorem, we characterize the set of solutions of problem (P).

THEOREM 2.2. *For $\lambda \in \mathbf{R}^q$ let $X^0(\lambda) := \{\ x^0 \in X \mid \lambda C x^0 \geq \lambda C x \text{ for } {}^\forall x \in X\}$. Then the set $N$ of the solutions of probrem(P) is equal to the union of $X^0(\lambda)$ with respect to all $\lambda \in ri\Lambda^*$, i.e.,*

$$N = \bigcup_{\lambda \in ri\Lambda^*} X^0(\lambda).$$

PROOF. We first show that

$$N \supset \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda).$$

For any $x \in \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda)$ , there exists $\lambda \in \mathrm{ri}\Lambda^*$ such that $x \in X^0(\lambda)$ . Then

$$\lambda C x \geq \lambda C y\ ,\ {}^\forall y \in X. \tag{1}$$

Now, we assume that $Cx - Cx' \in \Lambda$ is satisfied for some $x' \in X$ . Since there exists $\lambda^0 \in \Lambda$ such that $Cx - Cx' = \lambda^0$ ,

$$\lambda Cx - \lambda Cx' = \lambda\lambda^0 \leq 0.$$

If $\lambda\lambda^0 < 0$ , then $\lambda Cx < \lambda Cx'$ , which contradicts (1). Thus

$$\lambda\lambda^0 = 0.$$

By Lemma 2.1, $-\lambda^0 \in \Lambda$. Hence

$$Cx' - Cx = -\lambda^0 \in \Lambda.$$

This shows $x \in N$ .

We next show that

$$N \setminus \left( \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda) \right) = \emptyset.$$

Let us suppose that the left-hand side is not empty, and let

$$x \in N \setminus \left( \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda) \right).$$

Then for every $\lambda \in \mathrm{ri}\Lambda^*$ , $x \notin X^0(\lambda)$ , i.e. ,

$$\lambda Cx' > \lambda Cx \ , \ ^\forall x' \in X^0(\lambda). \tag{2}$$

Next, let

$$Y^0(\lambda) := \{Cx \mid x \in X^0(\lambda)\}.$$

Then we show that

$$\left( \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} Y^0(\lambda) \right) \bigcap (Cx - \Lambda) \neq \emptyset. \tag{3}$$

If the left-hand side of (3) is empty, in other words, for all $Cx' \in \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} Y^0(\lambda)$ ( $x' \in X^0(\lambda)$ ), $Cx'$ never belongs to the set $(Cx - \Lambda)$, then $Cx - Cx' \notin \Lambda$ . This implies that there exists $\lambda \in \mathrm{ri}\Lambda^*$ such that $\lambda(Cx - Cx') > 0$ . Hence $\lambda Cx > \lambda Cx'$ , which contradicts inequality (2).

Since it is proved that (3) holds,

$$^\exists Cx' \in \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} Y^0(\lambda) \text{ s.t. } Cx' \in Cx - \Lambda \ , \ x' \in X.$$

Hence $Cx - Cx' \in \Lambda$. Since $x$ belongs to $N$, $Cx' - Cx \in \Lambda$, which means that $\lambda Cx = \lambda Cx'$ for each $\lambda \in \mathrm{ri}\Lambda^*$. This contradicts inequality (2). Therefore $N \setminus \left( \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda) \right)$ is empty.

Thus Theorem 2.2 is completely proved. □

In the case that $\Lambda$ is acute, Yu ([2],p.218)gave the characterization of the solutions of problem (P). In the other case, he gave the following result.

Suppose that $\Lambda$ is a polyhedral cone. Then
1. $N \subset \bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda)$,
2. if $\Lambda$ is acute, then $N = \bigcup_{\lambda \in \mathrm{int}\Lambda^*} X^0(\lambda)$,
where int $\Lambda^*$ denotes the interior of $\Lambda^*$.

Our theorem is a generalization of above result 2, and our proof of 1 is simpler than that given by Yu([2]).

Next, we derive a way to obtain the solutions of problem (P). For this purpose, we assume that $\Lambda$ is presented as a system of inequalities, i.e. ,

$$\Lambda = \{\lambda \in \mathbf{R}^q \mid h^i\lambda \leq 0,\ i = 1, 2, \ldots, r\}, (h^1, h^2, \ldots, h^r \in \mathbf{R}^q).$$

The following lemma is well known.

LEMMA 2.3.

$$\Lambda^* = \left\{ \lambda \in \mathbf{R}^q \;\middle|\; \lambda = \sum_{i=1}^{r} \alpha_i h^i, \quad (\alpha_1, \alpha_2, \ldots, \alpha_r) \geq 0 \right\}$$

THEOREM 2.4. *Let* $H := (h^1, h^2, \ldots, h^r)^t$ . *Then problem (P) is equivalent to the problem:*

$$\Lambda^{\leq}\text{-}\mathit{maximize}\ \ HCx \\ \mathit{subject\ to}\ x \in X,$$

*where* $\Lambda^{\leq} := \{\lambda \in \mathbf{R}^r \mid \lambda \leq 0\}$.

PROOF. It suffices to show that the set of the solutions of problem (P) is equal to that of the problem in the theorem. By Lemma 2.2,

$$\mathrm{ri}\Lambda^* = \left\{ \sum_{i=1}^{r} \alpha_i h^i \;\middle|\; (\alpha_1, \alpha_2, \ldots, \alpha_r) > 0 \right\},$$

$$\mathrm{ri}\ (\Lambda^{\leq})^* = \{\lambda \in \mathbf{R}^r \mid \lambda > 0\} =: \Lambda^{>}.$$

Therefore, by Theorem 2.2,

$$\begin{aligned}
&\text{(all of the solutions of (P))} \\
= &\bigcup_{\lambda \in \mathrm{ri}\Lambda^*} X^0(\lambda) \\
= &\bigcup_{(\alpha_1, \ldots, \alpha_r) > 0} \left\{ x^0 \in X \;\middle|\; \left(\sum_{i=1}^{r} \alpha_i h^i\right) Cx^0 \geq \left(\sum_{i=1}^{r} \alpha_i h^i\right) Cx \text{ for } {}^\forall x \in X \right\} \\
= &\bigcup_{(\alpha_1, \ldots, \alpha_r) > 0} \{ x^0 \in X \mid (\alpha_1, \ldots, \alpha_r)HCx^0 \geq (\alpha_1, \ldots, \alpha_r)HCx \text{ for } {}^\forall x \in X\} \\
= &\bigcup_{\lambda \in \Lambda^{>}} \{ x^0 \in X \mid \lambda HCx^0 \geq \lambda HCx \text{ for } {}^\forall x \in X\}.
\end{aligned}$$

□

By Theorem 2.4, problem (P) can be converted to the problem with respect to the acute cone $\Lambda^{\leq}$. Consequently we may solve it as Yu ([2]). Moreover, by the following theorem, we can delete the redundant inequalities in the system of inequalities defining $\Lambda$. In fact, they are estimated many times in the process of getting solutions. Accordingly it would be worthy to delete the redundancy.

THEOREM 2.5. *Let* $\Lambda^j := \{\lambda \in \mathbf{R}^q \mid h^i\lambda \leq 0,\ i = 1, \ldots, j-1, j+1, \ldots, r\}$. *Then, the following conditions 1), 2) are equivalent.*
*1)* $\Lambda = \Lambda^j$
*2)* $\{\alpha \in \mathbf{R}^{r-1} |\ (h^1, \ldots, h^{j-1}, h^{j+1}, \ldots, h^r)\alpha = h^j,\ \alpha \geq 0\} \neq \emptyset$

PROOF.

$$\begin{aligned} \Lambda = \Lambda^j &\iff \Lambda^j \subset \{\lambda \mid h^j\lambda \leq 0\} \\ &\iff h^j\lambda \leq 0, \text{ for } {}^\forall\lambda \in \Lambda^j \\ &\iff h^j \in (\Lambda^j)^* \end{aligned}$$

By Lemma 2.1,

$$(\Lambda^j)^* = \left\{ \sum_{i=1, i\neq j}^{r} \alpha_i h^i \;\middle|\; \alpha^i \geq 0,\ i = 1, \ldots, j-1, j+1, \ldots, r \right\}.$$

Hence

$$\Lambda = \Lambda^j \iff^{\exists} \alpha^i \geq 0\ (i = 1, \ldots, j-1, j+1, \ldots, r)$$
$$\text{s.t.} \sum_{i=1, i\neq j}^{r} \alpha_i h^i = h^j.$$

□

For example, in order to verify that the inequality $h^j\lambda \leq 0$ is redundant, it is sufficient, by virtue of Theorem 2.5, to only check condition 2), i.e., to check whether the feasible set of the linear programming problem is empty or not.

EXAMPLE 2.6. Let $\Lambda := \{\ (\lambda_1, \lambda_2, \lambda_3) \mid 2\lambda_1 - \lambda_2 + \lambda_3 \leq 0,\ 2\lambda_1 - \lambda_2 - \lambda_3 \leq 0\ \}$ be a dominated cone, then we consider the following problem:

$$\begin{aligned} \Lambda\text{-maximize} \quad & (x_1 + x_2,\ 2x_1 - x_3,\ x_1 + x_2 + x_3) \\ \text{subject to} \quad & 2x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Since $\Lambda$ is not acute, by Theorem 2.4, we convert this problem into the following one:

$$\begin{array}{ll} \Lambda^{\leq}\text{-maximize} & HC\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \text{subject to} & 2x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0, \end{array}$$

where $H = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. This is the problem with respect to the acute cone $\Lambda^{\leq}$, so we can solve it as usual.

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