GIBBS $ p $-ADIC DERIVATIVES AND THEIR APPLICATIONS TO $ p $-ADIC STATIONARY PROCESSES

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GIBBS \( p \)-ADIC DERIVATIVES AND THEIR APPLICATIONS TO \( p \)-ADIC STATIONARY PROCESSES

By

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Abstract

In this paper the concept of Gibbs \( p \)-adic derivatives is introduced to the realm of stochastic processes. Differentiability of \( p \)-adic stationary processes both in almost sure convergence and in quadratic mean convergence are presented. A stochastic linear Gibbs \( p \)-adic differential equation is defined, and is solved for driving processes of harmonizable \( p \)-adic stationary processes.

1. Introduction

In pioneering work in 1967, J.E. Gibbs [7] introduced the finite Gibbs dyadic derivative in attempting to meet a desire to have a differential operator well suited to discrete functions defined on the finite dyadic group. Its characters or the discrete Walsh functions emerge as the eigenfunctions of the finite Gibbs dyadic derivative [7], i.e.,

\[
D\psi_k(x) = k\psi_k(x) \quad (x \in \{0, \ldots, 2^n - 1\}),
\]

where \( \psi_k \ (k \in \{0, \ldots, 2^n - 1\}) \) is the \( k \)-th Walsh function. It follows that the finite dyadic derivative is efficiently characterized by the Walsh-Fourier coefficients, i.e.,

\[
(\hat{Df})(k) = k\hat{f}(k) \quad (k \in \{0, \ldots, 2^n - 1\}),
\]

where the hat denotes the W-F transform operator.

The Gibbs derivative had been extended and generalized by some research workers. Among others, Butzer and Wagner[1] made a particularly important generalization for functions on the unit interval, that can be identified with the infinite dyadic group. We will introduce the concept of Gibbs \( p \)-adic derivative for functions on \( \mathbb{R}^+ := [0, \infty) \) (c.f.[10]) and for stochastic processes on the same interval. We also define a stochastic Gibbs \( p \)-adic differential equation and give its solution for a special driving process of a \( p \)-adic stationary process [5].

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2. Selfridge functions and Gibbs p-adic derivatives

Throughout this paper, $p$ is taken to be a fixed, but arbitrary greater than one. For the complete discussion, we introduce the Selfridge functions [9], or the Chrestenson functions [2], that are the extensions of the Walsh functions [11], [6].

Any $x \in \mathbb{R}^+$ with $p^M \leq x < p^{M+1}$ is uniquely expressed by the $p$-adic expansion,

$$x = \sum_{i=-M}^{\infty} x_ip^{-i},$$

provided that the finite expansion is assumed to be taken if it is a $p$-adic rational. The Selfridge function is defined by

$$\psi(t, x) = e^{tx},$$

where $t = \exp(2\pi\sqrt{-1}/p)$, and

$$t \odot x = \sum_{i=-M}^{T+1} x_it_{1-i} \quad (t, x \in \mathbb{R}^+)$$

with $t = \sum_{i=-T}^{\infty} t_ip^{-i}$ and $x = \sum_{i=-M}^{\infty} x_ip^{-i}$, respectively. It is noted that the Selfridge functions are symmetric in their variables, because of the symmetricity of $\odot$-operation.

It is easy to see that the following properties are satisfied:

$$\psi(t, x \odot y) = \psi(t, x)\psi(t, y),$$

$$\psi(t, x \odot y) = \psi(t, x)\psi(t, y),$$

for any $x \in \mathbb{R}^+$ and for a.a. $y \in \mathbb{R}^+$, which depends on $x$. (c.f. [6]).

For $f \in L^1(\mathbb{R}^+)$ its Selfridge-Fourier (S-F) transform is defined by

$$\hat{f}(x) = \int_0^\infty f(t)\psi(t, x)dt.$$  \hspace{1cm} (5)

For $f, g \in L^1(\mathbb{R}^+)$ the integral

$$f * g(t) = \int_0^\infty f(t \odot s)g(s)ds$$  \hspace{1cm} (6)

exists and is called the convolution of $f$ and $g$. Then we have that

$$(\hat{f} * \hat{g})(x) = \hat{f}(x)\hat{g}(x),$$

which is a convolustion theorem similar to the ordinary relationship between a convolution and its Fourier transform. The S-F transform of $f \in L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(x) = \lim_{A \to \infty} \int_0^A f(t)\psi(t, x)dt.$$  \hspace{1cm} (8)

The inverse S-F transform of $\hat{f} \in L^2(\mathbb{R}^+)$ can be defined, and is given by

$$f(t) = \lim_{A \to \infty} \int_0^\infty \hat{f}(x)\psi(t, x)dx.$$  \hspace{1cm} (9)
It is easy to see that
\begin{align}
(\tau_s f)(x) &= \hat{\psi}(s, x) \hat{f}(x), \\
(f \psi^s)(x) &= \tau_s f(x),
\end{align}
where \(\tau_s\) is a \(p\)-adic shift-operator, which is defined by
\[ \tau_s f(t) = f(t \odot s). \]

If a function \(f(t)\) is continuous at \(p\)-adic irrationals and right continuous at \(p\)-adic rationals, then it is said to be \(W\)-continuous. Note that Selfridge functions are uniform \(W\)-continuous.

Now we shall define the Gibbs \(p\)-adic derivative of a function \(f\) on \(\mathbb{R}^+\). If the limit of the sum
\[
\Delta_N f(t) = \sum_{k=-N}^{N} p^k \sum_{j=0}^{p-1} A_j f(t \odot j p^{-k-1})
\]
with \(A_0 = (p - 1)/2, \ A_j = c^j/(1 - c^j), \ j = 1, \ldots, p - 1\), converges as \(N \to \infty\), then the function \(f\) is called pointwise Gibbs \(p\)-adic differentiable at \(t \in \mathbb{R}^+\). The limit is denoted by \(Df(t)\) or \(f^{[1]}(t)\). If \(f\) is Gibbs \(p\)-adic differentiable at every \(x \in \mathbb{R}^+\), then it is simply called Gibbs \(p\)-adic differentiable.

For \(f \in L^r(\mathbb{R}^+), 1 \leq r < \infty\), the \(L^r\)-limit of \(\Delta_N f(t)\), if it exists, is called the \(L^r\)-strong Gibbs \(p\)-adic derivative. The higher derivatives are defined by induction. It follows from definition that the Selfridge functions are \(k\)-times differentiable in both the pointwise and the \(L^r\)-strong senses, and
\[
D^k \psi(t, x) = x^k \psi(t, x) \quad (t \in \mathbb{R}^+),
\]
where \(k \in \mathbb{P} := \{0, 1, \cdots\} \). Hence, putting \(P(s) = \sum_{k=0}^{m} a_k s^k\) and by linearity of the operator \(D\),
\[
P(D) \psi(t, x) = P(x) \psi(t, x).
\]
It is also shown [10] that if \(f, \ D^k f \in L^r(\mathbb{R}^+); \ k \in \mathbb{P}, \ 1 \leq r < \infty\), then
\[
(\widehat{D^k f})(x) = x^k \hat{f}(x) \ a.e.,
\]
which is a consequence of the fact that Selfridge functions are the eigenfunctions of this differential operator. This shows a characterization of the Gibbs \(p\)-adic derivatives. The Gibbs \(p\)-adic differentiator of order \(k\) can be considered as a convolution operator with a transformed function \(\hat{g}(x) = x^k\). It follows from (17) and the linearity of the S-F transform that
\[
(P(D)f)(x) = P(x)\hat{f}(x).
\]
In the \(p\)-adic group world the Selfridge functions and the Gibbs \(p\)-adic derivatives seem to play similar roles as the Euler functions and the Newton-Leibniz derivatives in real world. It is well known in real world that differentiable functions are continuous. However, in the \(p\)-adic group world Gibbs differentiable functions are not \(W\)-continuous, nor are \(W\)-continuous functions Gibbs differentiable.
3. Quadratic mean Gibbs differentiability of \( p \)-adic stationary processes

To begin with we shall consider two types of Gibbs differentiabilities of stochastic processes, according to the types of convergences. Let \( \{X(t, \omega); t \in \mathbb{R}^+\} \) be a stochastic process. If

\[
\Delta_N X(t) = \sum_{k=-N}^{N} p_k \sum_{j=0}^{p-1} A_j X(t \oplus jp^{-k-1})
\]

(18)

converges almost surely as \( N \to \infty \), then the limit is called the sample Gibbs derivative (sample G-d) of the process at \( t \) and is denoted by \( X^{[1]}(t) \) or \( DX(t) \). The higher order, say \( r \)-order, \( r \in \mathbb{N} \), sample Gibbs derivative is defined by induction. For second order process \( \{X(t, \omega); t \in \mathbb{R}^+\} \), if the mean square limit of \( \Delta_N S(t) \) as \( N \to \infty \) exists, then

\[
\lim_{N \to \infty} E|\Delta_N X(t)|^2
\]

(19)

is called the quadratic mean Gibbs derivative (q.m. G-d) of the process at \( t \) or the Gibbs derivative in the mean at \( t \), and is denoted by \( X^{[1]}(t) \) or \( DX(t) \).

Next we shall consider quadratic mean Gibbs differentiability of \( p \)-adic stationary processes. Let \( \{X(t, \omega); t \in \mathbb{R}^+\} \) be a second order process with zero mean. If its covariance function \( \text{Cov}(X(t), X(s)) = E X(t, \omega) X(s, \omega) \) satisfies that

\[
\text{Cov}(X(t), X(s)) = \text{Cov}(X(t \ominus s), X(0)) \quad (p(t) \ominus p(s) \in \mathbb{F}_c),
\]

(20)

then it is called a \( p \)-adic stationary process (\( p \)-SP) (for detail see [5]). Suppose that it is harmonizable, i.e., it is assumed to be expressible as

\[
X(t, \omega) = \int_0^\infty \psi(t, x) \zeta(dx, \omega) \quad (t \in \mathbb{R}^+),
\]

(21)

where \( \zeta \) is an orthogonal random measure with \( E\zeta(A) = 0 \) \( (A \in \mathcal{B}(\mathbb{R}^+)) \). Its covariance function is also expressed by

\[
\text{Cov}(X(t), X(s)) = \int_0^\infty \psi(t, x) \overline{\psi}(s, x) F(dx),
\]

(22)

where \( F \) is the spectral distribution function with

\[
F(dx) = E|\zeta(dx, \omega)|^2.
\]

(23)

A necessary and sufficient condition for harmonizability of a \( p \)-SP was given by the present author [5]. Note that harmonizable \( p \)-adic stationary processes are quadratic mean \( W \)-continuous.

Now we shall give a quadratic mean Gibbs differentiability condition for \( p \)-SP, which is a generalization of the result for dyadic stationary processes [4].

**Theorem 3.1.** Let \( r \in \mathbb{P} \). A harmonizable \( p \)-SP is \( r \)-times quadratic mean Gibbs differentiable, if and only if its spectral distribution has the \( 2r \)-th moment, i.e.,

\[
\int_0^\infty x^{2r} F(dx) < \infty.
\]

(24)
Then the quadratic mean r-th Gibbs derivative of the process is expressed by

$$DX(t, \omega) = \int_0^\infty x^r \psi(t, x) \zeta(dx, \omega).$$

(25)

Proof. Before proving this we note that (24) is sometimes written as $x \in L^{2r}(F)$. We will only prove it for $r = 1$, and the other cases are shown similarly.

(sufficiency.) After some familiar manipulations, we obtain that

$$E|\Delta_N X(t, \omega)|^2 = \int_0^\infty |\Delta_N \psi(t, x)|^2 F(dx).$$

(26)

Since

$$\lim_{N \to \infty} \Delta_N \psi(t, x) = x \psi(t, x),$$

(27)

uniformly, (26) and Lebesgue's convergence theorem show that

$$\lim_{N \to \infty} \int_0^\infty |\Delta_N \psi(t, x)|^2 F(dx) = \int_0^\infty |x \psi(t, x)|^2 F(dx)$$

(28)

$$= \int_0^\infty x^2 F(dx),$$

(29)

which is finite by (24).

(necessity.) If the process is Gibbs differentiable, i.e., the left side of (26) converges finitely as $N$ tends to infinity, then the integral in the right side of (29) is finite. In the same way as (26), we will have that

$$E|\Delta_N X(t, \omega) - \int_0^\infty x \psi(t, x) \zeta(dx, \omega)|^2 = \int_0^\infty |\Delta_N \psi(t, x) - x \psi(t, x)|^2 F(dx).$$

(30)

Thus the proof is completed by (27). □

Corollary 3.2. If the condition in Theorem 3.1 is satisfied, then the q.m. i-th (0 < i < r) Gibbs derivatives of the p-SP's are harmonizable, and their covariance functions are expressed by

$$\text{Cov}(D^{[i]} X(t), D^{[j]} X(s)) = \int_0^\infty \psi(t, x) \overline{\psi(s, x)} x^{2(i+j)} F(dx) \quad (0 \leq i, j \leq r).$$

(31)

This is a direct consequence of Theorem 3.1.

4. Sample Gibbs differentiability of p-adic stationary processes

In this section sample Gibbs differentiation of p-adic stationary processes will be considered. The idea comes from Kawata and Kubo's work on classical stationary processes[8]. Let us put that for $k, n \in \mathbb{N}_0$

$$\zeta_n, k = \int_{np^{-k}}^{(n+1)p^{-k}} \zeta(dx),$$

(32)
where \( \zeta \) is the spectral random measure of the \( p \)-adic stationary process \( X(t) \) and

\[
X_k(t) = \sum_{n=0}^{\infty} \psi(np^{-k}, t) \zeta_{n,k}.
\]

The series in the right side of (33) converges in \( L^2(\Omega) \)-norm, and the sum is an approximation of \( X(t) \) (see [3]). Let us introduce some lemmas for later use.

**Lemma 4.1.** If the spectral distribution function satisfies that for some \( \alpha > 1 \)

\[
\int_0^{\infty} x^\alpha F(dx) < \infty,
\]

then the series (33) absolutely converges almost surely, and \( X_k(t) \) converges uniformly for every finite interval \( 0 < t < A \) as \( k \to \infty \) almost surely to a \( p \)-adic stationary process \( \tilde{X}(t) \), which is sample \( W \)-continuous, and a version of \( X(t) \).

This is a \( p \)-adic extension of Theorem 6.1 in [3]. The proof of this lemma is similar to the theorem, so it is omitted here.

**Lemma 4.2.** Let \( r \in \mathbb{N}_0 \). If for some \( \alpha > 1 \) the spectral distribution function satisfies that

\[
\sum_{n=0}^{\infty} n^r (F(n+1) - F(n))^{1/2} < \infty.
\]

Hence

\[
\sum_{n=0}^{\infty} n^r |\zeta_{n,k}| < \infty \text{ a.s.}
\]

**Proof.** It is clear that by Schwartz's inequality

\[
\sum_{n=0}^{\infty} n^r (F(n+1) - F(n))^{1/2}
\leq \left( \sum_{n=0}^{\infty} n^{-\alpha} \right)^{1/2} \left( \sum_{n=0}^{\infty} n^{2r+\alpha} (F(n+1) - F(n)) \right)^{1/2}
\leq \left( \sum_{n=0}^{\infty} n^{-\alpha} \right)^{1/2} \left( \int_0^{\infty} n^{2r+\alpha} F(dx) \right)^{1/2} < \infty.
\]

To show (37) it is sufficient to show that

\[
\mathbb{E} \sum_{n=0}^{\infty} n^r |\zeta_{n,k}| < \infty.
\]
By Schwartz's inequality, we have that

$$
E \sum_{n=0}^{\infty} n^r |\zeta_{n,k}| \leq \sum_{n=0}^{\infty} n^r \left( E |\zeta_{n,k}|^2 \right)^{1/2} \\
\leq \sum_{n=0}^{\infty} n^r \left( F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
= S_k \text{ (say)}. \quad (40)
$$

Rewriting $S_k$, we see that

$$
S_k = \sum_{m=0}^{\infty} \sum_{n=mp^k}^{(m+1)p^k-1} n^r \left( F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
\leq \sum_{m=0}^{\infty} (m+1)^r \sum_{n=mp^k}^{(m+1)p^k-1} \left( F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
\leq \sum_{m=0}^{\infty} (m+1)^r p^r k \sum_{n=mp^k}^{(m+1)p^k-1} \left( F((n+1)p^{-k}) - F(np^{-k}) \right)^{1/2} \\
\leq p^{(r+1)/2} \sum_{m=0}^{\infty} (m+1)^r \left( F(m+1) - F(m) \right)^{1/2} \quad (41)
$$

Hence the proof is completed by (36).

**Lemma 4.3.** If (35) holds then $X_k(t)$ has a version which is a harmonizable $p$-adic stationary process with the almost sure $W$-continuous $r$-th Gibbs derivative.

**Proof.** We shall only prove it for $r = 1$, since the same argument will be applied to the other cases. Since by (37), for every $N \in \mathbb{N}_0$

$$
|\Delta_N X_k| = \left| \sum_{m=0}^{\infty} \Delta_N \psi(mp^{-k}, t) \zeta_{m,k} \right| \\
\leq \sum_{m=0}^{\infty} mp^{-k} |\zeta_{m,k}| < \infty \text{ a.s.} \quad (42)
$$

and

$$
\lim_{N \to \infty} \Delta_N \psi(mp^{-k}, t) = mp^{-k} \psi(mp^{-k}, t), \quad (43)
$$

uniformly, the limit of $\Delta_N X_k(t)$ as $N \to \infty$ should exist almost surely and is expressed by

$$
X^{[1]}(t) = p^{-k} \sum_{m=0}^{\infty} m \psi(mp^{-k}, t) \zeta_{m,k}, \quad (44)
$$

which is $W$-continuous almost surely.

Now we shall show a sample Gibbs differentiability condition for $p$-adic stationary processes.
THEOREM 4.4. If a harmonizable $p$-adic stationary process $X(t)$ satisfies (35), then it has a version which has the $W$-continuous $r$-th Gibbs derivative almost surely.

Proof. The proof of this theorem is demonstrated only for $r = 1$. First we show that for $0 < t < p^k$,

$$|\Delta_N X_{k+1}(t) - \Delta_N X_k(t)| \leq p^{-k} \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} |\zeta_{m_{p+l},k+1}|.$$  \hspace{1cm} (45)

Using the relations, $$\zeta_{m,k} = \sum_{l=0}^{p-1} \zeta_{m_{p+l},k+1},$$ and

$$\psi((mp)p^{-k-1}, t) = \psi(mp^{-k}, t),$$

we can rewrite that

$$\Delta_N X_{k+1}(t) - \Delta_N X_k(t)$$

$$= \sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \psi((mp + l)p^{-k-1}, t \oplus ip^{-j-1})\zeta_{m_{p+l},k+1}$$

$$- \psi(mp^{-k}, t \oplus ip^{-j-1}) \zeta_{m,k}$$

$$= \sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \left(\psi((mp + l)p^{-k-1}, t \oplus ip^{-j-1}) - \psi(mp^{-k}, t \oplus ip^{-j-1})\right)\zeta_{m_{p+l},k+1}$$

$$= \sum_{m=0}^{\infty} \sum_{l=0}^{p-1} \left[\psi((mp + l)p^{-k-1}, t) \sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \psi((mp + l)p^{-k-1}, ip^{-j-1})\right.$$

$$\left. - \psi(mp^{-k}, t) \sum_{j=-N}^{N} p^j \sum_{i=0}^{p-1} A_i \psi(mp^{-k}, ip^{-j-1})\right] \zeta_{m_{p+l},k+1}$$ \hspace{1cm} (46)

Since the Selfridge functions are Gibbs differentiable, and $\psi(lp^{-k}, t) = 1$ \hspace{1cm} (0 $\leq$ $t$ $<$ $p^k$), we have that for $0 \leq t < p^k$ and uniformly in $N$,

$$|\Delta_N X_{k+1}(t) - \Delta_N X_k(t)| \leq \sum_{m=0}^{\infty} \sum_{l=1}^{p-1} |\zeta_{m_{p+l},k+1}|$$

$$\leq p^{-k} \sum_{m=0}^{\infty} \sum_{l=1}^{p-1} |\zeta_{m_{p+l},k+1}|,$$ \hspace{1cm} (47)

hence we have shown (45). Since

$$\sum_{m=0}^{\infty} \left(F((m + 1)p^{-k-1}) - F(mp^{-k})\right)^{-1/2}$$
Gibbs $p$-adic derivatives and their applications to $p$-adic stationary processes

\[
\sum_{n=0}^{\infty} \left( p^{k+1} \sum_{m=np}^{(n+1)p^{k+1}+1} \left[ F((m+1)p^{-k-1}) - F(mp^{-k}) \right] \right)^{1/2}
\]
\[
= \sum_{n=0}^{\infty} \left( F(n+1) - F(n) \right)^{1/2},
\]

we have by (36) that for $\varepsilon_k > 0$ and $A < p^k$,

\[
P_k \equiv \Pr \left\{ \sup_{0 \leq t \leq A} |\Delta_N X_{k+1} - \Delta_N X_k| \geq \varepsilon_k \right\}
\]
\[
\leq \frac{1}{\varepsilon_k} p^{-k} \sum_{m=0}^{\infty} E|\zeta_{m,k+1}|
\]
\[
\leq \frac{1}{\varepsilon_k} p^{-k} \sum_{m=0}^{\infty} \left( F((m+1)p^{-k-1}) - F(mp^{-k}) \right)^{-1/2}
\]
\[
\leq C\varepsilon_k^{-1} p^{-k/2},
\]

where $C$ is a constant. If $\varepsilon_k$ is chosen to be $p^{-k/4}$, then $\sum_{k=0}^{\infty} \varepsilon_k < \infty$ and $\sum_{k=0}^{\infty} P_k < \infty$. Borel-Cantelli's lemma therefore shows that

\[
\Pr \left\{ \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \sup_{0 \leq t \leq A} |\Delta_N X_{k+1} - \Delta_N X_k| \geq \varepsilon_k \right\} = 0.
\]

Hence $\Delta_N X_k$ almost surely converges as $k \to \infty$ uniformly for $0 \leq t \leq A$ and uniformly in $N$. For any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}_0$ and $\{\varepsilon_k\}$ such that $\sum_{k=n_0}^{\infty} \varepsilon_k < \varepsilon$. On account of (50), if $m, n \geq n_0$ then

\[
\sup_{0 \leq t \leq A} |\Delta_N X_m(t) - \Delta_N X_n(t)| < \varepsilon \quad \text{a.s.}
\]

uniformly in $N$. Applying Lemma 4.1, and letting $n \to \infty$, we have that

\[
\sup_{0 \leq t \leq A} |\Delta_N X_m(t) - \Delta_N \bar{X}(t)| < \varepsilon \quad \text{a.s.}
\]

uniformly in $N$. Application of Lemma 4.3 with $r = 1$ shows that $\Delta_N X_m(t)$ converges almost surely as $N \to \infty$, and hence $\bar{X}(t)$, which is equivalent to $X(t)$, is Gibbs differentiable almost surely.

Note that the relation between sample differentiability condition and the mean square differentiability condition of $p$-adic stationary processes is obtained, i.e., the sample $r$-th Gibbs differentiability condition is $x \in L^{2r+\alpha}(F)$ ($\alpha > 1$), whereas that in the quadratic mean sense is $x \in L^{2r}(F)$. Hence a sample Gibbs differentiable $p$-adic stationary process $X(t)$ is quadratic mean Gibbs differentiable and

\[
X^{[r]}(t) = \int_0^\infty x^r \psi(x,t) \zeta(dx) \quad \text{a.s.}
\]

Hereafter we always choose from equivalent processes such a version that has well properties, such as sample $\mathbb{W}$-continuous, if it possesses.
5. Linear Gibbs differential equations

In this section we shall consider stochastic linear Gibbs differential equations, i.e., linear Gibbs differential equations with stochastic signals.

Let \( P(s) \) be an \( m \)-th order polynomial with complex coefficients,

\[
P(s) = \sum_{k=0}^{m} a_k s^k. \tag{52}
\]

If a harmonizable \( p \)-SP \( \{X(t, \omega)\} \) is \( m \)-times Gibbs differentiable almost surely then

\[
P(D)X(t, \omega) = \sum_{k=0}^{m} a_k DX(t, \omega) \tag{53}
\]

will be defined. If \( X(t, \omega) \) assume the representation (21), then by (51)

\[
P(D)X(t, \omega) = \int_0^{\infty} \psi(t, x)P(x)\zeta(dx, \omega). \tag{54}
\]

Its covariance function is also expressed by

\[
\text{Cov}(P(D)X(t), P(D)X(s)) = \int_0^{\infty} \psi(t, x)\overline{\psi(s, x)}|P(x)|^2 F(dx). \tag{55}
\]

The same argument is also valid in the quadratic mean convergence case.

Now let us define a stochastic linear Gibbs differential equation. An equation

\[
Q(D)Y(t, \omega) = P(D)X(t, \omega) \tag{56}
\]

is called a stochastic linear Gibbs differential equation, when \( \{X(t, \omega)\} \) is a given stochastic process, \( \{Y(t, \omega)\} \) is an unknown stochastic process, \( P(s) \) is a polynomial defined by (52), and \( Q(s) = \sum_{k=0}^{n} b_k s^k \). The process \( \{X(t, \omega)\} \) is sometimes called a driving process. If the process \( \{Y(t, \omega)\} \) satisfies (56) then it is called a solution of the stochastic linear Gibbs differential equation.

**Theorem 5.1.** Let a driving process \( \{X(t, \omega)\} \) be a harmonizable \( p \)-SP with the harmonic representation (21). Suppose that it is sample \( m \)-times Gibbs differentiable. If the equation \( Q(s) = 0 \) has no zeros on \( \mathbb{R}^+ \), and

\[
\int_0^{\infty} x^{2n+\alpha} \left| \frac{P(x)}{Q(x)} \right|^2 F(dx) < \infty \tag{57}
\]

for \( \alpha > 1 \), then

\[
Y(t, \omega) = \int_0^{\infty} \psi(t, x)\frac{P(x)}{Q(x)}\zeta(dx, \omega) \tag{58}
\]

satisfies the differential equation (56). Then its covariance function is given by

\[
\text{Cov}(Y(t), Y(s)) = \int_0^{\infty} \psi(t, x)\overline{\psi(s, x)}\left| \frac{P(x)}{Q(x)} \right|^2 F(dx). \tag{59}
\]
Proof. Remind that (58) is well-defined by (57). It follows from (57) and Theorem 4.1 that \( Y(t) \) or its version, more rigorously, has the sample \( W \)-continuous \( n \)-th Gibbs derivative, and

\[
Q(D)Y(t, \omega) = \int_0^\infty \psi(t, x)Q(x)\frac{P(x)}{Q(x)}\zeta(dx, \omega) \\
= \int_0^\infty \psi(t, x)P(x)\zeta(dx, \omega) \\
= P(D)X(t, \omega).
\]

Hence, \( \{Y(t, \omega)\} \) satisfies (56).

Notice that Theorem 5.1 is valid with \( \alpha = 0 \) for the quadratic mean case.

Example. For fixed \( x \in \mathbb{R}^+ \) define

\[
X(t, \omega) = \psi(t, x)X(\omega) \quad (t \in \mathbb{R}^+),
\]

where \( X(\omega) \) is a random variable with zero mean and finite variance. Then \( \{X(t, \omega); t \in \mathbb{R}^+\} \) thus defined is a harmonizable \( p \)-SP, which is expressed by \( X(t, \omega) = \psi(t, x)X(\omega) \). With \( \{X(t, \omega)\} \) for a driving process, (56) has a solution of the form,

\[
Y(t, \omega) = \frac{\psi(t, x)}{Q(x)}X(\omega) \quad (t \in \mathbb{R}^+),
\]

provided that \( Q(x) \neq 0 \). It is clear that \( Y(t) \) is sample uniform \( W \)-continuous.

References


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