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By

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Abstract

This paper analyses transitive closures of fuzzy relations on a compact metric space with a contraction property in Kurano et al. (1992). We show that the transitive closure is a unique solution of a fuzzy relational equation and also has the same contraction property.

1. Introduction and notations

Let E be a compact metric space and d be a metric on E . $\mathcal{C}(E)$ denotes the collection of all non-empty closed subsets of E . We put $d(x, D) := \inf_{y \in D} d(x, y)$, $x \in E, D \in \mathcal{C}(E)$. Let ρ be the Hausdorff metric on $\mathcal{C}(E)$. Then it is well-known ([1]) that $(\mathcal{C}(E), \rho)$ is a compact metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\tilde{s} : E \mapsto [0, 1]$ which are upper semi-continuous and satisfy $\sup_{x \in E} \tilde{s}(x) = 1$.

For a fuzzy relation $\tilde{r} : E \times E \mapsto [0, 1]$ satisfying $\tilde{r}(x, \cdot) \in \mathcal{F}(E)$ ($x \in E$), we define a map $\tilde{r}_\alpha : \mathcal{C}(E) \mapsto \mathcal{C}(E)$ ($\alpha \in [0, 1]$) by

$$\tilde{r}_\alpha(D) := \begin{cases} \{y \mid \tilde{r}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha \neq 0, D \in \mathcal{C}(E) \\ \text{cl}\{y \mid \tilde{r}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, D \in \mathcal{C}(E), \end{cases} \quad (1.1)$$

where cl denotes the closure of a set. Let $\mathcal{R}(E)$ be the set of all fuzzy relations $\tilde{r} : E \times E \mapsto [0, 1]$ which satisfy $\tilde{r}(x, \cdot) \in \mathcal{F}(E)$ ($x \in E$) and

$$\sup_{\alpha \in [0, 1]} \rho(\tilde{r}_\alpha(y), \tilde{r}_\alpha(x)) \rightarrow 0 \quad (y \rightarrow x) \quad \text{for } x \in E. \quad (1.2)$$

We denote the maximum operation and the minimum operation by $\vee$ and $\wedge$, respectively. Let $\tilde{q} \in \mathcal{R}(E)$ be a continuous fuzzy relation. We define sequences of fuzzy relations $\{\tilde{q}^n\}_{n=1}^\infty$ and $\{\tilde{c}^m\}_{m=1}^\infty$ by

$$\tilde{q}^1 := \tilde{q} \quad \text{and} \quad \tilde{q}^{n+1}(x, y) := \sup_{z \in E} \{\tilde{q}(x, z) \wedge \tilde{q}^n(z, y)\}, \quad x, y \in E, \quad n = 1, 2, \dots; \quad (1.3)$$

$$\tilde{c}^m(x, y) := \bigvee_{n=1, 2, \dots, m} \tilde{q}^n(x, y), \quad x, y \in E, \quad m = 1, 2, \dots. \quad (1.4)$$

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If E is finite, then the transitive closure of the fuzzy relation $\tilde{q}$ is given ([5]) by

$$\bigvee_{n=1,2,\dots} \tilde{q}^n(x, y), \quad x, y \in E. \quad (1.5)$$

This paper discusses the transitive closure when E is a general compact metric space. From now on we assume the following contraction property.

ASSUMPTION (Contraction property, [2]). There exists a real number β ($0 < \beta < 1$) satisfying the following condition :

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta \rho(A, B) \quad \text{for all } A, B \in \mathcal{C}(E) \text{ and all } \alpha \in [0, 1]. \quad (1.6)$$

We call β a contraction factor.

LEMMA 1. *The condition (1.6) is equivalent to the following condition :*

$$\rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)) \leq \beta d(x, y) \quad \text{for all } x, y \in E \text{ and all } \alpha \in [0, 1], \quad (1.7)$$

where $\tilde{q}_\alpha(x) := \tilde{q}_\alpha(\{x\})$ ($x \in E, \alpha \in [0, 1]$).

PROOF. We can obtain (1.7) from (1.6), taking $A = \{x\}$, $B = \{y\}$. Conversely we assume (1.7). Let $A, B \in \mathcal{C}(E)$. Since $\tilde{q}_\alpha(x) \subset \tilde{q}_\alpha(A)$ for all $x \in A$ and $\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(B)$ for all $y \in B$,

$$\begin{aligned} \rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) &= \max\left\{ \max_{x' \in \tilde{q}_\alpha(A)} d(x', \tilde{q}_\alpha(B)), \max_{y' \in \tilde{q}_\alpha(B)} d(\tilde{q}_\alpha(A), y') \right\} \\ &\leq \max\left\{ \max_{x' \in \tilde{q}_\alpha(A)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)), \max_{y' \in \tilde{q}_\alpha(B)} \min_{x \in A} d(\tilde{q}_\alpha(x), y') \right\}. \end{aligned}$$

Here since $\tilde{q}_\alpha(A) = \bigcup_{x \in A} \tilde{q}_\alpha(x)$, we have

$$\max_{x' \in \tilde{q}_\alpha(A)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)) = \max_{x \in A} \max_{x' \in \tilde{q}_\alpha(x)} \min_{y \in B} d(x', \tilde{q}_\alpha(y)) \leq \max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)).$$

Similarly

$$\max_{y' \in \tilde{q}_\alpha(B)} \min_{x \in A} d(\tilde{q}_\alpha(x), y') \leq \max_{y \in B} \min_{x \in A} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)).$$

Therefore we obtain

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \max\left\{ \max_{x \in A} \min_{y \in B} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)), \max_{y \in B} \min_{x \in A} \rho(\tilde{q}_\alpha(x), \tilde{q}_\alpha(y)) \right\}.$$

From (1.7), we get

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta \max\left\{ \max_{x \in A} \min_{y \in B} d(x, y), \max_{y \in B} \min_{x \in A} d(x, y) \right\} = \beta \rho(A, B).$$

Therefore we obtain (1.6). □

DEFINITION 1 ([3]). For $\tilde{r}_n, \tilde{r} \in \mathcal{R}(E)$,

$$\lim_{n \rightarrow \infty} \tilde{r}_n = \tilde{r}$$

means

$$\sup_{\alpha \in [0,1]} \rho(\tilde{r}_{n,\alpha}(D), \tilde{r}_\alpha(D)) \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{for } D \in \mathcal{C}(E),$$

where $\tilde{r}_{n,\alpha}, \tilde{r}_\alpha$ are defined by (1.1) for the fuzzy relations $\tilde{r}_n, \tilde{r}$, respectively.

LEMMA 2 ([2, Lemma 2]). *Suppose that a family of subsets $\{D_\alpha \mid \alpha \in [0, 1]\} \subset \mathcal{C}(E)$ satisfies the following conditions:*

- (i) $D_\alpha \subset D_{\alpha'}$ for $\alpha' \leq \alpha$.
- (ii) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'} = D_\alpha$, i.e., $\lim_{\alpha' \uparrow \alpha} \rho(D_{\alpha'}, D_\alpha) = 0$ for $\alpha \in (0, 1]$.

Then it holds that

$$\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(D_{\alpha'}) = \tilde{q}_\alpha(D_\alpha) \quad \text{for } \alpha \in (0, 1]. \quad (1.8)$$

LEMMA 3. *We suppose that a family of subsets $\{D_\alpha(x) \mid x \in E, \alpha \in [0, 1]\} \subset \mathcal{C}(E)$ satisfies the following conditions (i) – (iii) :*

- (i) $D_\alpha(x) \subset D_{\alpha'}(x)$ for $x \in E, 0 \leq \alpha' < \alpha \leq 1$.
- (ii) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'}(x) = D_\alpha(x)$ for $x \in E, \alpha \in (0, 1]$.
- (iii) $\sup_{\alpha \in [0,1]} \rho(D_\alpha(y), D_\alpha(x)) \rightarrow 0 \quad (y \rightarrow x)$ for $x \in E$.

Then

$$\tilde{r}(x, y) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{D_\alpha(x)}(y)\}, \quad x, y \in E,$$

satisfies $\tilde{r} \in \mathcal{R}(E)$ and $\tilde{r}_\alpha(x) = D_\alpha(x)$ for all $x \in E, \alpha \in [0, 1]$.

PROOF. Fix any $x \in E$. By [4], from the conditions (i) and (ii), we have $\tilde{r}(x, \cdot) \in \mathcal{F}(E)$ and $\tilde{r}_\alpha(x) = \{y \in E \mid \tilde{r}(x, y) \geq \alpha\} = D_\alpha(x)$ for $\alpha \in (0, 1]$. Therefore (1.2) holds from (iii). Thus we get $\tilde{r} \in \mathcal{R}(E)$. $\square$

We define maps $\tilde{q}_\alpha^n : \mathcal{C}(E) \mapsto \mathcal{C}(E)$ ($n = 1, 2, \dots, \alpha \in [0, 1]$) by $\tilde{q}_\alpha^1 := \tilde{q}_\alpha$ and $\tilde{q}_\alpha^{n+1} := \tilde{q}_\alpha(\tilde{q}_\alpha^n)$ ($n = 1, 2, \dots$).

LEMMA 4. *Let $\alpha \in [0, 1]$. Then :*

- (i) $(\tilde{q}^n)_\alpha(D) = \tilde{q}_\alpha^n(D), \quad D \in \mathcal{C}(E) \quad \text{for } n = 1, 2, \dots;$
- (ii) $(\tilde{c}^m)_\alpha(D) = \bigcup_{n=1,2,\dots,m} \tilde{q}_\alpha^n(D), \quad D \in \mathcal{C}(E) \quad \text{for } m = 1, 2, \dots.$

PROOF. We have (i) from [2, Lemma 1] and (1.3). Further (ii) is trivial from (1.4). $\square$

LEMMA 5. It holds that

$$\rho(A \cup C, B \cup D) \leq \max\{\rho(A, B), \rho(C, D)\} \quad \text{for } A, B, C, D \in \mathcal{C}(E).$$

PROOF. Let $A, B, C, D \in \mathcal{C}(E)$. Then

$$\begin{aligned}
 \rho(A \cup C, B \cup D) &= \max\left\{\max_{x \in A \cup C} d(x, B \cup D), \max_{y \in B \cup D} d(A \cup C, y)\right\} \\
 &= \max\left\{\max_{x \in A} d(x, B \cup D), \max_{y \in B} d(A \cup C, y), \max_{x \in C} d(x, B \cup D), \max_{y \in D} d(A \cup C, y)\right\} \\
 &\leq \max\left\{\max_{x \in A} d(x, B), \max_{y \in B} d(A, y), \max_{x \in C} d(x, D), \max_{y \in D} d(C, y)\right\} \\
 &= \max\{\rho(A, B), \rho(C, D)\}.
 \end{aligned}$$

Therefore we obtain this lemma. $\square$

2. Main results

We discuss the convergence of the sequence of fuzzy relations $\{\tilde{c}^m\}_{m=1}^\infty$.

THEOREM 1.

(i) *There exists a unique solution $\tilde{c} \in \mathcal{R}(E)$ of the following fuzzy relational equation:*

$$\tilde{c}(x, y) = \tilde{q}(x, y) \vee \max_{z \in E} \{\tilde{c}(x, z) \wedge \tilde{q}(z, y)\} \quad x, y \in E. \quad (2.1)$$

(ii) *The fuzzy relation $\tilde{c}$ also has the contraction property with the same contraction factor β .*

(iii) *The fuzzy relation $\tilde{c}$ equals to the limit of $\{\tilde{c}^m\}_{m=1}^\infty$:*

$$\tilde{c} = \lim_{m \rightarrow \infty} \tilde{c}^m. \quad (2.2)$$

PROOF. Define a map $T_{x,\alpha} : \mathcal{C}(E) \rightarrow \mathcal{C}(E)$ ($x \in E, \alpha \in [0, 1]$) by

$$T_{x,\alpha}(D) := \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D), \quad D \in \mathcal{C}(E).$$

From Lemmas 1 and 5,

$$\begin{aligned}
 \rho(T_{x,\alpha}(D), T_{x,\alpha}(D')) &= \rho(\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D), \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D')) \\
 &\leq \rho(\tilde{q}_\alpha(D), \tilde{q}_\alpha(D')) \\
 &\leq \beta \rho(D, D'), \quad D, D' \in \mathcal{C}(E), \quad x \in E, \quad \alpha \in [0, 1].
 \end{aligned}$$

Since the metric space $(\mathcal{C}(E), \rho)$ is compact, from the Banach's fixed point theorem, there exists a family $\{A_\alpha(x) \mid x \in E, \alpha \in [0, 1]\} \subset \mathcal{C}(E)$ such that

$$\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(A_\alpha(x)) = T_{x,\alpha}(A_\alpha(x)) = A_\alpha(x), \quad x \in E, \quad \alpha \in [0, 1],$$

and $\lim_{n \rightarrow \infty} T_{x,\alpha}^n(D) = A_\alpha(x)$ for any $D \in \mathcal{C}(E)$. From the definition of $\tilde{q}_\alpha$, $T_{x,\alpha'}(D) = \tilde{q}_{\alpha'}(x) \cup \tilde{q}_{\alpha'}(D) \supset \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D) = T_{x,\alpha}(D)$ for $\alpha' \leq \alpha$. Inductively we have $T_{x,\alpha'}^n(D) \supset T_{x,\alpha}^n(D)$ for $n = 1, 2, \dots$. By letting $n \rightarrow \infty$, we obtain

$$A_{\alpha'}(x) \supset A_\alpha(x) \quad \text{for } \alpha' \leq \alpha. \quad (2.3)$$

Let $\alpha' \leq \alpha$. Inductively we have

$$\begin{aligned} \rho(A_\alpha(x), A_{\alpha'}(x)) &= \rho(T_{x,\alpha}^n(A_\alpha(x)), T_{x,\alpha'}^n(A_{\alpha'}(x))) \\ &\leq \rho(T_{x,\alpha'}^n(A_\alpha(x)), T_{x,\alpha}^n(A_\alpha(x))) + \rho(T_{x,\alpha'}^n(A_{\alpha'}(x)), T_{x,\alpha'}^n(A_\alpha(x))) \\ &\leq \rho(T_{x,\alpha'}^n(A_\alpha(x)), T_{x,\alpha}^n(A_\alpha(x))) + \beta^n \rho(A_{\alpha'}(x), A_\alpha(x)), \quad n = 1, 2, \dots \end{aligned}$$

Then $\rho(A_{\alpha'}(x), A_\alpha(x))$ is uniformly bounded since E is compact. We put $\rho(A_{\alpha'}(x), A_\alpha(x)) \leq M$ for some $M > 0$. Therefore

$$\rho(A_\alpha(x), A_{\alpha'}(x)) \leq \rho(T_{x,\alpha'}^n(A_\alpha(x)), T_{x,\alpha}^n(A_\alpha(x))) + \beta^n M, \quad n = 1, 2, \dots \quad (2.4)$$

By Lemma 2, we have $\lim_{\alpha' \uparrow \alpha} T_{x,\alpha'}(A_\alpha(x)) = \lim_{\alpha' \uparrow \alpha} \{\tilde{q}_{\alpha'}(x) \cup \tilde{q}_{\alpha'}(A_\alpha(x))\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(A_\alpha(x)) = T_{x,\alpha}(A_\alpha(x))$. Repeating these arguments inductively, we have

$$\lim_{\alpha' \uparrow \alpha} T_{x,\alpha'}^n(A_\alpha(x)) = T_{x,\alpha}^n(A_\alpha(x)), \quad n = 1, 2, \dots$$

Therefore (2.4) implies

$$\lim_{\alpha' \uparrow \alpha} \rho(A_\alpha(x), A_{\alpha'}(x)) \leq \beta^n M, \quad n = 1, 2, \dots$$

By letting $n \rightarrow \infty$, we obtain

$$\lim_{\alpha' \uparrow \alpha} A_{\alpha'}(x) = A_\alpha(x). \quad (2.5)$$

Let $\alpha \in [0, 1]$ and $x, y \in E$. From Lemma 5 and the contraction property of $\tilde{q}$, we have

$$\begin{aligned} \rho(T_{y,\alpha}(D), T_{x,\alpha}(D')) &= \rho(\tilde{q}_\alpha(y) \cup \tilde{q}_\alpha(D), \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(D')) \\ &\leq \max\{\rho(\tilde{q}_\alpha(y), \tilde{q}_\alpha(x)), \rho(\tilde{q}_\alpha(D), \tilde{q}_\alpha(D'))\} \\ &\leq \beta \max\{d(y, x), \rho(D, D')\}, \quad D, D' \in \mathcal{C}(E). \end{aligned}$$

Repeating these arguments inductively, we have

$$\rho(T_{y,\alpha}^n(D), T_{x,\alpha}^n(D')) \leq \max\{\beta d(y, x), \beta^n \rho(D, D')\}, \quad D, D' \in \mathcal{C}(E), \quad n = 1, 2, \dots$$

Since $\rho(D, D')$ is uniformly bounded, letting $n \rightarrow \infty$, we obtain

$$\rho(A_\alpha(y), A_\alpha(x)) \leq \beta d(y, x) \quad \text{for } x, y \in E. \quad (2.6)$$

Therefore

$$\sup_{\alpha \in [0,1]} \rho(A_\alpha(y), A_\alpha(x)) \rightarrow 0 \quad (y \rightarrow x) \quad \text{for } x \in E.$$

Thus the family $\{A_\alpha(x) \mid x \in E, \alpha \in [0, 1]\}$ satisfies the conditions (i) – (iii) of Lemma 3. By Lemma 3, we can define a fuzzy relation $\tilde{c} \in \mathcal{R}(E)$ by

$$\tilde{c}(x, y) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha(x)}(y)\}, \quad x, y \in E.$$

Then $\tilde{c}_\alpha(x) = A_\alpha(x)$ ($x \in E, \alpha \in [0, 1]$). Since $A_\alpha(x)$ is a unique fixed point of $T_{x,\alpha}$,

$$\lim_{n \rightarrow \infty} (\tilde{c}^n)_\alpha(x) = \lim_{n \rightarrow \infty} T_{x,\alpha}^n(\{x\}) = A_\alpha(x) = \tilde{c}_\alpha(x), \quad \alpha \in [0, 1].$$

We get (iii) since the convergence is uniform in $\alpha \in [0, 1]$.

Next we show that $\tilde{c}$ is a solution of (2.1). Since $\tilde{c}_\alpha(x) = A_\alpha(x)$, we note that

$$\tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}_\alpha(x)) = T_\alpha(\tilde{c}_\alpha(x)) = \tilde{c}_\alpha(x), \quad \alpha \in [0, 1]. \quad (2.7)$$

If $\alpha > 0$, then we have

$$\left\{ y \in E \mid \tilde{q}(x, y) \vee \max_{z \in E} \{ \tilde{c}(x, z) \wedge \tilde{q}(z, y) \} \geq \alpha \right\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}_\alpha(x)).$$

If $\alpha = 0$, then in a similar way to the proof of [2, Lemma 1] we have

$$\text{cl} \left\{ y \in E \mid \max_{x \in E} \{ \tilde{q}(x, y) \vee \max_{z \in E} \{ \tilde{c}(x, z) \wedge \tilde{q}(z, y) \} \} > 0 \right\} = \tilde{q}_0(x) \cup \tilde{q}_0(\tilde{c}_0(x)).$$

Therefore

$$\left\{ y \in E \mid \tilde{q}(x, y) \vee \max_{z \in E} \{ \tilde{c}(x, z) \wedge \tilde{q}(z, y) \} \geq \alpha \right\} = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}_\alpha(x)) \quad \text{for } \alpha \in [0, 1].$$

Together with (2.7), we get

$$\left\{ y \in E \mid \tilde{q}(x, y) \vee \max_{z \in E} \{ \tilde{c}(x, z) \wedge \tilde{q}(z, y) \} \geq \alpha \right\} = \tilde{c}_\alpha(x) \quad \text{for } \alpha \in [0, 1].$$

Therefore $\tilde{c}$ satisfies (2.1). We prove the uniqueness of solution of (2.1). Let us denote by $\tilde{c}' \in \mathcal{R}(E)$ another solution of (2.1). For $x \in E$, $\alpha \in [0, 1]$, it is shown similarly that $\tilde{c}'_\alpha(x) = \tilde{q}_\alpha(x) \cup \tilde{q}_\alpha(\tilde{c}'_\alpha(x))$. That is, $\tilde{c}'_\alpha(x)$ is a fixed point of $T_{x,\alpha} : \mathcal{C}(E) \mapsto \mathcal{C}(E)$. From the uniqueness of the fixed point, we get $\tilde{c}'_\alpha(x) = \tilde{c}_\alpha(x)$ for $x \in E$, $\alpha \in [0, 1]$. By Lemma 3, $\tilde{c}' = \tilde{c}$. Thus we get (i).

Finally (ii) is trivial from (2.6), using Lemma 1 and $\tilde{c}_\alpha(x) = A_\alpha(x)$ for $x \in E$, $\alpha \in [0, 1]$. Thus the proof is completed. $\square$

THEOREM 2. *It holds that*

$$\tilde{c}(x, y) = \bigvee_{n=1,2,\dots} \tilde{q}^n(x, y), \quad x, y \in E. \quad (2.8)$$

Further $\tilde{c}$ is the transitive closure of the fuzzy relation $\tilde{q}$, namely $\tilde{c}$ satisfies (i) – (iii) :

(i) $\tilde{c} \geq \tilde{q}$.

(ii) $\tilde{c}$ has the transitive property :

$$\tilde{c}(x, y) \geq \sup_{z \in E} \{ \tilde{c}(x, z) \wedge \tilde{c}(z, y) \}, \quad x, y \in E. \quad (2.9)$$

(iii) If $\tilde{r} \in \mathcal{R}(E)$ satisfies $\tilde{r} \geq \tilde{q}$ and has the transitive property, then $\tilde{r} \geq \tilde{c}$.

PROOF. Let $\tilde{r}(x, y) := \bigvee_{n=1,2,\dots} \tilde{q}^n(x, y)$, $x, y \in E$. Then we have $\tilde{r} \geq \bigvee_{n=1,2,\dots,m} \tilde{q}^n = \tilde{c}^m$ for $m = 1, 2, \dots$. Therefore $\tilde{r}_\alpha(x) \supset (\tilde{c}^m)_\alpha(x)$ for $x \in E, \alpha \in [0, 1], m = 1, 2, \dots$. From (2.2) we obtain $\tilde{r}_\alpha(x) \supset \tilde{c}_\alpha(x)$ for $x \in E, \alpha \in [0, 1]$. Thus we get $\tilde{r} \geq \tilde{c}$.

On the other hand, from (2.1), we obtain $\tilde{c} \geq \tilde{q}$ and

$$\tilde{c}(x, y) \geq \sup_{z \in E} \{\tilde{c}(x, z) \wedge \tilde{q}(z, y)\} \geq \sup_{z \in E} \{\tilde{q}(x, z) \wedge \tilde{q}(z, y)\} = \tilde{q}^2(x, y), \quad x, y \in E.$$

Repeating this argument inductively, we obtain $\tilde{c} \geq \tilde{q}^n$ for $n = 1, 2, \dots$. Therefore $\tilde{c}(x, y) \geq \tilde{c}^n(x, y)$ for $x, y \in E, n = 1, 2, \dots$. Thus we get $\tilde{c} \geq \tilde{r}$. Therefore we obtain (2.8).

Next we prove (i) – (iii). (i) is trivial from (2.1). From (2.8), we have

$$\begin{aligned} \tilde{c}(x, y) &\geq \bigvee_{n=1,2,\dots,m; n'=1,2,\dots,m'} \tilde{q}^{n+n'}(x, y) \\ &= \sup_{z \in E} \left\{ \bigvee_{n=1,2,\dots,m} \tilde{q}^n(x, z) \wedge \bigvee_{n'=1,2,\dots,m'} \tilde{q}^{n'}(z, y) \right\} \\ &= \sup_{z \in E} \{\tilde{c}^m(x, z) \wedge \tilde{c}^{m'}(z, y)\}, \quad x, y \in E. \end{aligned}$$

Taking the supremum over $m = 1, 2, \dots$ and $m' = 1, 2, \dots$, we obtain (ii). Finally let $\tilde{r} \in \mathcal{R}(E)$ satisfy $\tilde{r} \geq \tilde{q}$ and have the transitive property. Then

$$\tilde{r}(x, y) \geq \sup_{z \in E} \{\tilde{r}(x, z) \wedge \tilde{r}(z, y)\} \geq \sup_{z \in E} \{\tilde{q}(x, z) \wedge \tilde{q}(z, y)\} = \tilde{q}^2(x, y), \quad x, y \in E.$$

Repeating this argument inductively, we obtain $\tilde{r} \geq \tilde{q}^n$ for $n = 1, 2, \dots$. Therefore $\tilde{r}(x, y) \geq \tilde{c}^n(x, y)$ for $x, y \in E, n = 1, 2, \dots$. Thus we get $\tilde{r} \geq \tilde{c}$. Therefore (iii) holds. The proof is completed. $\square$

3. Numerical example

Let $E = [-2, 2]$ be a space of states. We consider a fuzzy relation (see [2, Figure 1])

$$\tilde{q}(x, y) = \left\{ 1 - \left| y - \left(\frac{x}{2} + \frac{1}{4} \right) \right| \right\} \vee 0, \quad x, y \in E. \quad (3.1)$$

Then we have

$$\tilde{q}^n(x, y) = \left\{ 1 - \left| y - \left(\frac{1}{2^n}x + \frac{1}{2} - \frac{1}{2^{n+1}} \right) \right| \right\} \vee 0, \quad x, y \in E, n = 1, 2, \dots. \quad (3.2)$$

From Theorem 2, we obtain

$$\tilde{c}(x, y) = \bigvee_{n=1,2,\dots} \left\{ 1 - \left| y - \left(\frac{1}{2^n}x + \frac{1}{2} - \frac{1}{2^{n+1}} \right) \right| \right\} \vee 0, \quad x, y \in E. \quad (3.3)$$

Then, (3.3) is the unique solution of Theorem 1 (see Figure 1).

Fig. 1 : The transitive closure $\tilde{c}(x, y)$.

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