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Endow, Yasushi
Department of Industrial and Systems Engineering, Chuo University

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GIBBS p -ADIC DERIVATIVES AND THEIR APPLICATIONS TO p -ADIC STATIONARY PROCESSES

By

Yasushi ENDOW*

Abstract

In this paper the concept of Gibbs p -adic derivatives is introduced to the realm of stochastic processes. Differentiability of p -adic stationary processes both in almost sure convergence and in quadratic mean convergence are presented. A stochastic linear Gibbs p -adic differential equation is defined, and is solved for driving processes of harmonizable p -adic stationary processes.

1. Introduction

In pioneering work in 1967, J.E. Gibbs [7] introduced the finite Gibbs dyadic derivative in attempting to meet a desire to have a differential operator well suited to discrete functions defined on the finite dyadic group. Its characters or the discrete Walsh functions emerge as the eigenfunctions of the finite Gibbs dyadic derivative [7], i.e.,

$$D\psi_k(x) = k\psi_k(x) \quad (x \in \{0, \dots, 2^n - 1\}),$$

where ψ_k ($k \in \{0, \dots, 2^n - 1\}$) is the k -th Walsh function. It follows that the finite dyadic derivative is efficiently characterized by the Walsh-Fourier coefficients, i.e.,

$$(\widehat{Df})(k) = k\widehat{f}(k) \quad (k \in \{0, \dots, 2^n - 1\}),$$

where the hat denotes the W-F transform operator.

The Gibbs derivative had been extended and generalized by some research workers. Among others, Butzer and Wagner[1] made a particularly important generalization for functions on the unit interval, that can be identified with the infinite dyadic group. We will introduce the concept of Gibbs p -adic derivative for functions on $\mathbf{R}^+ := [0, \infty)$ (c.f.[10]) and for stochastic processes on the same interval. We also define a stochastic Gibbs p -adic differential equation and give its solution for a special driving process of a p -adic stationary process [5].

* Department of Industrial and Systems Engineering, Chuo University, 1-13-27 Kasuga Bunkyo-ku Tokyo, 112 Japan.

2. Selfridge functions and Gibbs p -adic derivatives

Throughout this paper p is taken to be a fixed, but arbitrary greater than one. For the complete discussion, we introduce the Selfridge functions[9], or the Chrestenson functions[2], that are the extensions of the Walsh functions [11],[6].

Any $x \in \mathbf{R}^+$ with $p^M \leq x < p^{M+1}$ is uniquely expressed by the p -adic expansion,

$$x = \sum_{i=-M}^{\infty} x_i p^{-i},$$

provided that the finite expansion is assumed to be taken if it is a p -adic rational. The Selfridge function is defined by

$$\psi(t, x) = \epsilon^{t \odot x}, \quad (1)$$

where $\epsilon = \exp(2\pi\sqrt{-1}/p)$, and

$$t \odot x = \sum_{i=-M}^{T+1} x_i t_{1-i} \quad (t, x \in \mathbf{R}^+) \quad (2)$$

with $t = \sum_{i=-T}^{\infty} t_i p^{-i}$ and $x = \sum_{i=-M}^{\infty} x_i p^{-i}$, respectively. It is noted that the Selfridge functions are symmetric in their variables, because of the symmetricity of $\odot$ -operation. It is easy to see that the following properties are satisfied:

$$\psi(t, x \oplus y) = \psi(t, x)\psi(t, y), \quad (3)$$

$$\psi(t, x \ominus y) = \psi(t, x)\overline{\psi(t, y)}, \quad (4)$$

for any $x \in \mathbf{R}^+$ and for a.a. $y \in \mathbf{R}^+$, which depends on x . (c.f. [6]).

For $f \in L^1(\mathbf{R}^+)$ its Selfridge-Fourier (S-F) transform is defined by

$$\widehat{f}(x) = \int_0^{\infty} f(t) \overline{\psi(t, x)} dt. \quad (5)$$

For $f, g \in L^1(\mathbf{R}^+)$ the integral

$$f * g(t) = \int_0^{\infty} f(t \ominus s) g(s) ds \quad (6)$$

exists and is called the convolution of f and g . Then we have that

$$(\widehat{f * g})(x) = \widehat{f}(x) \widehat{g}(x), \quad (7)$$

which is a convolution theorem similar to the ordinary relationship between a convolution and its Fourier transform. The S-F transform of $f \in L^2(\mathbf{R}^+)$ is defined by

$$\widehat{f}(x) = \text{l.i.m.}_{A \rightarrow \infty} \int_0^A f(t) \overline{\psi(t, x)} dt. \quad (8)$$

The inverse S-F transform of $\widehat{f} \in L^2(\mathbf{R}^+)$ can be defined, and is given by

$$f(t) = \text{l.i.m.}_{A \rightarrow \infty} \int_0^{\infty} \widehat{f}(x) \psi(t, x) dx. \quad (9)$$

It is easy to see that

$$(\widehat{\tau_s f})(x) = \widehat{\psi}(s, x) \widehat{f}(x), \quad (10)$$

$$(\widehat{f\psi(\cdot, y)})(x) = \tau_y \widehat{f}(x), \quad (11)$$

where τ_s is a p -adic shift-operator, which is defined by

$$\tau_s f(t) = f(t \ominus s). \quad (12)$$

If a function $f(t)$ is continuous at p -adic irrationals and right continuous at p -adic rationals, then it is said to be W-continuous. Note that Selfridge functions are uniform W-continuous.

Now we shall define the Gibbs p -adic derivative of a function f on $\mathbf{R}^+$. If the limit of the sum

$$\Delta_N f(t) = \sum_{k=-N}^N p^k \sum_{j=0}^{p-1} A_j f(t \oplus jp^{-k-1}) \quad (13)$$

with $A_0 = (p-1)/2$, $A_j = \epsilon^j/(1-\epsilon^j)$, $j = 1, \dots, p-1$, converges as $N \rightarrow \infty$, then the function f is called pointwise Gibbs p -adic differentiable at $t \in \mathbf{R}^+$. The limit is denoted by $Df(t)$ or $f^{[1]}(t)$. If f is Gibbs p -adic differentiable at every $x \in \mathbf{R}^+$, then it is simply called Gibbs p -adic differentiable.

For $f \in L^r(\mathbf{R}^+)$, $1 \leq r < \infty$, the L^r -limit of $\Delta_N f(t)$, if it exists, is called the L^r -strong Gibbs p -adic derivative. The higher derivatives are defined by induction. It follows from definition that the Selfridge functions are k -times differentiable in both the pointwise and the L^r -strong senses, and

$$D^k \psi(t, x) = x^k \psi(t, x) \quad (t \in \mathbf{R}^+), \quad (14)$$

where $k \in \mathbf{P} := \{0, 1, \dots\}$. Hence, putting $P(s) = \sum_{k=0}^m a_k s^k$ and by linearity of the operator D ,

$$P(D)\psi(t, x) = P(x)\psi(t, x). \quad (15)$$

It is also shown [10] that if f , $D^k f \in L^r(\mathbf{R}^+)$; $k \in \mathbf{P}$, $1 \leq r < \infty$, then

$$(\widehat{D^k f})(x) = x^k \widehat{f}(x) \text{ a.e.}, \quad (16)$$

which is a consequence of the fact that Selfridge functions are the eigenfunctions of this differential operator. This shows a characterization of the Gibbs p -adic derivatives. The Gibbs p -adic differentiator of order k can be considered as a convolution operator with a transformed function $\widehat{g}(x) = x^k$. It follows from (17) and the linearity of the S-F transform that

$$(P(\widehat{D})f)(x) = P(x)\widehat{f}(x). \quad (17)$$

In the p -adic group world the Selfridge functions and the Gibbs p -adic derivatives seem to play similar roles as the Euler functions and the Newton-Leibniz derivatives in real world. It is well known in real world that differentiable functions are continuous. However, in the p -adic group world Gibbs differentiable functions are not W-continuous, nor are W-continuous functions Gibbs differentiable.

3. Quadratic mean Gibbs differentiability of p -adic stationary processes

To begin with we shall consider two types of Gibbs differentiabilitys of stochastic processes, according to the types of convergences. Let $\{X(t, \omega); t \in \mathbf{R}^+\}$ be a stochastic process. If

$$\Delta_N X(t) = \sum_{k=-N}^N p_k \sum_{j=0}^{p-1} A_j X(t \oplus jp^{-k-1}) \quad (18)$$

converges almost surely as $N \rightarrow \infty$, then the limit is called the sample Gibbs derivative (sample G-d) of the process at t and is denoted by $X^{[1]}(t)$ or $DX(t)$. The higher order, say r -order, $r \in \mathbf{N}$, sample Gibbs derivative is defined by induction. For second order process $\{X(t, \omega); t \in \mathbf{R}^+\}$, if the mean square limit of $\Delta_N X(t)$ as $N \rightarrow \infty$ exists, then the limit,

$$\lim_{N \rightarrow \infty} \mathbf{E} |\Delta_N X(t)|^2 \quad (19)$$

is called the quadratic mean Gibbs derivative (q.m. G-d) of the process at t or the Gibbs derivative in the mean at t , and is denoted by $X^{[1]}(t)$ or $DX(t)$.

Next we shall consider quadratic mean Gibbs differentiability of p -adic stationary processes. Let $\{X(t, \omega); t \in \mathbf{R}^+\}$ be a second order process with zero mean. If its covariance function $\text{Cov}(X(t), X(s)) = \mathbf{E} X(t, \omega) \overline{X(s, \omega)}$ satisfies that

$$\text{Cov}(X(t), X(s)) = \text{Cov}(X(t \ominus s), X(0)) \quad (\mu(t) \ominus \mu(s) \in \mathbf{F}^c), \quad (20)$$

then it is called a p -adic stationary process (p -SP) (for detail see [5]). Suppose that it is harmonizable, i.e., it is assumed to be expressible as

$$X(t, \omega) = \int_0^\infty \psi(t, x) \zeta(dx, \omega) \quad (t \in \mathbf{R}^+), \quad (21)$$

where ζ is an orthogonal random measure with $\mathbf{E} \zeta(A) = 0$ ($A \in \mathcal{B}(\mathbf{R}^+)$). Its covariance function is also expressed by

$$\text{Cov}(X(t), X(s)) = \int_0^\infty \psi(t, x) \overline{\psi(s, x)} F(dx), \quad (22)$$

where F is the spectral distribution function with

$$F(dx) = \mathbf{E} |\zeta(dx, \omega)|^2. \quad (23)$$

A necessary and sufficient condition for harmonizability of a p -SP was given by the present author [5]. Note that harmonizable p -adic stationary processes are quadratic mean W-continuous.

Now we shall give a quadratic mean Gibbs differentiability condition for p -SP, which is a generalization of the result for dyadic stationary processes [4].

THEOREM 3.1. *Let $r \in \mathbf{P}$. A harmonizable p -SP is r -times quadratic mean Gibbs differentiable, if and only if its spectral distribution has the $2r$ -th moment, i.e.,*

$$\int_0^\infty x^{2r} F(dx) < \infty. \quad (24)$$

Then the quadratic mean r -th Gibbs derivative of the process is expressed by

$$DX(t, \omega) = \int_0^\infty x^r \psi(t, x) \zeta(dx, \omega). \quad (25)$$

Proof. Before proving this we note that (24) is sometimes written as $x \in L^{2r}(F)$. We will only prove it for $r = 1$, and the other cases are shown similarly. (sufficiency.) After some familiar manipulations, we obtain that

$$\mathbf{E}|\Delta_N X(t, \omega)|^2 = \int_0^\infty |\Delta_N \psi(t, x)|^2 F(dx). \quad (26)$$

Since

$$\lim_{N \rightarrow \infty} \Delta_N \psi(t, x) = x\psi(t, x), \quad (27)$$

uniformly, (26) and Lebesgue's convergence theorem show that

$$\lim_{N \rightarrow \infty} \int_0^\infty |\Delta_N \psi(t, x)|^2 F(dx) = \int_0^\infty |x\psi(t, x)|^2 F(dx) \quad (28)$$

$$= \int_0^\infty x^2 F(dx), \quad (29)$$

which is finite by (24).

(necessity.) If the process is Gibbs differentiable, i.e., the left side of (26) converges finitely as N tends to infinity, then the integral in the right side of (29) is finite. In the same way as (26), we will have that

$$\mathbf{E}|\Delta_N X(t, \omega) - \int_0^\infty x\psi(t, x)\zeta(dx, \omega)|^2 = \int_0^\infty |\Delta_N \psi(t, x) - x\psi(t, x)|^2 F(dx). \quad (30)$$

Thus the proof is completed by (27). $\square$

COROLLARY 3.2. *If the condition in Theorem 3.1 is satisfied, then the q.m. i -th ($0 \leq i \leq r$) Gibbs derivatives of the p -SP's are harmonizable, and their covariance functions are expressed by*

$$\text{Cov}(D^{[i]}X(t), D^{[j]}X(s)) = \int_0^\infty \psi(t, x) \overline{\psi(s, x)} x^{2(i+j)} F(dx) \quad (0 \leq i, j \leq r). \quad (31)$$

This is a direct consequence of Theorem 3.1.

4. Sample Gibbs differentiability of p -adic stationary processes

In this section sample Gibbs differentiation of p -adic stationary processes will be considered. The idea comes from Kawata and Kubo's work on classical stationary processes[8]. Let us put that for $k, n \in \mathbf{N}_0$

$$\zeta_{n,k} = \int_{np^{-k}}^{(n+1)p^{-k}} \zeta(dx), \quad (32)$$

where ζ is the spectral random measure of the p -adic stationary process $X(t)$ and

$$X_k(t) = \sum_{n=0}^{\infty} \psi(np^{-k}, t) \zeta_{n,k}. \quad (33)$$

The series in the right side of (33) converges in $L^2(\Omega)$ -norm, and the sum is an approximation of $X(t)$ (see [3]). Let us introduce some lemmas for later use.

LEMMA 4.1. *If the spectral distribution function satisfies that for some $\alpha > 1$*

$$\int_0^{\infty} x^{\alpha} F(dx) < \infty, \quad (34)$$

then the series (33) absolutely converges almost surely, and $X_k(t)$ converges uniformly for every finite interval $0 \leq t \leq A$ as $k \rightarrow \infty$ almost surely to a p -adic stationary process $\tilde{X}(t)$, which is sample W -continuous, and a version of $X(t)$.

This is a p -adic extension of Theorem 6.1 in [3]. The proof of this lemma is similar to the theorem, so it is omitted here.

LEMMA 4.2. *Let $r \in \mathbb{N}_0$. If for some $\alpha > 1$ the spectral distribution function satisfies that*

$$\int_0^{\infty} x^{2r+\alpha} F(dx) < \infty, \quad (35)$$

then

$$\sum_{n=0}^{\infty} n^r (F(n+1) - F(n))^{1/2} < \infty. \quad (36)$$

Hence

$$\sum_{n=0}^{\infty} n^r |\zeta_{n,k}| < \infty \text{ a.s.} \quad (37)$$

Proof. It is clear that by Schwartz's inequality

$$\begin{aligned} & \sum_{n=0}^{\infty} n^r (F(n+1) - F(n))^{1/2} \\ & \leq \left(\sum_{n=0}^{\infty} n^{-\alpha} \right)^{1/2} \left(\sum_{n=0}^{\infty} n^{2r+\alpha} (F(n+1) - F(n)) \right)^{1/2} \\ & \leq \left(\sum_{n=0}^{\infty} n^{-\alpha} \right)^{1/2} \left(\int_0^{\infty} n^{2r+\alpha} F(dx) \right)^{1/2} < \infty. \end{aligned} \quad (38)$$

To show (37) it is sufficient to show that

$$\mathbf{E} \sum_{n=0}^{\infty} n^r |\zeta_{n,k}| < \infty. \quad (39)$$

By Schwartz's inequality, we have that

$$\begin{aligned}
 \mathbf{E} \sum_{n=0}^{\infty} n^r |\zeta_{n,k}| &\leq \sum_{n=0}^{\infty} n^r \left(\mathbf{E} |\zeta_{n,k}|^2 \right)^{1/2} \\
 &\leq \sum_{n=0}^{\infty} n^r \left(F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
 &= S_k \text{ (say)}.
 \end{aligned} \tag{40}$$

Rewriting S_k , we see that

$$\begin{aligned}
 S_k &= \sum_{m=0}^{\infty} \sum_{n=mp^k}^{(m+1)p^k-1} n^r \left(F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
 &\leq \sum_{m=0}^{\infty} ((m+1)p^k)^r \sum_{n=mp^k}^{(m+1)p^k-1} \left(F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
 &\leq \sum_{m=0}^{\infty} (m+1)^r p^{rk} \left(p^k \sum_{n=mp^k}^{(m+1)p^k-1} \left(F((n+1)p^{-k}) - F(np^{-k}) \right) \right)^{1/2} \\
 &\leq p^{(r+1/2)k} \sum_{m=0}^{\infty} (m+1)^r \left(F(m+1) - F(m) \right)^{1/2}
 \end{aligned} \tag{41}$$

Hence the proof is completed by (36). $\square$

LEMMA 4.3. *If (35) holds then $X_k(t)$ has a version which is a harmonizable p -adic stationary process with the almost sure W -continuous r -th Gibbs derivative.*

Proof. We shall only prove it for $r = 1$, since the same argument will be applied to the other cases. Since by (37), for every $N \in \mathbb{N}_0$

$$\begin{aligned}
 |\Delta_N X_k| &= \left| \sum_{m=0}^{\infty} \Delta_N \psi(mp^{-k}, t) \zeta_{m,k} \right| \\
 &\leq \sum_{m=0}^{\infty} mp^{-k} |\zeta_{m,k}| < \infty \text{ a.s.}
 \end{aligned} \tag{42}$$

and

$$\lim_{N \rightarrow \infty} \Delta_N \psi(mp^{-k}, t) = mp^{-k} \psi(mp^{-k}, t), \tag{43}$$

uniformly, the limit of $\Delta_N X_k(t)$ as $N \rightarrow \infty$ should exist almost surely and is expressed by

$$X^{[1]}(t) = p^{-k} \sum_{m=0}^{\infty} m \psi(mp^{-k}, t) \zeta_{m,k}, \tag{44}$$

which is W -continuous almost surely. $\square$

Now we shall show a sample Gibbs differentiability condition for p -adic stationary processes.

THEOREM 4.4. *If a harmonizable p -adic stationary process $X(t)$ satisfies (35), then it has a version which has the W -continuous r -th Gibbs derivative almost surely.*

Proof. The proof of this theorem is demonstrated only for $r = 1$. First we show that for $0 \leq t < p^k$,

$$|\Delta_N X_{k+1}(t) - \Delta_N X_k(t)| \leq p^{-k} \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} |\zeta_{mp+l,k+1}|. \quad (45)$$

Using the relations,

$$\zeta_{m,k} = \sum_{l=0}^{p-1} \zeta_{mp+l,k+1},$$

and

$$\psi((mp)p^{-k-1}, t) = \psi(mp^{-k}, t),$$

we can rewrite that

$$\begin{aligned} & \Delta_N X_{k+1}(t) - \Delta_N X_k(t) \\ = & \sum_{j=-N}^N p^j \sum_{i=0}^{p-1} A_i \sum_{m=0}^{\infty} \left[\sum_{l=0}^{p-1} \psi((mp+l)p^{-k-1}, t \oplus ip^{-j-1}) \zeta_{mp+l,k+1} \right. \\ & \quad \left. - \psi(mp^{-k}, t \oplus ip^{-j-1}) \zeta_{m,k} \right] \\ = & \sum_{j=-N}^N p^j \sum_{i=0}^{p-1} A_i \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \left(\psi((mp+l)p^{-k-1}, t \oplus ip^{-j-1}) \right. \\ & \quad \left. - \psi(mp^{-k}, t \oplus ip^{-j-1}) \right) \zeta_{mp+l,k+1} \\ = & \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \left[\psi((mp+l)p^{-k-1}, t) \sum_{j=-N}^N p^j \sum_{i=0}^{p-1} A_i \psi((mp+l)p^{-k-1}, ip^{-j-1}) \right. \\ & \quad \left. - \psi(mp^{-k}, t) \sum_{j=-N}^N p^j \sum_{i=0}^{p-1} A_i \psi(mp^{-k}, ip^{-j-1}) \right] \zeta_{mp+l,k+1} \end{aligned} \quad (46)$$

Since the Selfridge functions are Gibbs differentiable, and $\psi(lp^{-k}, t) = 1$ ($0 \leq t < p^k$), we have that for $0 \leq t < p^k$ and uniformly in N ,

$$\begin{aligned} |\Delta_N X_{k+1}(t) - \Delta_N X_k(t)| & \leq \sum_{m=0}^{\infty} \sum_{l=1}^{p-1} |(mp+l)p^{-k-1} - mp^{-k}| |\zeta_{mp+l,k+1}| \\ & \leq p^{-k} \sum_{m=0}^{\infty} \sum_{l=1}^{p-1} |\zeta_{mp+l,k+1}|, \end{aligned} \quad (47)$$

hence we have shown (45). Since

$$\sum_{m=0}^{\infty} \left(F((m+1)p^{-k-1}) - F(mp^{-k}) \right)^{-1/2}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \left(p^{k+1} \sum_{m=np^{k+1}}^{(n+1)p^{k+1}-1} \left[F((m+1)p^{-k-1}) - F(mp^{-k}) \right] \right)^{1/2} \\
&= p^{(k+1)/2} \sum_{n=0}^{\infty} \left(F(n+1) - F(n) \right)^{1/2}, \tag{48}
\end{aligned}$$

we have by (36) that for $\varepsilon_k > 0$ and $A < p^k$,

$$\begin{aligned}
P_k &\equiv \Pr \left\{ \sup_{0 \leq t \leq A} |\Delta_N X_{k+1} - \Delta_N X_k| \geq \varepsilon_k \right\} \\
&\leq \frac{1}{\varepsilon_k} p^{-k} \sum_{m=0}^{\infty} \mathbf{E} |\zeta_{m,k+1}| \\
&\leq \frac{1}{\varepsilon_k} p^{-k} \sum_{m=0}^{\infty} \left(F((m+1)p^{-k-1}) - F(mp^{-k}) \right)^{-1/2} \\
&\leq C \varepsilon_k^{-1} p^{-k/2}, \tag{49}
\end{aligned}$$

where C is a constant. If ε_k is chosen to be $p^{-k/4}$, then $\sum_{k=0}^{\infty} \varepsilon_k < \infty$ and $\sum_{k=0}^{\infty} P_k < \infty$. Borel-Cantelli's lemma therefore shows that

$$\Pr \left\{ \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \sup_{0 \leq t \leq A} |\Delta_N X_{k+1} - \Delta_N X_k| \geq \varepsilon_k \right\} = 0. \tag{50}$$

Hence $\Delta_N X_k$ almost surely converges as $k \rightarrow \infty$ uniformly for $0 \leq t \leq A$ and uniformly in N . For any $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}_0$ and $\{\varepsilon_k\}$ such that $\sum_{k=n_0}^{\infty} \varepsilon_k < \varepsilon$. On account of (50), if $m, n \geq n_0$ then

$$\sup_{0 \leq t \leq A} |\Delta_N X_m(t) - \Delta_N X_n(t)| < \varepsilon \quad \text{a.s.}$$

uniformly in N . Applying Lemma 4.1, and letting $n \rightarrow \infty$, we have that

$$\sup_{0 \leq t \leq A} |\Delta_N X_m(t) - \Delta_N \tilde{X}(t)| < \varepsilon \quad \text{a.s.}$$

uniformly in N . Application of Lemma 4.3 with $r = 1$ shows that $\Delta_N X_m(t)$ converges almost surely as $N \rightarrow \infty$, and hence $\tilde{X}(t)$, which is equivalent to $X(t)$, is Gibbs differentiable almost surely. $\square$

Note that the relation between sample differentiability condition and the mean square differentiability condition of p -adic stationary processes is obtained, i.e., the sample r -th Gibbs differentiability condition is $x \in L^{2r+\alpha}(F)$ ($\alpha > 1$), whereas that in the quadratic mean sense is $x \in L^{2r}(F)$. Hence a sample Gibbs differentiable p -adic stationary process $X(t)$ is quadratic mean Gibbs differentiable and

$$X^{[r]}(t) = \int_0^{\infty} x^r \psi(x, t) \zeta(dx) \quad \text{a.s.} \tag{51}$$

Hereafter we always choose from equivalent processes such a version that has well properties, such as sample W -continuous, if it possesses.

5. Linear Gibbs differential equations

In this section we shall consider stochastic linear Gibbs differential equations, i.e., linear Gibbs differential equations with stochastic signals.

Let $P(s)$ be an m -th order polynomial with complex coefficients,

$$P(s) = \sum_{k=0}^m a_k s^k. \quad (52)$$

If a harmonizable p -SP $\{X(t, \omega)\}$ is m -times Gibbs differentiable almost surely then

$$P(D)X(t, \omega) = \sum_{k=0}^m a_k DX(t, \omega) \quad (53)$$

will be defined. If $X(t, \omega)$ assume the representation (21), then by (51)

$$P(D)X(t, \omega) = \int_0^\infty \psi(t, x) P(x) \zeta(dx, \omega). \quad (54)$$

Its covariance function is also expressed by

$$\text{Cov}(P(D)X(t), P(D)X(s)) = \int_0^\infty \psi(t, x) \overline{\psi(s, x)} |P(x)|^2 F(dx). \quad (55)$$

The same argument is also valid in the quadratic mean convergence case.

Now let us define a stochastic linear Gibbs differential equation. An equation

$$Q(D)Y(t, \omega) = P(D)X(t, \omega) \quad (56)$$

is called a stochastic linear Gibbs differential equation, when $\{X(t, \omega)\}$ is a given stochastic process, $\{Y(t, \omega)\}$ is an unknown stochastic process, $P(s)$ is a polynomial defined by (52), and $Q(s) = \sum_{k=0}^n b_k s^k$. The process $\{X(t, \omega)\}$ is sometimes called a driving process. If the process $\{Y(t, \omega)\}$ satisfies (56) then it is called a solution of the stochastic linear Gibbs differential equation.

THEOREM 5.1. *Let a driving process $\{X(t, \omega)\}$ be a harmonizable p -SP with the harmonic representation (21). Suppose that it is sample m -times Gibbs differentiable. If the equation $Q(s) = 0$ has no zeros on $\mathbf{R}^+$, and*

$$\int_0^\infty x^{2n+\alpha} \left| \frac{P(x)}{Q(x)} \right|^2 F(dx) < \infty \quad (57)$$

for $\alpha > 1$, then

$$Y(t, \omega) = \int_0^\infty \psi(t, x) \frac{P(x)}{Q(x)} \zeta(dx, \omega) \quad (58)$$

satisfies the differential equation (56). Then its covariance function is given by

$$\text{Cov}(Y(t), Y(s)) = \int_0^\infty \psi(t, x) \overline{\psi(s, x)} \left| \frac{P(x)}{Q(x)} \right|^2 F(dx). \quad (59)$$

Proof. Remind that (58) is well-defined by (57). It follows from (57) and Theorem 4.1 that $Y(t)$ or its version, more rigorously, has the sample W -continuous n -th Gibbs derivative, and

$$\begin{aligned} Q(D)Y(t, \omega) &= \int_0^\infty \psi(t, x) Q(x) \frac{P(x)}{Q(x)} \zeta(dx, \omega) \\ &= \int_0^\infty \psi(t, x) P(x) \zeta(dx, \omega) \\ &= P(D)X(t, \omega). \end{aligned}$$

Hence, $\{Y(t, \omega)\}$ satisfies (56). □

Notice that Theorem 5.1 is valid with $\alpha = 0$ for the quadratic mean case.

Example. For fixed $x \in \mathbf{R}^+$ define

$$X(t, \omega) = \psi(t, x)X(\omega) \quad (t \in \mathbf{R}^+), \quad (60)$$

where $X(\omega)$ is a random variable with zero mean and finite variance. Then $\{X(t, \omega); t \in \mathbf{R}^+\}$ thus defined is a harmonizable p -SP, which is expressed by $X(t, \omega) = \psi(t, x)X(\omega)$. With $\{X(t, \omega)\}$ for a driving process, (56) has a solution of the form,

$$Y(t, \omega) = \frac{\psi(t, x)}{Q(x)} X(\omega) \quad (t \in \mathbf{R}^+), \quad (61)$$

provided that $Q(x) \neq 0$. It is clear that $Y(t)$ is sample uniform W -continuous.

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