# GIBBS $p$-ADIC DERIVATIVES AND THEIR APPLICATIONS TO $p$-ADIC STATIONARY PROCESSES

Endow, Yasushi
Department of Industrial and Systems Engineering, Chuo University

https://doi.org/10.5109/13448

出版情報：Bulletin of informatics and cybernetics. 27 (1), pp.147-158, 1995-03. Research Association of Statistical Sciences
バージョン：
権利関係：

Bulletin of Informatics and Cybernetics, Vol. 27, No. 1, 1995

# GIBBS $p$-ADIC DERIVATIVES AND THEIR APPLICATIONS TO $p$-ADIC STATIONARY PROCESSES

By

Yasushi ENDOW*

**Abstract**

In this paper the concept of Gibbs $p$-adic derivatives is introduced to the realm of stochastic processes. Differentiability of $p$-adic stationary processes both in almost sure convergence and in quadratic mean convergence are presented. A stochastic linear Gibbs $p$-adic differential equation is defined, and is solved for driving processes of harmonizable $p$-adic stationary porcesses.

## 1. Introduction

In pioneering work in 1967, J.E.Gibbs [7] introduced the finite Gibbs dyadic derivative in attempting to meet a desire to have a differential operator well suited to discrete functions defined on the finite dyadic group. Its characters or the discrete Walsh functions emarge as the eigenfunctions of the finite Gibbs dyadic derivative [7], i.e.,

$$D\psi_k(x) = k\psi_k(x) \quad (x \in \{0, \cdots, 2^n - 1\}),$$

where $\psi_k$ $(k \in \{0, \cdots, 2^n - 1\})$ is the $k$-th Walsh function. It follows that the finite dyadic derivative is efficiently characterized by the Walsh-Fourier coefficients, i.e.,

$$\widehat{(Df)}(k) = k\hat{f}(k) \quad (k \in \{0, \cdots, 2^n - 1\}),$$

where the hat denotes the W-F transform operator.

The Gibbs derivative had been extended and generalized by some research workers. Among others, Butzer and Wagner[1] made a particularly important generalization for functions on the unit interval, that can be identified with the infinite dyadic group. We will introduce the concept of Gibbs $p$-adic derivative for functions on $\mathbf{R}^+ := [0, \infty)$ (c.f.[10]) and for stochasitic processes on the same interval. We also define a stochastic Gibbs $p$-adic differential equation and give its solution for a special driving process of a $p$-adic stationary process [5].

* Department of Industrial and Systems Engineering, Chuo University, 1-13-27 Kasuga Bunkyou-ku Tokyo, 112 Japan.

## 2. Selfridge functions and Gibbs $p$-adic derivatives

Throughout this paper $p$ is taken to be a fixed, but arbitrary greater than one. For the complete discussion, we introduce the Selfridge functions[9], or the Chrestenson functions[2], that are the extensions of the Walsh functions [11],[6].

Any $x \in \mathbf{R}^+$ with $p^M \leq x < p^{M+1}$ is uniquely expressed by the $p$-adic expansion,

$$x = \sum_{i=-M}^{\infty} x_i p^{-i},$$

provided that the finite expansion is assumed to be taken if it is a $p$-adic rational. The Selfridge function is defined by

$$\psi(t,x) = \epsilon^{t \odot x}, \tag{1}$$

where $\epsilon = \exp(2\pi\sqrt{-1}/p)$, and

$$t \odot x = \sum_{i=-M}^{T+1} x_i t_{1-i} \quad (t, x \in \mathbf{R}^+) \tag{2}$$

with $t = \sum_{i=-T}^{\infty} t_i p^{-i}$ and $x = \sum_{i=-M}^{\infty} x_i p^{-i}$, respectively. It is noted that the Selfridge functions are symmetric in their variables, because of the symmetricity of $\odot$-oparation. It is easy to see that the following properties are satisfied:

$$\psi(t, x \oplus y) = \psi(t,x)\psi(t,y), \tag{3}$$
$$\psi(t, x \ominus y) = \psi(t,x)\overline{\psi(t,y)}, \tag{4}$$

for any $x \in \mathbf{R}^+$ and for a.a. $y \in \mathbf{R}^+$, which depends on $x$. (c.f. [6]).

For $f \in L^1(\mathbf{R}^+)$ its Selfridge-Fourier (S-F) transform is defined by

$$\widehat{f}(x) = \int_0^{\infty} f(t)\overline{\psi(t,x)}dt. \tag{5}$$

For $f,\ g \in L^1(\mathbf{R}^+)$ the integral

$$f * g(t) = \int_0^{\infty} f(t \ominus s)g(s)ds \tag{6}$$

exists and is called the convolution of $f$ and $g$. Then we have that

$$(\widehat{f * g})(x) = \widehat{f}(x)\widehat{g}(x), \tag{7}$$

which is a convolustion theorem similar to the ordinary relationship between a convolution and its Fourier transform. The S-F transform of $f \in L^2(\mathbf{R}^+)$ is defined by

$$\widehat{f}(x) = \underset{A\to\infty}{\text{l.i.m.}} \int_0^{A} f(t)\overline{\psi(t,x)}dt. \tag{8}$$

The inverse S-F transform of $\widehat{f} \in L^2(\mathbf{R}^+)$ can be defined, and is given by

$$f(t) = \underset{A\to\infty}{\text{l.i.m.}} \int_0^{\infty} \widehat{f}(x)\psi(t,x)dx. \tag{9}$$

It is easy to see that

$$\widehat{(\tau_s f)}(x) = \widehat{\psi}(s,x)\widehat{f}(x), \tag{10}$$

$$(\widehat{f\psi}(\cdot,y))(x) = \tau_y\widehat{f}(x), \tag{11}$$

where $\tau_s$ is a $p$-adic shift-operator, which is defined by

$$\tau_s f(t) = f(t \ominus s). \tag{12}$$

If a function $f(t)$ is continuous at $p$-adic irrationals and right continuous at $p$-adic rationals, then it is said to be W-continuous. Note that Selfridge functions are uniform W-continuous.

Now we shall define the Gibbs $p$-adic derivative of a function $f$ on $\mathbf{R}^+$. If the limit of the sum

$$\Delta_N f(t) = \sum_{k=-N}^{N} p^k \sum_{j=0}^{p-1} A_j f(t \oplus jp^{-k-1}) \tag{13}$$

with $A_0 = (p-1)/2$, $A_j = \epsilon^j/(1-\epsilon^j)$, $j = 1, \cdots, p-1$, converges as $N \to \infty$, then the function $f$ is called pointwise Gibbs $p$-adic defferentiable at $t \in \mathbf{R}^+$. The limit is denoted by $Df(t)$ or $f^{[1]}(t)$. If $f$ is Gibbs $p$-adic differentiable at every $x \in \mathbf{R}^+$, then it is simply called Gibbs $p$-adic differentiable.

For $f \in L^r(\mathbf{R}^+)$, $1 \leq r < \infty$, the $L^r$-limit of $\Delta_N f(t)$, if it exists, is called the $L^r$-strong Gibbs $p$-adic derivative. The higher derivatives are defined by induction. It follows from definition that the Selfridge functions are $k$-times differentiable in both the pointwise and the $L^r$-strong senses, and

$$D^k\psi(t,x) = x^k\psi(t,x) \quad (t \in \mathbf{R}^+), \tag{14}$$

where $k \in \mathbf{P} := \{0, 1, \cdots\}$. Hence, putting $P(s) = \sum_{k=0}^{m} a_k s^k$ and by linearlity of the operator $D$,

$$P(D)\psi(t,x) = P(x)\psi(t,x). \tag{15}$$

It is also shown [10] that if $f$, $D^k f \in L^r(\mathbf{R}^+)$; $k \in \mathbf{P}$, $1 \leq r < \infty$, then

$$(\widehat{D^k f})(x) = x^k\widehat{f}(x) \text{ a.e.}, \tag{16}$$

which is a consequence of the fact that Selfridge functions are the eigenfunctions of this differential operator. This shows a characterization of the Gibbs $p$-adic derivatives. The Gibbs $p$-adic differentiator of order $k$ can be considered as a convolution operator with a transformed function $\widehat{g}(x) = x^k$. It follows from (17) and the linearily of the S-F transform that

$$(\widehat{P(D)}f)(x) = P(x)\widehat{f}(x). \tag{17}$$

In the $p$-adic group world the Selfridge functions and the Gibbs $p$-adic derivatives seem to play similar roles as the Euler functions and the Newton-Leibniz derivatives in real world. It is well know in real world that differentiable functions are continuous. However, in the $p$-adic group world Gibbs differentiable functions are not W-continuous, nor are W-continuous funcions Gibbs differentiable.

## 3. Quadratic mean Gibbs differentiability of $p$-adic stationary processes

To begin with we shall consider two types of Gibbs differentiabilities of stochastic processes, according to the types of convergences. Let $\{X(t,\omega);\ t \in \mathbf{R}^+\}$ be a stochastic process. If

$$\Delta_N X(t) = \sum_{k=-N}^{N} p_k \sum_{j=0}^{p-1} A_j X(t \oplus jp^{-k-1}) \tag{18}$$

converges almost surely as $N \to \infty$, then the limit is called the sample Gibbs derivative (sample G-d) of the process at $t$ and is denoted by $X^{[1]}(t)$ or $DX(t)$. The higher order, say $r$-order, $r \in \mathbf{N}$, sample Gibbs derivative is defined by induction. For second order process $\{X(t,\omega);\ t \in \mathbf{R}^+\}$, if the mean square limit of $\Delta_N S(t)$ as $N \to \infty$ exists, then the limit,

$$\lim_{N\to\infty} \mathbf{E}\left|\Delta_N X(t)\right|^2 \tag{19}$$

is called the quadratic mean Gibbs derivative (q.m. G-d) of the process at $t$ or the Gibbs derivative in the mean at $t$, and is denoted by $X^{[1]}(t)$ or $DX(t)$.

Next we shall consider quadratic mean Gibbs differentiability of $p$-adic stationary processes. Let $\{X(t,\omega);\ t \in \mathbf{R}^+\}$ be a second order process with zero mean. If its covariance function $\mathrm{Cov}(X(t), X(s)) = \mathbf{E}X(t,\omega)\overline{X(s,\omega)}$ satisfies that

$$\mathrm{Cov}(X(t), X(s)) = \mathrm{Cov}(X(t \ominus s), X(0)) \quad (\mu(t) \ominus \mu(s) \in \mathbf{F}^c), \tag{20}$$

then it is called a $p$-adic stationary process ($p$-SP) (for detail see [5]). Suppose that it is harmonizable, i.e., it is assumed to be expressible as

$$X(t,\omega) = \int_0^\infty \psi(t,x)\zeta(dx,\omega) \quad (t \in \mathbf{R}^+), \tag{21}$$

where $\zeta$ is an orthogonal random measure with $\mathbf{E}\zeta(A) = 0$ $(A \in \mathcal{B}(\mathbf{R}^+))$. Its covariance function is also expressed by

$$\mathrm{Cov}(X(t), X(s)) = \int_0^\infty \psi(t,x)\overline{\psi(s,x)}F(dx), \tag{22}$$

where $F$ is the spectral distribution function with

$$F(dx) = \mathbf{E}|\zeta(dx,\omega)|^2. \tag{23}$$

A necessary and sufficient condition for harmonizability of a $p$-SP was given by the present author [5]. Note that harmonizable $p$-adic stationary processs are quadratic mean W-continuous.

Now we shall give a quadratic mean Gibbs differentiability condtion for $p$-SP, which is a generalization of the result for dyadic stationary processes [4].

THEOREM 3.1. *Let $r \in \mathbf{P}$. A harmonizable p-SP is r-times quadratic mean Gibbs differentiable, if and only if its spectral distribution has the 2r-th moment, i.e.,*

$$\int_0^\infty x^{2r}F(dx) < \infty. \tag{24}$$

*Then the quadratic mean r-th Gibbs derivative of the process is expressed by*

$$DX(t,\omega) = \int_0^\infty x^r \psi(t,x)\zeta(dx,\omega). \tag{25}$$

*Proof.* Before proving this we note that (24) is sometimes written as $x \in L^{2r}(F)$. We will only prove it for $r = 1$, and the other cases are shown similarly.
(sufficiency.) After some familiar manipulations, we obtain that

$$\mathbf{E}|\Delta_N X(t,\omega)|^2 = \int_0^\infty |\Delta_N \psi(t,x)|^2 F(dx). \tag{26}$$

Since

$$\lim_{N\to\infty} \Delta_N \psi(t,x) = x\psi(t,x), \tag{27}$$

uniformly, (26) and Lebesgue's convergence theorem show that

$$\lim_{N\to\infty} \int_0^\infty |\Delta_N \psi(t,x)|^2 F(dx) = \int_0^\infty |x\psi(t,x)|^2 F(dx) \tag{28}$$

$$= \int_0^\infty x^2 F(dx), \tag{29}$$

which is finite by (24).
(necessity.) If the process is Gibbs differentiable, i.e., the left side of (26) converges finitely as $N$ tends to infinity, then the integral in the right side of (29) is finite. In the same way as (26), we will have that

$$\mathbf{E}\left|\Delta_N X(t,\omega) - \int_0^\infty x\psi(t,x)\zeta(dx,\omega)\right|^2 = \int_0^\infty |\Delta_N \psi(t,x) - x\psi(t,x)|^2 F(dx). \tag{30}$$

Thus the proof is completed by (27). □

COROLLARY 3.2. *If the condition in Theorem 3.1 is satisfied, then the q.m. i-th* $(0 \le i \le r)$ *Gibbs derivatives of the p-SP's are harmonizable, and their covariance functions are expressed by*

$$\mathrm{Cov}(D^{[i]}X(t), D^{[j]}X(s)) = \int_0^\infty \psi(t,x)\overline{\psi(s,x)} x^{2(i+j)} F(dx) \quad (0 \le i,\ j \le r). \tag{31}$$

This is a direct consequence of Theorem 3.1.

## 4. Sample Gibbs differentiability of $p$-adic stationary processes

In this section sample Gibbs differentiation of $p$-adic stationary processes will be considered. The idea comes from Kawata and Kubo's work on classical stationary processes[8]. Let us put that for $k,\ n \in \mathbf{N}_0$

$$\zeta_{n,k} = \int_{np^{-k}}^{(n+1)p^{-k}} \zeta(dx), \tag{32}$$

where $\zeta$ is the spectral random measure of the $p$-adic staionary porcess $X(t)$ and

$$X_k(t) = \sum_{n=0}^{\infty} \psi(np^{-k}, t)\zeta_{n,k}. \tag{33}$$

The series in the right side of (33) converges in $L^2(\Omega)$-norm, and the sum is an approximation of $X(t)$(see [3]). Let us introduce some lemmas for later use.

LEMMA 4.1. *If the spectral distribution function satisfies that for some $\alpha > 1$*

$$\int_0^\infty x^\alpha F(dx) < \infty, \tag{34}$$

*then the series (33) absolutely converges almost surely, and $X_k(t)$ converges uniformly for every finite interval $0 \leq t \leq A$ as $k \to \infty$ almost surely to a p-adic sationary process $\widetilde{X}(t)$, which is sample W-continuous, and a version of $X(t)$.*

This is a $p$-adic extension of Theorem 6.1 in [3]. The proof of this lemma is similar to the theorem, so it is omitted here.

LEMMA 4.2. *Let $r \in \mathbf{N}_0$. If for some $\alpha > 1$ the spectral distribution function satisfies that*

$$\int_0^\infty x^{2r+\alpha} F(dx) < \infty, \tag{35}$$

*then*

$$\sum_{n=0}^{\infty} n^r \big(F(n+1) - F(n)\big)^{1/2} < \infty. \tag{36}$$

*Hence*

$$\sum_{n=0}^{\infty} n^r |\zeta_{n,k}| < \infty \text{ a.s.} \tag{37}$$

*Proof.* It is clear that by Schwartz's inequatity

$$\begin{aligned} &\sum_{n=0}^{\infty} n^r \big(F(n+1) - F(n)\big)^{1/2} \\ \leq\ & \Big(\sum_{n=0}^{\infty} n^{-\alpha}\Big)^{1/2} \Big(\sum_{n=0}^{\infty} n^{2r+\alpha}(F(n+1) - F(n))\Big)^{1/2} \\ \leq\ & \Big(\sum_{n=0}^{\infty} n^{-\alpha}\Big)^{1/2} \Big(\int_0^\infty n^{2r+\alpha} F(dx)\Big)^{1/2} < \infty. \end{aligned} \tag{38}$$

To show (37) it is sufficient to show that

$$\mathbf{E} \sum_{n=0}^{\infty} n^r |\zeta_{n,k}| < \infty. \tag{39}$$

By Schwartz's inequality, we have that

$$\begin{aligned} \mathbf{E}\sum_{n=0}^{\infty} n^r|\zeta_{n,k}| &\le \sum_{n=0}^{\infty} n^r\left(\mathbf{E}|\zeta_{n,k}|^2\right)^{1/2} \\ &\le \sum_{n=0}^{\infty} n^r\left(F((n+1)p^{-k}) - F(np^{-k})\right)^{1/2} \\ &= S_k \text{ (say)}. \end{aligned} \tag{40}$$

Rewriting $S_k$, we see that

$$\begin{aligned} S_k &= \sum_{m=0}^{\infty}\sum_{n=mp^k}^{(m+1)p^k-1} n^r\left(F((n+1)p^{-k}) - F(np^{-k})\right)^{1/2} \\ &\le \sum_{m=0}^{\infty}\left((m+1)p^k\right)^r \sum_{n=mp^k}^{(m+1)p^k-1}\left(F((n+1)p^{-k}) - F(np^{-k})\right)^{1/2} \\ &\le \sum_{m=0}^{\infty}(m+1)^r p^{rk}\left(p^k \sum_{n=mp^k}^{(m+1)p^k-1}\left(F((n+1)p^{-k}) - F(np^{-k})\right)\right)^{1/2} \\ &\le p^{(r+1/2)k}\sum_{m=0}^{\infty}(m+1)^r\left(F(m+1) - F(m)\right)^{1/2} \end{aligned} \tag{41}$$

Hence the proof is completed by (36). □

LEMMA 4.3. *If (35) holds then $X_k(t)$ has a version which is a harmonizable $p$-adic stationary process with the almost sure W-continuous $r$-th Gibbs derivative.*

*Proof.* We shall only porve it for $r = 1$, since the same argument will be applied to the other cases. Since by (37), for every $N \in \mathbf{N}_0$

$$\begin{aligned} |\Delta_N X_k| &= \left|\sum_{m=0}^{\infty}\Delta_N\psi(mp^{-k}, t)\zeta_{m,k}\right| \\ &\le \sum_{m=0}^{\infty} mp^{-k}|\zeta_{m,k}| < \infty \text{ a.s.} \end{aligned} \tag{42}$$

and

$$\lim_{N\to\infty}\Delta_N\psi(mp^{-k}, t) = mp^{-k}\psi(mp^{-k}, t), \tag{43}$$

unifromly, the limit of $\Delta_N X_k(t)$ as $N \to \infty$ should exist almost surely and is expressed by

$$X^{[1]}(t) = p^{-k}\sum_{m=0}^{\infty} m\psi(mp^{-k}, t)\zeta_{m,k}, \tag{44}$$

which is W-continuous almost surely. □

Now we shall show a sample Gibbs differentiability condition for $p$-adic stationary processes.

THEOREM 4.4. *If a harmonizable p-adic stationary process $X(t)$ satisfies (35), then it has a version which has the W-continuous r-th Gibbs derivative almost surely.*

*Proof.* The proof of this theorem is demonstrated only for $r = 1$. First we show that for $0 \le t < p^k$,

$$\left|\Delta_N X_{k+1}(t) - \Delta_N X_k(t)\right| \le p^{-k} \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \left|\zeta_{mp+l,k+1}\right|. \tag{45}$$

Using the relations,

$$\zeta_{m,k} = \sum_{l=0}^{p-1} \zeta_{mp+l,k+1},$$

and

$$\psi((mp)p^{-k-1}, t) = \psi(mp^{-k}, t),$$

we can rewrite that

$$\begin{aligned}
&\Delta_N X_{k+1}(t) - \Delta_N X_k(t) \\
= &\sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \sum_{m=0}^{\infty} \Big[ \sum_{l=0}^{p-1} \psi((mp+l)p^{-k-1}, t \oplus ip^{-j-1}) \zeta_{mp+l,k+1} \\
&\qquad - \psi(mp^{-k}, t \oplus ip^{-j-1}) \zeta_{m,k} \Big] \\
= &\sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \Big( \psi((mp+l)p^{-k-1}, t \oplus ip^{-j-1}) \\
&\qquad - \psi(mp^{-k}, t \oplus ip^{-j-1}) \Big) \zeta_{mp+l,k+1} \\
= &\sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \Big[ \psi((mp+l)p^{-k-1}, t) \sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \psi((mp+l)p^{-k-1}, ip^{-j-1}) \\
&\qquad - \psi(mp^{-k}, t) \sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \psi(mp^{-k}, ip^{-j-1}) \Big] \zeta_{mp+l,k+1}
\end{aligned} \tag{46}$$

Since the Selfridge functions are Gibbs differentiable, and $\psi(lp^{-k}, t) = 1 \ \ (0 \le t < p^k)$, we have that for $0 \le t < p^k$ and uniformly in $N$ ,

$$\begin{aligned}
\left|\Delta_N X_{k+1}(t) - \Delta_N X_k(t)\right| &\le \sum_{m=0}^{\infty} \sum_{l=1}^{p-1} \left|(mp+l)p^{-k-1} - mp^{-k}\right| \left|\zeta_{mp+l,k+1}\right| \\
&\le p^{-k} \sum_{m=0}^{\infty} \sum_{l=1}^{p-1} \left|\zeta_{mp+l,k+1}\right|,
\end{aligned} \tag{47}$$

hence we have shown (45). Since

$$\sum_{m=0}^{\infty} \Big( F\big((m+1)p^{-k-1}\big) - F\big(mp^{-k}\big) \Big)^{-1/2}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty}\Big(p^{k+1}\sum_{m=np^{k+1}}^{(n+1)p^{k+1}-1}\Big[F\big((m+1)p^{-k-1}\big)-F\big(mp^{-k}\big)\Big]\Big)^{1/2} \\
&= p^{(k+1)/2}\sum_{n=0}^{\infty}\Big(F(n+1)-F(n)\Big)^{1/2}, \qquad (48)
\end{aligned}$$

we have by (36) that for $\varepsilon_k > 0$ and $A < p^k$,

$$\begin{aligned}
P_k &\equiv \Pr\Big\{\sup_{0\leq t\leq A}\big|\Delta_N X_{k+1}-\Delta_N X_k\big|\geq \varepsilon_k\Big\} \\
&\leq \frac{1}{\varepsilon_k}p^{-k}\sum_{m=0}^{\infty}\mathbf{E}\big|\zeta_{m,k+1}\big| \\
&\leq \frac{1}{\varepsilon_k}p^{-k}\sum_{m=0}^{\infty}\Big(F\big((m+1)p^{-k-1}\big)-F\big(mp^{-k}\big)\Big)^{-1/2} \\
&\leq C\varepsilon_k^{-1}p^{-k/2}, \qquad (49)
\end{aligned}$$

where $C$ is a constant. If $\varepsilon_k$ is choson to be $p^{-k/4}$, then $\sum_{k=0}^{\infty}\varepsilon_k < \infty$ and $\sum_{k=0}^{\infty}P_k < \infty$. Borel-Cantelli's lemma therefore shows that

$$\Pr\Big\{\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\sup_{0\leq t\leq A}\big|\Delta_N X_{k+1}-\Delta_N X_k\big|\geq\varepsilon_k\Big\}=0. \qquad (50)$$

Hence $\Delta_N X_k$ almost surely converges as $k\to\infty$ uniformly for $0\leq t\leq A$ and uniformly in $N$. For any $\varepsilon>0$ there exists $n_0\in\mathbf{N}_0$ and $\{\varepsilon_k\}$ such that $\sum_{k=n_0}^{\infty}\varepsilon_k<\varepsilon$. On account of (50), if $m,n\geq n_0$ then

$$\sup_{0\leq t\leq A}\big|\Delta_N X_m(t)-\Delta_N X_n(t)\big|<\varepsilon \ \text{ a.s.}$$

uniformly in $N$. Applying Lemma 4.1, and letting $n\to\infty$, we have that

$$\sup_{0\leq t\leq A}\big|\Delta_N X_m(t)-\Delta_N\widetilde{X}(t)\big|<\varepsilon \ \text{ a.s.}$$

uniformly in $N$. Application of Lemma 4.3 with $r=1$ shows that $\Delta_N X_m(t)$ converges almost surely as $N\to\infty$, and hence $\widetilde{X}(t)$, which is equivalent to $X(t)$, is Gibbs differentiable almost surely. □

Note that the relation between sample differntiability condition and the mean square differentiability condition of $p$-adic stationary processes is obtained, i.e., the sample $r$-th Gibbs differntiability condition is $x\in L^{2r+\alpha}(F)$ $(\alpha>1)$, whereas that in the quadratic mean sense is $x\in L^{2r}(F)$. Hence a sample Gibbs differntiable $p$-adic stationary process $X(t)$ is quadratic mean Gibbs differentiable and

$$X^{[r]}(t)=\int_0^{\infty}x^r\psi(x,t)\zeta(dx) \ \text{ a.s.} \qquad (51)$$

Hereafter we always choose from equivalent processes such a version that has well properties, such as sample W-continuous, if it posseses.

## 5. Linear Gibbs differential equations

In this section we shall consider stochastic linear Gibbs differential equations, i.e., linear Gibbs differential equations with stochastic signals.

Let $P(s)$ be an $m$-th order polynomial with complex coefficients,

$$P(s) = \sum_{k=0}^{m} a_k s^k. \tag{52}$$

If a harmonizable $p$-SP $\{X(t,\omega)\}$ is $m$-times Gibbs differentiable almost surely then

$$P(D)X(t,\omega) = \sum_{k=0}^{m} a_k DX(t,\omega) \tag{53}$$

will be defined. If $X(t,\omega)$ assume the representation (21) , then by (51)

$$P(D)X(t,\omega) = \int_0^\infty \psi(t,x)P(x)\zeta(dx,\omega). \tag{54}$$

Its covariance function is also expressed by

$$\mathrm{Cov}(P(D)X(t), P(D)X(s)) = \int_0^\infty \psi(t,x)\overline{\psi(s,x)}\left|P(x)\right|^2 F(dx). \tag{55}$$

The same aregument is also valid in the quadratic mean convergence case.

Now let us define a stochastic linear Gibbs differential equation. An equation

$$Q(D)Y(t,\omega) = P(D)X(t,\omega) \tag{56}$$

is called a stochastic linear Gibbs differential equation, when $\{X(t,\omega)\}$ is a given stochastic process, $\{Y(t,\omega)\}$ is an unknown stochastic process, $P(s)$ is a polynomial defined by (52), and $Q(s) = \sum_{k=0}^{n} b_k s^k$. The process $\{X(t,\omega)\}$ is sometimes called a driving process. If the process $\{Y(t,\omega)\}$ satisfies (56) then it is called a solution of the stochastic linear Gibbs differential equation.

THEOREM 5.1. *Let a driving process $\{X(t,\omega)\}$ be a harmonizable p-SP with the harmonic representation (21) . Suppose that it is sample m-times Gibbs differentiable. If the equation $Q(s) = 0$ has no zeros on $\mathbf{R}^+$, and*

$$\int_0^\infty x^{2n+\alpha}\left|\frac{P(x)}{Q(x)}\right|^2 F(dx) < \infty \tag{57}$$

*for $\alpha > 1$, then*

$$Y(t,\omega) = \int_0^\infty \psi(t,x)\frac{P(x)}{Q(x)}\zeta(dx,\omega) \tag{58}$$

*satisfies the differetial equation (56). Then its covariance function is given by*

$$\mathrm{Cov}(Y(t), Y(s)) = \int_0^\infty \psi(t,x)\overline{\psi(s,x)}\left|\frac{P(x)}{Q(x)}\right|^2 F(dx). \tag{59}$$

*Proof.* Remind that (58) is well-defined by (57). It follows from (57) and Theorem 4.1 that $Y(t)$ or its version, more rigorously, has the sample W-continuous $n$-th Gibbs derivative, and

$$\begin{aligned} Q(D)Y(t,\omega) &= \int_0^\infty \psi(t,x)Q(x)\frac{P(x)}{Q(x)}\zeta(dx,\omega) \\ &= \int_0^\infty \psi(t,x)P(x)\zeta(dx,\omega) \\ &= P(D)X(t,\omega). \end{aligned}$$

Hence, $\{Y(t,\omega)\}$ satisfies (56). □

Notice that Theorem 5.1 is valid with $\alpha = 0$ for the quadratic mean case.

**Example.** For fixed $x \in \mathbf{R}^+$ define

$$X(t,\omega) = \psi(t,x)X(\omega) \quad (t \in \mathbf{R}^+), \tag{60}$$

where $X(\omega)$ is a random variable with zero mean and finte variance. Then $\{X(t,\omega); t \in \mathbf{R}^+\}$ thus defined is a harmonizable $p$-SP, which is expressed by $X(t,\omega) = \psi(t,x)X(\omega)$. With $\{X(t,\omega)\}$ for a driving process, (56) has a solution of the form,

$$Y(t,\omega) = \frac{\psi(t,x)}{Q(x)}X(\omega) \quad (t \in \mathbf{R}^+), \tag{61}$$

provided that $Q(x) \neq 0$. It is clear that $Y(t)$ is sample uniform W-continuous.

## References

[1] Butzer, P. L. and Wagner, H.J.: *Walsh-Fourier series and the concept of a derivative*, Applicable Anal., **3** (1973), 29-46.

[2] Chrestenson, H. E.: *A class of generalized Walsh functions*, Pacific J. Math., **5** (1955), 17-31.

[3] Endow, Y.: *The Walsh series of a syadic stationary process*, Tôhoku Math. J., **38** (1986), 501-512.

[4] Endow, Y.: *On a condition for the dyadic differentiability of dyadic stationary processes*, Rep. Stat. Appl. Res., JUSE, **34** (1987), 1-5.

[5] Endow, Y.: *Harmonizability of a p-adic stationary process*, Bull. Information and Cyberntics, **26** (1994), 75-86.

[6] Fine, N. J.: *The generalized Walsh functions*, Trans. Amer. Math. Soc., **69** (1950), 66-77.

[7] Gibbs, J. E.: *Walsh spectrometry, a form of spectral analysis well suited to binary digital computation*, NPL DES Rept., in UK, (1967).

[8] Kawata, T., and Kubo, I.: *Sample properties of weakly stationary processes*, Nagoya Math. J., **39**(1970), 7-21.

[9] Selfridge, R. G.: *Generalized Walsh transform*, Pacific. J. Math., **5** (1955), 451-490.

[10] Su Weiyi: *Gibbs derivatives and their applications*, Rept. Inst. Math., Nanjing Univ., (1991).

[11] Walsh, J. L.: *A closed set of normal orthogonal functions*, Amer. J. Math., **45** (1923), 5-24.

*Received July 1, 1994*
*Revised January 12, 1995*