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Iwasa, Manabu
Department of Mathematical Science, Faculty of Engineering Science, Osaka University

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ONE-SIDED TEST OF THE MINIMUM PARAMETER

By

Manabu Iwasa*

Abstract

When independent random variables X_i $(i = 1, 2, \cdots, k)$ have probability density f_{θ_i} with monotone likelihood ratio respectively, we consider testing $H_0 : \min(\theta_1, \theta_2, \cdots, \theta_k) = \theta^*$ vs. $H_1 : \min(\theta_1, \theta_2, \cdots, \theta_k) > \theta^*$ for a constant θ^*. We give a class of similar tests and find an unbiased test in this class. We apply and extend the arrangement ordering arguments. It is also proved that this unbiased test has a monotone power function. A modification to testing $H_0^* : \min(\theta_1, \theta_2, \cdots, \theta_k) \le \theta^*$ vs. H_1 is also considered.

1. Introduction

Let $X_1, X_2, \cdots, X_k$ $(k \ge 2)$ be independent random variables and each X_i have probability density $f_{\theta_i}(x)$ with respect to Lebesgue measure. We assume that $f_\theta(x)$ has monotone likelihood ratio, i.e. $f_{\theta'}(x) f_\theta(x') \le f_\theta(x) f_{\theta'}(x')$ for all $x \le x'$ and $\theta \le \theta'$. Then we consider testing

$$H_0 : \min(\theta_1, \theta_2, \cdots, \theta_k) = \theta^* \text{ and each } \theta_i \in \Theta$$

vs.

$$H_1 : \min(\theta_1, \theta_2, \cdots, \theta_k) > \theta^* \text{ and each } \theta_i \in \Theta$$

where Θ $(\subset \mathbf{R})$ is an open set and θ^* is a constant in Θ. For these hypotheses, the likelihood ratio test rejects H_0 iff $\min(X_1, X_2, \cdots, X_k) > c$ for a certain constant c.

In this paper, we shall call a test whose critical region is given by $\{\min(X_1, X_2, \cdots, X_k) > c\}$ as a min-test. Min-tests are natural and possess good properties. It was shown under some conditions that min-tests are admissible and uniformly most powerful monotone tests (cf. Lehmann [10], Cohen, Gatsonis and Marden [3], Nomakuchi and Sakata [13], Gutmann [6]). However min-tests are not unbiased. In particular, if

$$\sup_{\theta \ge \theta^*, \theta \in \Theta} Pr_\theta\{X_1 > c\} = 1, \tag{1.1}$$

and $\Pr_\theta\{X_1 > c\}$ is continuous in θ, then we have $\sup_{H_0} Pr\{\min(X_1, \cdots, X_k) > c\} = Pr_{\theta^*}\{X_i > c\}$ and

$$\inf_{H_1} Pr\{\min(X_1, \cdots, X_k) > c\} = [\sup_{H_0} Pr\{\min(X_1, \cdots, X_k) > c\}]^k. \tag{1.2}$$

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* Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka 560, Japan

The relation (1.2) implies that the power of a min-test is little, especially in the neighborhood of $(\theta^*, \cdots, \theta^*)$, where the infimum of (1.2) is attained. The similar phenomenon occurs also in testing $H_0^* : \min(\theta_1, \theta_2, \cdots, \theta_k) \leq \theta^*$ and each $\theta_i \in \Theta$ vs. H_1.

This shortcoming of min-tests was studied by Nomakuchi and Sakata [13], Gutmann [6], Berger [1], Zelterman [16], Shirley [15] and Li [11]. They gave some tests which are uniformly more powerful than min-tests in some situations satisfying (1.1). The critical regions of these tests are constructed by adding suitable regions to that of min-tests. It is important that these extensions of the critical region never change the size and such extensions of the critical region might be continued until we obtain an unbiased test. In this sense, our consideration on unbiasedness may be natural and significant. In such contexts, Lehmann [10], Nomakuchi and Sakata [13] and Zelterman [16] discussed unbiasedness.

Lehmann [10] gave an important result concerning the unbiasedness for H_0^* vs. H_1. His result implies that there exist no unbiased test functions for H_0^* vs. H_1 except the constant test functions when $\{f_\theta\}$ is an exponential family. On the other hand, when $k = 2$, Nomakuchi and Sakata [13] showed that there exist non-trivial unbiased tests for H_0 vs. H_1 under the normal distribution $N(\theta, 1)$. They gave a sufficient condition for a test to be unbiased for H_0 vs. H_1 and showed that a similar test proposed by Lehmann[10] satisfies the condition. When the level $\alpha = 1/n$, their test rejects H_0 iff

$$(x_1, x_2) \in \bigcup_{i=1}^{n} \{(x_1, x_2); c(\frac{i-1}{n}) < x_j - \theta^* < c(\frac{i}{n}), j = 1, 2\}$$

where $c(i/n)$ is the lower i/n point of the standard normal distribution. The critical region of this test is shaped by adding $(n-1)$ square regions diagonally to the critical region of the min-test. Zelterman [16] also discussed the unbiasedness. He considered testing a hypothesis $H_2 : \min(\theta_1, \theta_2, \cdots, \theta_k) \geq 0$ or $\max(\theta_1, \theta_2, \cdots, \theta_k) \leq 0$ vs. not H_2. The likelihood ratio test for these hypotheses, which is derived by Gail and Simon [5], also has low power near the origin. When $k = 2$, their hypotheses are closely related to our hypotheses H_0 vs. H_1 and the argument of Nomakuchi and Sakata [13] can be applied to their hypotheses. Zelterman [16] discussed locally most powerful unbiased tests, and gave a numerical example.

Gutmann [6], Berger [1], Shirley [15] and Li [11] discussed the improvements of min-tests for H_0^* vs. H_1.

Berger [1] and Li [11] proposed tests whose critical regions are shaped by adding some *cubic* regions *diagonally* to that of the min-test. Berger [1] discussed the case in which $X_1, X_2, \cdots, X_k$ are dependent. He considered testing a null hypothesis $H_0^{**} : b_i'\theta \leq 0$ for some $i = 1, \cdots, p$ vs. $H_1^{**} : b_i'\theta > 0$ for all $i = 1, \cdots, p$ under $X \sim N((\theta_1, \cdots, \theta_k)', \Sigma)$, where Σ and k-dimensional vectors $b_1, \cdots, b_p$ are known. He showed that if for each $i = 1, \cdots, p$, there exists an $m \in \{1, \cdots, p\}$ such that $b_i'\Sigma b_m \leq 0$, a test whose critical region is given by

$$\bigcup_{i=1}^{i_0} R_i, \text{ where } R_i = \{x; c_i < \frac{b_j' x}{(b_j'\Sigma b_j)^{\frac{1}{2}}} < c_{i-1}, j = 1, \cdots, p\},$$

$c_0, \cdots, c_{i_0}$ are certain standard normal quantiles and R_1 is the critical region of the likelihood ratio test, has the same size as the likelihood ratio test. Li [11] generalized Berger's idea to the case of an exponential family by assuming the independence of $X_1, \cdots, X_k$. Although they gave the tests for arbitrary k, their idea is short of considerations on dimensionality.

Gutmann [6] considered testing H_0^* vs. H_1 under a general location model. The critical region of his test is constructed by adding one cubic region to that of the min-test. Shirley [15] discussed the maximum size of the cube added in Gutmann's method. Moreover, Shirley [15] considered an alternative method to extend the critical region and proposed a test whose critical region is given by

$$\{\min(X_1, \cdots, X_k) > c_1\} \cup \{c_2 \leq \text{ any two } X_i \leq c_1, \text{ other } X_i > c_2, i = 1, \cdots, k\},$$

where $c_1 > c_2$ are suitable constants. It is noteworthy that the dimensionality is taken into consideration in this idea. But this test is not very powerful near $(\theta^*, \cdots, \theta^*)$ as far as we refer to examples given by Shirley [15].

Our original purpose is to extend the result of Nomakuchi and Sakata [13] with respect to the hypotheses H_0 vs. H_1 to the case of arbitrary k. Although we do not give unbiased tests for $k \geq 4$, our idea can be applied to that case.

In the next section, we give a simple class of similar tests. When $k = 2$, this class contains the unbiased test given by Nomakuchi and Sakata [13]. This class is identified respectively with the set of permutations when $k = 2$ and with the set of Latin squares when $k = 3$. Our investigation is done within this class. In Section 3, we discuss the case of $k = 2$. We show that the unbiased test given by Nomakuchi and Sakata [13] is uniformly most powerful in this class by using arrangement ordering arguments on the set of permutations. In Section 4, we discuss the case of $k = 3$. We consider an extension of the arrangement ordering to an ordering on the set of Latin squares. This partial ordering is a useful criterion to compare power functions in the class. We give some basic properties of this partial order, and specify an unbiased test for H_0 vs. H_1. In Section 5, we modify our arguments to testing H_0^* vs. H_1.

2. A class of similar test functions

Throughout this paper, we consider only level-$1/n$ tests ($n = 2, 3, \cdots$).

To begin with, we construct a class of similar test functions investigated in this work. F_θ denotes the cumulative distribution function corresponding to f_θ. To begin with, we divide the sample space $\mathbf{R}^k$ into n^k regions

$$A(i_1, \cdots, i_k) = \{(x_1, \cdots, x_k) \in \mathbf{R}^k; z(\frac{i_j - 1}{n}) < x_j < z(\frac{i_j}{n}), j = 1, \cdots, k\}$$

where $i_j = 1, \cdots, n$ and $z(c)$ is the lower c point of F_{θ^*}.

We denote by $G^{k,n}$ the set of all functions g from $\{1, 2, \cdots, n\}^k$ to $\{0, 1\}$ such that

$$\sum_{i_j=1}^{n} g(i_1, \cdots, i_k) = 1 \qquad \text{for all } j = 1, \cdots, k. \tag{2.1}$$

It is easily verified that $g \in G^{2,n}$ is identified with a permutation matrix. Moreover, when $k = 3$ and ≥ 4, $g \in G^{k,n}$ is regarded as a Latin square and a $(k-1)$-dimensional permutation cube respectively (cf. Dénes and Keedwell [4]). Here a Latin square of order n is defined as an $n \times n$ matrix where all rows and columns contain each number of $\{1, 2, \cdots, n\}$ only once. For $g \in G^{3,n}$, let l_{ij} be the number a satisfying $g(i, j, a) = 1$. Since $\sum_{i=1}^{n} g(i, j, a) = 1$ for fixed j and a, $l_{i_1 j} \neq l_{i_2 j}$ for all $i_1 \neq i_2$. Similarly $l_{ij_1} \neq l_{ij_2}$ for all $j_1 \neq j_2$. Hence $L = (l_{ij})$ is a Latin square. Conversely, if we define a function g as $g(i, j, a) = 1$ if $a = l_{ij}$ and $= 0$ otherwise for a Latin square $L = (l_{ij})$, then $g \in G^{3,n}$. Thus $G^{3,n}$ is identified with the set of Latin squares.

For $g \in G^{k,n}$, let φ_g be a test function such that

$$\varphi_g(x_1, \cdots, x_k) = \sum_{i_1, \cdots, i_k = 1}^{n} \chi_{A(i_1, \cdots, i_k)}(x_1, \cdots, x_k) g(i_1, \cdots, i_k)$$

where $\chi_A(x) = 1$ if $x \in A$ and $= 0$ otherwise. The critical region of φ_g is given by $\bigcup_{g(i_1, \cdots, i_k)=1} A(i_1, \cdots, i_k)$.

Let $\Phi^{k,n} = \{\varphi_g; g \in G^{k,n}\}$.

We define

$$a_i(\theta) = F_\theta(z(\frac{i}{n})) - F_\theta(z(\frac{i-1}{n})) \qquad \text{for } i = 1, 2, \cdots, n.$$

Obviously,

$$a_i(\theta^*) = \frac{1}{n} \qquad \text{for all } i = 1, 2, \cdots, n. \tag{2.2}$$

THEOREM 2.1. *$\varphi_g \in \Phi^{k,n}$ is a similar test of size $1/n$ for any $k \geq 2$.*

PROOF. Without loss of generality, we assume that $\theta_1 = \theta^*$. From (2.1) and (2.2), we have

$$\begin{aligned}
\int \varphi_g dF_{\theta_1} \cdots dF_{\theta_k} &= \sum_{i_1, \cdots, i_k = 1}^{n} a_{i_1}(\theta_1) \cdots a_{i_k}(\theta_k) g(i_1, \cdots, i_k) \\
&= \sum_{i_2, \cdots, i_k = 1}^{n} a_{i_2}(\theta_2) \cdots a_{i_k}(\theta_k) \sum_{i_1 = 1}^{n} a_{i_1}(\theta^*) g(i_1, \cdots, i_k) \\
&= \frac{1}{n} \sum_{i_2, \cdots, i_k = 1}^{n} a_{i_2}(\theta_2) \cdots a_{i_k}(\theta_k) \\
&= \frac{1}{n} \prod_{j=2}^{n} \sum_{i_j = 1}^{n} a_{i_j}(\theta_j) \; = \; \frac{1}{n}.
\end{aligned}$$

□

REMARK 2.1. If $\sup_{\theta \geq \theta^*, \theta \in \Theta} a_n(\theta) = 1$, $A(n, \cdots, n)$ is the critical region of the size-$\frac{1}{n}$ min-test. Then, any φ_g satisfying $g(n, \cdots, n) = 1$ is uniformly more powerful than the min-test.

Let $D_n = \{(x_1, \cdots, x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq \cdots \leq x_n\}$. For $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in D_n$, x is said to be majorized by y, denoted by $x <^M y$, if $\sum_{i=j}^n x_i \leq \sum_{i=j}^n y_i$ for all $j = 2, \cdots, n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Note that the above definition is given for $x, y \in D_n$.

LEMMA 2.1. *For all* $\theta \geq \theta^*$, $a(\theta) = (a_1(\theta), \cdots, a_n(\theta)) \in D_n$. *Furthermore, if* $\theta^* \leq \theta \leq \theta'$, *then* $a(\theta) <^M a(\theta')$.

PROOF. We have

$$a_i(\theta) - a_{i-1}(\theta) = \int g_1(x) f_\theta(x) dx, \text{ where } g_1(x) = \begin{cases} 1 & z(\frac{i-1}{n}) < x < z(\frac{i}{n}) \\ -1 & z(\frac{i-2}{n}) < x < z(\frac{i-1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

Since $a_i(\theta^*) - a_{i-1}(\theta^*) = 0$ and $f_\theta(x)$ has monotone likelihood ratio, we obtain $a_i(\theta) - a_{i-1}(\theta) \geq 0$ for $\theta \geq \theta^*$ by variation-diminishing arguments (cf. Karlin [9] and Brown, Johnstone and MacGibbon [2]). Thus $a(\theta) \in D_n$ for $\theta \geq \theta^*$. Moreover, since

$$\sum_{i=j}^n a_i(\theta) = \int g_2(x) f_\theta(x) dx, \text{ where } g_2(x) = \begin{cases} 1 & x > z(\frac{i-1}{n}) \\ 0 & \text{otherwise,} \end{cases}$$

it is also shown by variation-diminishing arguments that $\sum_{i=j}^n a_i(\theta)$ is a non-decreasing function of θ for any j. It is obvious that $\sum_{i=1}^n a_i(\theta) = \sum_{i=1}^n a_i(\theta') = 1$. Therefore $a(\theta) <^M a(\theta')$ for $\theta^* \leq \theta \leq \theta'$. □

3. The uniformly most powerful test in $\Phi^{2,n}$

In this section, we show that the unbiased test given by Nomakuchi and Sakata [13] is uniformly most powerful in $\Phi^{2,n}$ and has a monotone power function.

S_n denotes the set of all permutations of $\{1, 2, \cdots, n\}$. When a permutation π assigns π_i for i, we use a representation as $\pi = (\pi_1, \cdots, \pi_n)$. We consider the arrangement ordering discussed in Hollander, Proschan and Sethuraman [7] and Pečarić, Proschan and Tong [14].

For $\pi = (\pi_1, \cdots, \pi_n)$, $\sigma = (\sigma_1, \cdots, \sigma_n) \in S_n$, we define $\sigma <^a \pi$ if for some $i < j$ satisfying $\pi_i < \pi_j$, it holds that

$$\sigma_i = \pi_j, \sigma_j = \pi_i \text{ and } \sigma_h = \pi_h \text{ for all } h \neq i, j.$$

Moreover, $\sigma <^A \pi$ if there exist some elements $\pi^1, \cdots, \pi^r$ of S_n satisfying

$$\sigma = \pi^1 <^a \cdots <^a \pi^r = \pi.$$

We assume that $\pi <^A \pi$ for all $\pi \in S_n$. S_n is partially ordered by the binary relation $<^A$, and $\pi^* = (1, 2, \cdots, n)$ is the maximum element in S_n with respect to this partial order, that is, $\pi <^A \pi^*$ for all $\pi \in S_n$.

We define a function f_2 of $x, y \in \mathbf{R}^n$ and $\pi \in S_n$ as

$$f_2(x, y, \pi) = \sum_{i=1}^n x_i y_{\pi_i}.$$

LEMMA 3.1. *(i) If $\sigma <^A \pi$, then $f_2(x,y,\sigma) \le f_2(x,y,\pi)$ for $x,y \in D_n$.*
(ii) If $x,x',y,y' \in D_n$ satisfy $x <^M x', y <^M y'$, then $f_2(x,y,\pi^*) \le f_2(x',y',\pi^*)$.

PROOF. (i) It is sufficient to prove for $\sigma <^a \pi$. By the definition of $<^a$, there exist i and j satisfying $i < j$, $\sigma_j = \pi_i < \sigma_i = \pi_j$ and $\sigma_h = \pi_h$ for all $h \ne i,j$. Then $f_2(x,y,\pi) - f_2(x,y,\sigma) = (x_j - x_i)(y_{\pi_j} - y_{\pi_i}) \ge 0$ for $x,y \in D_n$.
(ii) Since $\frac{d}{dx_i} f(x,y,\pi^*) = y_i$, we have $\frac{d}{dx_1} f(x,y,\pi^*) \le \cdots \le \frac{d}{dx_n} f(x,y,\pi^*)$ for $y \in D_n$. From Theorem A.3 of Marshall and Olkin ([12], p.56), we have $f_2(x,y,\pi^*) \le f_2(x',y,\pi^*)$ for $x <^M x'$ and $y \in D_n$. Similarly, $f_2(x',y,\pi^*) \le f_2(x',y',\pi^*)$ for $x' \in D_n$ and $y <^M y'$. Thus we obtain (ii). □

Nomakuchi and Sakata [13] proved that the test function

$$\varphi_{NS}(x_1,x_2) = \begin{cases} 1 & \text{if } (x_1,x_2) \in A(i,i) \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

is unbiased under an exponential family with Schur concave densities. Admissibility of φ_{NS} is discussed by Iwasa [8].

For $\pi = (\pi_1,\cdots,\pi_n) \in S_n$, let $g_\pi(i,j) = 1$ if $j = \pi_i$ and $= 0$ otherwise. Then $G^{2,n} = \{g_\pi : \pi \in S_n\}$ and $\Phi^{2,n} = \{\varphi_{g_\pi} : \pi \in S_n\}$. The power function of φ_{g_π} is given by $\sum_{i=1}^n a_i(\theta_1)a_{\pi_i}(\theta_2) = f_2(a(\theta_1),a(\theta_2),\pi)$. Since φ_{NS} is identified with $\varphi_{g_{\pi^*}}$, we obtain the following theorems from Lemmas 2.1 and 3.1.

THEOREM 3.1. *φ_{NS} is uniformly most powerful in $\Phi^{2,n}$.*

THEOREM 3.2. *The power function of φ_{NS}, denoted by $\beta_{NS}(\theta_1,\theta_2)$, satisfies that*

$$\beta_{NS}(\theta_1,\theta_2) \le \beta_{NS}(\theta_1',\theta_2') \qquad \textit{for } \theta^* \le \theta_1 \le \theta_1', \theta^* \le \theta_2 \le \theta_2'.$$

4. The case of $k = 3$ — an arrangement ordering on the set of Latin squares

We discuss the case of $k = 3$. Although our idea can be extended to the higher dimensions, we do not deal with the case of $k \ge 4$ in this work. As noted before, a test $\varphi_g \in \Phi^{3,n}$ is identified with a Latin square of order n. We denote by $\mathbf{L}_n$ the set of all Latin squares of order n.

Two permutations $\pi = (\pi_1,\cdots,\pi_n)$ and $\sigma = (\sigma_1,\cdots,\sigma_n)$ are said to be discordant if $\pi_i \ne \sigma_i$ for all $i = 1,\cdots,n$, and n permutations $\pi^1,\cdots,\pi^n$ of order n are said to be n-discordant if π^i and π^j are discordant for any $i \ne j$. We denote by $\mathbf{P}_n$ the set of all ordered n-discordant permutations of order n. We distinguish, for example, $(\pi^1,\pi^2,\pi^3,\cdots,\pi^n)$ and $(\pi^2,\pi^1,\pi^3,\cdots,\pi^n)$ in $\mathbf{P}_n$. $\mathbf{P}_n$ is not empty for all $n \ge 2$.

To begin with, we define an ordering on $\mathbf{P}_n$. Before giving the definition, we shall present a result on n-discordant permutations of order n.

For the sake of convenience and simplicity, we consider a matrix representation of a permutation π. For $\pi = (\pi_1,\cdots,\pi_n)$, we define a matrix $\Lambda(\pi) = (\lambda_{ij}(\pi))$ as $\lambda_{ij}(\pi) = 1$ if $j = \pi_i$, and $= 0$ otherwise. We denote by I_n the identity matrix of order n and by

$Q_n = (q_{ij})$ the matrix of order n such that $q_{ij} = 1$ if $i + j = n + 1$, and $= 0$ otherwise. $A \otimes B$ denotes the Kronecker product of $A = (a_{ij})$ and B, i.e. $A \otimes B = (a_{ij}B)$.

For $n = 2^p$ $(p = 1, 2, \cdots)$, we define n permutations $\tau^{n,1}, \cdots, \tau^{n,n}$ as

$$\begin{aligned}
\Lambda(\tau^{n,1}) &= Q_2 \otimes \cdots \otimes Q_2 \quad (= Q_{2^p}), \\
\Lambda(\tau^{n,2}) &= Q_2 \otimes \cdots \otimes Q_2 \otimes I_2 \quad (= Q_{2^{p-1}} \otimes I_2), \\
&\vdots \\
\Lambda(\tau^{n,n}) &= I_2 \otimes \cdots \otimes I_2 \quad (= I_{2^p}).
\end{aligned}$$

Identifying I_2 and Q_2 with 1 and 0 respectively, $\Lambda(\tau^{n,i})$ gives a representation of $i - 1$ by the binary system. Let $P_n^* = (\tau^{n,1}, \cdots, \tau^{n,n})$.

LEMMA 4.1. $P_n^* \in \mathbf{P}_n$.

PROOF. It is sufficient to show

$$\sum_{\mu=1}^{n} \lambda_{ij}(\tau^{n,\mu}) = 1 \qquad \text{for all } 1 \leq i, j \leq n. \tag{4.1}$$

When $n = 2$, (4.1) is obvious. Assume that (4.1) is true for $n = 2^p$. Let $n = 2^{p+1}$. Note that

$$\Lambda(\tau^{2^{p+1},i}) = \begin{cases} Q_2 \otimes \Lambda(\tau^{2^p,i}) & 1 \leq i \leq 2^p \\ I_2 \otimes \Lambda(\tau^{2^p,i-2^p}) & 2^p + 1 \leq i \leq 2^{p+1}. \end{cases}$$

Therefore, if $1 \leq i, j \leq 2^p$, we have

$$\sum_{\mu=1}^{2^{p+1}} \lambda_{ij}(\tau^{2^{p+1},\mu}) = \sum_{\mu=2^p+1}^{2^{p+1}} \lambda_{ij}(\tau^{2^{p+1},\mu}) = \sum_{\mu=1}^{2^p} \lambda_{ij}(\tau^{2^p,\mu}) = 1.$$

The other cases are similar. The proof is completed by induction. □

A proof of the following theorem is given in Appendix.

THEOREM 4.1. *$(\pi^1, \pi^2, \cdots, \pi^n) \in \mathbf{P}_n$ satisfying*

$$\pi^1 <^A \pi^2 <^A \cdots <^A \pi^n \tag{4.2}$$

exists if and only if $n = 2^p$ $(p = 1, 2, \cdots)$, and it is uniquely given by P_n^*.

Theorem 4.1 implies that n-discordant permutations of order n are not totally ordered by $<^A$ except the special cases. Thus we consider the following extension of the arrangement ordering.

DEFINITION 4.1. Let $P = (\pi^1, \cdots, \pi^n), P' = (\sigma^1, \cdots, \sigma^n) \in \mathbf{P}_n$. We define $P <^b P'$ if for some $i < j$,

$$\sigma^i <^A \pi^i, \ \pi^j <^A \sigma^j \text{ and } \pi^h = \sigma^h \text{ for all } h \neq i, j.$$

Furthermore, $P <^B P'$ if there exist some elements $P^1, \cdots, P^r \in \mathbf{P}_n$ satisfying $P = P^1 <^b \cdots <^b P^r = P'$.

We note that although $\mathbf{L}_n$ is identified with $\mathbf{P}_n$, the one-to-one correspondence between $\mathbf{L}_n$ and $\mathbf{P}_n$ is not unique. Here we consider three one-to-one mappings $\Delta^{<r>}$, $\Delta^{<c>}$ and $\Delta^{<n>}$ from $\mathbf{L}_n$ to $\mathbf{P}_n$ defined as for a Latin square $L = (l_{ij})$

$$\begin{aligned}\Delta^{<r>}(L) &= ((l_{11}, \cdots, l_{1n}), \cdots, (l_{n1}, \cdots, l_{nn})),\\ \Delta^{<c>}(L) &= ((l_{11}, \cdots, l_{n1}), \cdots, (l_{1n}, \cdots, l_{nn})),\\ \Delta^{<n>}(L) &= ((\bar{l}_{11}, \cdots, \bar{l}_{1n}), \cdots, (\bar{l}_{n1}, \cdots, \bar{l}_{nn})),\end{aligned}$$

where $\bar{l}_{ij}$ is ν satisfying $l_{j\nu} = i$. For example, for a Latin square

$$L = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix},$$

we have

$$\begin{aligned}\Delta^{<r>}(L) &= ((3,1,2),(2,3,1),(1,2,3)),\\ \Delta^{<c>}(L) &= ((3,2,1),(1,3,2),(2,1,3)),\\ \Delta^{<n>}(L) &= ((2,3,1),(3,1,2),(1,2,3)).\end{aligned}$$

We define an ordering on $\mathbf{L}_n$ through $\Delta^{<r>}, \Delta^{<c>}, \Delta^{<n>}$.

DEFINITION 4.2. Let $L, L' \in \mathbf{L}_n$. We define $L <^c L'$ if at least one of

$$\Delta^{<r>}(L) <^B \Delta^{<r>}(L'), \Delta^{<c>}(L) <^B \Delta^{<c>}(L'), \Delta^{<n>}(L) <^B \Delta^{<n>}(L')$$

holds. Furthermore, $L <^C L'$ if there exist $L^1, L^2, \cdots, L^r \in \mathbf{L}_n$ satisfying $L = L^1 <^c L^2 <^c \cdots <^c L^r = L'$.

REMARK 4.1. Even if we replace $<^B$ with $<^b$ in the definition of $<^c$, the ordering $<^C$ does not change. Such is not the case in Definition 4.1.

The following relations are induced through $\Delta^{<r>}$, $\Delta^{<c>}$, $\Delta^{<n>}$ respectively.

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} <^c \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

A 2×2 submatrix of a Latin square which consists of only two different numbers is called as an intercalate (cf. Dénes and Keedwell [4]). Intercalates play an important role with respect to $<^c$. By exchanging the two numbers within an intercalate, the binary relation $<^c$ is induced as follows.

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & i & \cdots & j & \cdots \\ & \vdots & & \vdots & \\ \cdots & j & \cdots & i & \cdots \\ & \vdots & & \vdots & \end{pmatrix} <^c \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & j & \cdots & i & \cdots \\ & \vdots & & \vdots & \\ \cdots & i & \cdots & j & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \qquad \text{for } i < j.$$

Let L_n^* be the Latin square of order n obtained uniquely from P_n^* by the inverse mappings of $\Delta^{<r>}, \Delta^{<c>}, \Delta^{<n>}$.

The structure of $\mathbf{L}_n$ induced by $<^C$ is very complex. The maximum element does not always exist in $\mathbf{L}_n$. But, when $n = 2$ or 4, the maximum element in $\mathbf{L}_n$ is given by

$$L_2^* = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ or } L_4^* = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

respectively. The maximality of L_4^* is verified by the intercalate-argument above.

For $x, y, z \in \mathbf{R}^n$ and $L = (l_{ij}) \in \mathbf{L}_n$, let

$$f_3(x, y, z, L) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j z_{l_{ij}}.$$

The following lemma is an extension of Lemma 3.1 to the case of $k = 3$.

LEMMA 4.2. *(i) If $L <^C L'$, then $f_3(x,y,z,L) \le f_3(x,y,z,L')$ for $x,y,z \in D_n$. (ii) If $x, x', y, y', z, z' \in D_n$ satisfy $x <^M x', y <^M y', z <^M z'$, then $f_3(x,y,z,L_n^*) \le f_3(x',y',z',L_n^*)$.*

PROOF. (*i*) It is sufficient to prove the case of $L = (l_{ij}) <^c L' = (l'_{ij})$. We show the case of $\Delta^{<r>}(L) <^b \Delta^{<r>}(L')$ (cf. Remark 4.1). The other cases are shown similarly since

$$f_3(x,y,z,L) = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j z_{l_{ij}} = \sum_{j=1}^{n} y_j \sum_{i=1}^{n} z_{l_{ij}} x_i = \sum_{h=1}^{n} z_h \sum_{i=1}^{n} x_i y_{\bar{l}_{hi}}$$

where $\bar{l}_{hi}$ is j satisfying $l_{ij} = h$. Let $\pi^i = (l_{i1}, \cdots, l_{in})$ and $\sigma^i = (l'_{i1}, \cdots, l'_{in})$ for $i = 1, \cdots, n$. From the definition of $<^b$, there exist $i_1 < i_2$ satisfying $\sigma^{i_1} <^A \pi^{i_1}, \pi^{i_2} <^A \sigma^{i_2}$ and $\sigma^i = \pi^i$ for $i \ne i_1, i_2$. Then, since $\{l'_{i_1 j}, l'_{i_2 j}\} = \{l_{i_1 j}, l_{i_2 j}\}$ for all j, it holds that

$$\sum_{j=1}^{n} y_j(z_{l'_{i_2 j}} - z_{l_{i_2 j}}) + \sum_{j=1}^{n} y_j(z_{l'_{i_1 j}} - z_{l_{i_1 j}}) = \sum_{j=1}^{n} y_j(z_{l'_{i_2 j}} - z_{l_{i_2 j}} + z_{l'_{i_1 j}} - z_{l_{i_1 j}}) = 0.$$

Therefore, we have from Lemma 3.1 (*i*)

$$\begin{aligned} f_3(x,y,z,L') - f_3(x,y,z,L) &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j(z_{l'_{ij}} - z_{l_{ij}}) \\ &= x_{i_2} \sum_{j=1}^{n} y_j(z_{l'_{i_2 j}} - z_{l_{i_2 j}}) + x_{i_1} \sum_{j=1}^{n} y_j(z_{l'_{i_1 j}} - z_{l_{i_1 j}}) \\ &= (x_{i_2} - x_{i_1}) \sum_{j=1}^{n} y_j(z_{l'_{i_2 j}} - z_{l_{i_2 j}}) \\ &= (x_{i_2} - x_{i_1})\{f(y,z,\sigma^{i_2}) - f(y,z,\pi^{i_2})\} \ge 0. \end{aligned}$$

(*ii*) From the definition of L_n^*, we have

$$f_3(x,y,z,L_n^*) = \sum_{i=1}^{n} x_i f_2(y,z,\tau^{n,i}) = \sum_{i=1}^{n} y_i f_2(z,x,\tau^{n,i}) = \sum_{i=1}^{n} z_i f_2(x,y,\tau^{n,i}).$$

Since $\tau^{n,1} <^A \cdots <^A \tau^{n,n}$, we have $f_2(y,z,\tau^{n,1}) \le \cdots \le f_2(y,z,\tau^{n,n})$ from Lemma 3.1 (*i*). Therefore, from Lemma 3.1 (*ii*), it holds that $f_3(x,y,z,L_n^*) \le f_3(x',y,z,L_n^*)$ for $x <^M x'$. Similarly, $f_3(x,y,z,L_n^*) \le f_3(x,y',z,L_n^*)$ for $y <^M y'$ and $f_3(x,y,z,L_n^*) \le f_3(x,y,z',L_n^*)$ for $z <^M z'$. This completes the proof. □

For $L = (l_{ij}) \in \mathbf{L}_n$, let $g_L(i,j,h) = 1$ if $h = l_{ij}$ and $= 0$ otherwise. Then $G^{3,n} = \{g_L : L \in \mathbf{L}_n\}$ and $\Phi^{3,n} = \{\varphi_{g_L} : L \in \mathbf{L}_n\}$. The power function of φ_{g_L}, where $L = (l_{ij})$, is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i(\theta_1)a_j(\theta_2)a_{l_{ij}}(\theta_3).$$

Therefore, we obtain the following theorems from Lemmas 2.1 and 4.2.

THEOREM 4.2. *If $L <^C L'$, $\varphi_{g_{L'}}$ is uniformly more powerful than φ_{g_L}. In particular, when $n = 2$ or 4, $\varphi_{g_{L_n^*}}$ is uniformly most powerful in $\Phi^{3,n}$.*

THEOREM 4.3. *For $n = 2^p$ ($p = 1,2,\cdots$), the power function of $\varphi_{g_{L_n^*}}$, denoted by $\beta_{L_n^*}(\theta)$, is monotone in the sense that*

$$\beta_{L_n^*}(\theta) \le \beta_{L_n^*}(\theta') \quad \text{if } \theta^* \le \theta_i \le \theta_i' \text{ for } i = 1,2,3.$$

In particular, $\varphi_{g_{L_n^*}}$ is unbiased for H_0 vs. H_1.

REMARK 4.2. We note that a maximal Latin square does not necessary correspond to a test which is uniformly more powerful than the min-test even if $\sup_{\theta \ge \theta^*, \theta \in \Theta} a_n(\theta) = 1$. For example, although a Latin square

$$L = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

is a maximal element in $\mathbf{L}_n$, the critical region of φ_{g_L} does not contain that of the min-test.

5. Modification for H_0^* vs. H_1

In this section, we assume that $\{f_\theta(x)\}$ is an exponential family, that is $f_\theta(x) = \eta(\theta)h(x)\exp(x\theta)$.

As mentioned in the introduction, there exist no non-trivial unbiased tests for H_0^* vs. H_1. Thus our unbiased test for H_0 vs. H_1 is not a level $1/n$-test for H_0^* any longer. We propose a method to construct level-$1/n$ tests for H_0^* by combining results of Li [11] and our results in the previous sections.

Let $m = \int x f_{\theta^*}(x)dx$ and n_0 be the number satisfying $z(\frac{n_0}{n}) \leq m < z(\frac{n_0+1}{n})$. Assume that $n - n_0 \geq 2$. Then, we define as

$$z^*(0) = m, z^*(\frac{i}{n}) = z(\frac{i+n_0}{n}), i = 1, \cdots, n - n_0,$$
$$a_i^*(\theta) = F_\theta(z^*(\frac{i}{n})) - F_\theta(z^*(\frac{i-1}{n})), i = 1, \cdots, n - n_0,$$
$$A^*(i_1, \cdots, i_n) = \{(x_1, \cdots, x_k) \in \mathbf{R}^k : z^*(\frac{i_j - 1}{n}) < x_j < z^*(\frac{i_j}{n}), j = 1, \cdots, k\}.$$

For $g \in G^{k,n-n_0}$, let φ_g^* be a test function defined as

$$\varphi_g^*(x_1, \cdots, x_k) = \sum_{i_1,\cdots,i_k=1}^{n-n_0} g(i_1, \cdots, i_k)\chi_{A^*(i_1,\cdots,i_k)}(x_1, \cdots, x_k).$$

Let $\Phi^{*k,n-n_0} = \{\varphi_g^* : g \in G^{k,n-n_0}\}$.

THEOREM 5.1. *$\varphi_g^* \in \Phi^{*k,n-n_0}$ is a level-$1/n$ test.*

PROOF.

$$\int \varphi_g^* dF_{\theta_1} \cdots dF_{\theta_k} = \sum_{i_1,\cdots,i_k=1}^{n-n_0} a_{i_1}^*(\theta_1) \cdots a_{i_k}^*(\theta_k) g(i_1, \cdots, i_k). \tag{5.1}$$

By Lemma 1 of Li [11], $a_i^*(\theta)$ is a non-decreasing function of θ in $\{\theta \leq \theta^*\}$ for all $i = 1, \cdots, n - n_0$. Therefore, the supremum of (5.1) under H_0^* is equal to that under H_0. It is shown as Theorem 2.1 that the supremum under H_0 is not more than $1/n$. □

For $x, y \in D_n$, x is said to be weakly submajorized by y, denoted by $x <^W y$, if $\sum_{i=j}^n x_i \leq \sum_{i=j}^n x_i$ for all $j = 1, \cdots, n$.

LEMMA 5.1. *(i) For all $\theta > \theta^*$, $a^*(\theta) = (a_1^*(\theta), \cdots, a_{n-n_0}^*(\theta)) \in D_{n-n_0}$.*
(ii) If $\theta^* \leq \theta \leq \theta'$, then $a^*(\theta) <^W a^*(\theta')$.

PROOF. Noting that $a_1^*(\theta) \leq a_{n_0+1}(\theta)$ and $a_i^*(\theta) = a_{n_0+i}(\theta)$ for $i = 2, \cdots, n - n_0$, the proof is an analogy of that of Lemma 2.1. □

Lemma 5.1 (i) implies that the arguments based on the arrangement orderings discussed in Sections 3 and 4 are also available for comparisons of power functions of tests in $\Phi^{*2,n-n_0}$ and $\Phi^{*3,n-n_0}$ (cf. Lemma 3.1 (i) and Lemma 4.2 (i)).

The next lemma is a generalization of Lemma 3.1 (ii).

LEMMA 5.2. *If $x, x', y, y' \in D_n$ satisfy $x <^W x', y <^W y'$, then $f_2(x, y, \pi^*) \leq f_2(x', y', \pi^*)$.*

PROOF. It is proved similarly as Lemma 3.1 (ii) by Theorem A.7 of Marshall and Olkin ([12], p.59). □

For the power function of $\varphi^*_{g_{\pi^*}} \in \Phi^{*2,n-n_0}$ and $\varphi^*_{g_{L^*_{n-n_0}}} \in \Phi^{*3,n-n_0}$, we can prove a monotonicity result analogous to Theorems 3.2 and 4.3 by using Lemmas 5.1 (*ii*) and 5.2.

Li [11] proposed a test φ^* whose critical region is given by $\bigcup_{i=1}^{n-n_0} A^*(i,\cdots,i)$ for any k. When $k=2$, φ^* belongs to $\Phi^{*2,n-n_0}$ and is uniformly most powerful in $\Phi^{*2,n-n_0}$. However, when $k=3$, φ^* does not belongs to $\Phi^{*3,n-n_0}$. And, if $\sup_{\theta\geq\theta^*,\theta\in\Theta} a^*_{n-n_0}(\theta)=1$, we can find a test which is uniformly more powerful than φ^* in $\Phi^{*3,n-n_0}$. Let $L=(l_{ij})$ be a Latin square of order $n-n_0$ such that $l_{i(n-n_0)}=i$ for all $i=1,\cdots,n-n_0$. Certainly such a square is constructed by rearrangement of rows of a Latin square. Then it holds from Lemma 5.1 (*i*) that

$$\begin{aligned}\int \varphi^* dF_{\theta_1}dF_{\theta_2}dF_{\theta_3} &= \sum_{i=1}^{n-n_0} a_i^*(\theta_1)a_i^*(\theta_2)a_i^*(\theta_3) \leq \sum_{i=1}^{n-n_0} a_i^*(\theta_1)a_{n-n_0}^*(\theta_2)a_i^*(\theta_3)\\ &\leq \sum_{i,j=1}^{n-n_0} a_i^*(\theta_1)a_j^*(\theta_2)a_{l_{ij}}^*(\theta_3) = \int \varphi^*_{g_L} dF_{\theta_1}dF_{\theta_2}dF_{\theta_3}\end{aligned}$$

under H_1. On the other hand, from $\sup_{\theta\geq\theta^*,\theta\in\Theta} a^*_{n-n_0}(\theta)=1$ and Theorem 5.1, we have

$$\sup_{H_0^*} E[\varphi^*] = \sup_{H_0^*} E[\varphi^*_{g_L}] = 1/n.$$

Furthermore, in some cases, it is verified that $\varphi^*_g \in \Phi^{*k,n-n_0}$ is more powerful than other tests presented in the introduction near the point $(\theta^*,\cdots,\theta^*)$. For example, let $X_i \sim N(\theta_i,1)$, $\Theta=\mathbf{R}$, $k=3$ and $n=20$. Then $m=\theta^*$ and $n_0=10$. Therefore, at $(\theta^*,\theta^*,\theta^*)$, the power of $\varphi^*_g \in \Phi^{*3,10}$ is 0.0125, which is 100 times that of the min-test, 10 times that of φ^* by Li [11] and 1.25 times that of a test T^* by Shirley [15].

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A. Appendix

We give a proof of Theorem 4.1. To begin with, we give some notations. For a permutation $\pi = (\pi_1, \cdots, \pi_n)$, let $C(\pi)$ and $c(\pi)$ be the set of all concordant pairs and the number of the concordant pairs respectively, that is,

$$C(\pi) = \{(i,j); i < j \text{ and } \pi_i < \pi_j\} \quad \text{and} \quad c(\pi) = \sharp C(\pi).$$

Obviously $0 \leq c(\pi) \leq n(n-1)/2$.

When $\sigma <^a \pi$, let $\pi \wedge \sigma$ and $\pi \vee \sigma$ be the integers i and j satisfying $i < j$,

$$\pi_i = \sigma_j < \pi_j = \sigma_i \quad \text{and} \quad \pi_h = \sigma_h \text{ for all } h \neq i, j$$

respectively and

$$d(\pi,\sigma) = \sharp\{i; \pi\wedge\sigma < i < \pi\vee\sigma, \pi_{\pi\wedge\sigma} < \pi_i < \pi_{\pi\vee\sigma}\}.$$

Then we have the following.

LEMMA A.1. *If* $\sigma <^a \pi$, $c(\pi) - c(\sigma) = 2d(\pi,\sigma) + 1$.

PROOF. We need to examine only the pairs containing $\pi\wedge\sigma$ or $\pi\vee\sigma$. For $i < \pi\wedge\sigma$,

$$\begin{aligned}(i, \pi\vee\sigma) \in C(\pi) &\iff (i, \pi\wedge\sigma) \in C(\sigma),\\ (i, \pi\wedge\sigma) \in C(\pi) &\iff (i, \pi\vee\sigma) \in C(\sigma).\end{aligned}$$

The case of $\pi\vee\sigma < i$ is similar. When $\pi\wedge\sigma < i < \pi\vee\sigma$,

$$\begin{aligned}(i, \pi\vee\sigma) \in C(\pi) &\iff (\pi\wedge\sigma, i) \notin C(\sigma),\\ (\pi\wedge\sigma, i) \in C(\pi) &\iff (i, \pi\vee\sigma) \notin C(\sigma).\end{aligned}$$

Moreover $(\pi\wedge\sigma, \pi\vee\sigma)$ belongs to $C(\pi)$ but not to $C(\sigma)$. Hence, noting that $(\pi\wedge\sigma, i) \notin C(\pi)$ and $(i, \pi\vee\sigma) \notin C(\pi)$ never occur at the same time, we have

$$c(\pi) - c(\sigma) = 2 \times \sharp\{i; (\pi\wedge\sigma, i) \in C(\pi) \text{ and } (i, \pi\vee\sigma) \in C(\pi)\} + 1 = 2d(\pi,\sigma) + 1. \quad \square$$

When π and σ are discordant, we use a symbol $\sigma \perp \pi$.

COROLLARY A.1. *If* $\sigma <^A \pi$ *and* $\pi \perp \sigma$,

$$c(\pi) - c(\sigma) \geq \left[\frac{n}{2}\right]$$

where $[n/2]$ *is the minimum integer not less than* $n/2$.

PROOF. Let $\sigma^0, \sigma^1, \cdots, \sigma^r$ satisfy $\sigma = \sigma^0 <^a \cdots <^a \sigma^r = \pi$. Then, since $\sharp\{i : \pi_i \neq \sigma_i\} \leq 2r$, $\pi \perp \sigma$ implies $2r \geq n$, that is, $r \geq [n/2]$. Therefore, from Lamma A.1,

$$c(\pi) - c(\sigma) = \sum_{i=1}^{r} c(\sigma^{i-1}) - c(\sigma^i) \geq r \geq \left[\frac{n}{2}\right]. \quad \square$$

LEMMA A.2. *If* $(\pi^1, \cdots, \pi^n) \in \mathbf{P}_n$ *satisfies (4.2), it holds that*
(i) $\pi^1 = (n, n-1, \cdots, 1), \pi^n = (1, 2, \cdots, n)$,
(ii) $c(\pi^i) - c(\pi^{i-1}) = n/2$ *for all* $i = 2, \cdots, n$, *and* n *is even.*

PROOF. Since $\pi^i \perp \pi^{i-1}$ $(i = 2, \cdots, n)$, from Corollary A.1,

$$c(\pi^n) - c(\pi^1) = \sum_{i=2}^{n} c(\pi^i) - c(\pi^{i-1}) \geq (n-1)\left[\frac{n}{2}\right] \geq \frac{n(n-1)}{2}. \tag{A.1}$$

Since $0 \leq c(\pi) \leq n(n-1)/2$, the equalities must hold in (A.1). This leads to the conclusions. $\square$

From now on, we assume that n is even, i.e. $n = 2\kappa$.

For a partition $T = \{(i_1, j_1), \cdots, (i_\kappa, j_\kappa)\}$ of $\{1, 2, \cdots, n\}$ and a permutation $\pi \in S_n$, we define permutations $\pi(T)^h$ $(h = 0, 1, \cdots, \kappa)$ recursively as $\pi(T)^0 = \pi$ and

$$\pi(T)^h_{i_h} = \pi(T)^{h-1}_{j_h}, \pi(T)^h_{j_h} = \pi(T)^{h-1}_{i_h} \text{ and } \pi(T)^h_l = \pi(T)^{h-1}_l \text{ for all } l \neq i_h, j_h,$$

for $h = 1, \cdots, \kappa$.

The problem to find σ satisfying

$$\begin{cases} (1) & \sigma <^A \pi \\ (2) & \sigma \perp \pi \\ (3) & c(\pi) - c(\sigma) = \kappa \end{cases} \tag{A.2}$$

for a given π is reduced as follows.

LEMMA A.3. *For a given π, there exists a permutation σ satisfying (A.2) if and only if there exists a partition $T = \{(i_1, j_1), \cdots, (i_\kappa, j_\kappa)\}$ satisfying that*

$$(i_h, j_h) \in C(\pi) \textit{ for all } h = 1, \cdots, \kappa \tag{A.3}$$

and that

$$d(\pi(T)^{h-1}, \pi(T)^h) = 0 \textit{ for all } h = 1, \cdots, \kappa. \tag{A.4}$$

And then, $\pi(T)^\kappa$ is a permutation satisfying (A.2).

PROOF. If σ satisfies (A.2), there exist permutations $\sigma^0, \cdots, \sigma^r$ such that $\sigma = \sigma^r <^a \cdots <^a \sigma^1 <^a \sigma^0 = \pi$ from (1). Then $r = \kappa$ from (2) and (3). Therefore, $T_\sigma = \{(\sigma^0 \wedge \sigma^1, \sigma^0 \vee \sigma^1), \cdots, (\sigma^{\kappa-1} \wedge \sigma^\kappa, \sigma^{\kappa-1} \vee \sigma^\kappa)\}$ is a partition of $\{1, 2, \cdots, n\}$ from (2) and $d(\sigma^{i-1}, \sigma^i) = 0$ for all $i = 1, \cdots, \kappa$ from (3). Thus T_σ satisfies (A.3) and (A.4). The converse is obvious. □

REMARK A.1. Although $\pi(T)^1, \cdots, \pi(T)^{\kappa-1}$ depend on the arrangement of the pairs in a partition T, it does not depend on the arrangement whether the condition (A.4) holds or not because $\pi(T)^\kappa$ does not.

REMARK A.2. σ satisfying (A.2) is not always unique. For example, both $(3, 1, 4, 2)$ and $(2, 4, 1, 3)$ satisfy (A.2) for $(1, 3, 2, 4)$.

LEMMA A.4. *(i) σ satisfying (A.2) is uniquely given by $A \otimes Q_2 \otimes I_p$ when $\pi = A \otimes I_2 \otimes Q_p$, where A is a permutation matrix and $p \geq 1$.*
(ii) No permutation satisfies (A.2) when $\pi = I_q \otimes Q_p$, where p is even and q is odd.

PROOF. Let $I(s) = \{p(s-1)+1, \cdots, ps\}$ and $J(s) = I(2s-1) \cup I(2s)$.
(*i*) *Step 1* : First we prove the case of $A = I_1$, i.e. $\pi = I_2 \otimes Q_p$. Let a partition $T = \{(i_1, j_1), \cdots, (i_p, j_p)\}$ satisfy (A.3). Then $i_h \in I(1)$ and $j_h \in I(2)$ for any h. Suppose that there exist (i_{h_1}, j_{h_1}) and (i_{h_2}, j_{h_2}) satisfying $i_{h_1} < i_{h_2}, j_{h_1} < j_{h_2}$. Without loss of generality, we can assume that $h_1 = 1, h_2 = 2$ (cf. Remark A.1). Then we have

$\pi(T)^1_{i_2} < \pi(T)^1_{j_1} < \pi(T)^1_{j_2} < \pi(T)^1_{i_1}$ from $\pi_{i_2} < \pi_{i_1} < \pi_{j_2} < \pi_{j_1}$. On the other hand, $i_2 < j_1 < j_2$. Hence we have $d(\pi(T)^1, \pi(T)^2) \neq 0$, which implies that (A.4) is not satisfied. Therefore T must be $T_0 = \{(h, 2p-h+1); h = 1, 2, \cdots, p\}$ in order to satisfy (A.4). T_0 certainly satisfies (A.3) and (A.4), and $\pi(T_0)^p = Q_2 \otimes I_p$.

Step 2 : Next we prove the general case. Let a be the order of A. From Step 1, $T_1 = \{(2p(s-1)+h, 2ps-h+1); s = 1, \cdots, a, h = 1, \cdots, p\}$ is the unique partition satisfying (A.3) and (A.4) among all partitions $\{(i_1, j_1), \cdots, (i_{ap}, j_{ap})\}$ such that there exists $J(s_h)$ containing both i_h and j_h for all $h = 1, \cdots, ap$, and then $\pi(T_1)^{ap}$ is given by $A \otimes Q_2 \otimes I_p$. Therefore, the proof is completed if we show that any partition T', which contains a pair (i_h, j_h) such that

$$i_h \in J(s), j_h \in J(s') \quad \text{for some } s < s', \tag{A.5}$$

does not satisfy both (A.3) and (A.4). Assume that T' satisfies (A.3). Without loss of generality, we can suppose that (i_1, j_1) has the smallest first-entry among the pairs satisfying (A.5) (cf. Remark A.1). If $i_h < i_1$ and $i_h \in I(2s-1)$, then $j_h \in I(2s)$ by (A.3). Thus $i_1 \in I(2s)$ leads to a contradiction that $I(2s)$ contains more than p numbers. Therefore, $i_1 \in I(2s-1)$ for some s. Then, since $i_1 < i < j_1$ and $\pi_{i_1} < \pi_i < \pi_{j_1}$ for any $i \in I(2s)$, we have $d(\pi(T')^0, \pi(T')^1) \neq 0$. Hence the proof is completed by Lemma A.3.

(*ii*) When $q = 1$, it is obvious. We prove the case of $p = 2p_0$ and $q \geq 3$. Suppose that $T = \{(i_1, j_1), \cdots, (i_{p_0 q}, j_{p_0 q})\}$ satisfies (A.3). Then, for all $h = 1, \cdots, p_0 q$, there exist s_{i_h} and s_{j_h} satisfying $1 \leq s_{i_h} < s_{j_h} \leq q$, $i_h \in I(s_{i_h})$ and $j_h \in I(s_{j_h})$. If $s_{j_h} - s_{i_h} > 1$, it is shown by the same argument as Step 2 of (*i*) that (A.4) is not satisfied. However, since q is odd, it is impossible that $s_{j_h} - s_{i_h} = 1$ for all h. By Lemma A.3, no permutation satisfies (A.2). □

PROOF OF THEOREM 4.1. Let $n = q \cdot 2^p$, where $p \geq 0$ and q is an odd number. Assume that $P = (\pi^1, \cdots, \pi^n) \in \mathbf{P}_n$ satisfies (4.2). Then, from Lemma A.2,

$$\pi^n = (1, 2, \cdots, n) = I_q \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{p}$$

and π^{i-1} $(i = 2, \cdots, n)$ is a permutation σ satisfying (A.2) for $\pi = \pi^i$. Therefore, by Lemma A4 (*i*), $\pi^n, \pi^{n-1}, \cdots, \pi^{n-2^p+1}$ are uniquely determined by $I_q \otimes I_{2^p}, I_q \otimes I_{2^{p-1}} \otimes Q_2, \cdots, I_q \otimes Q_{2^p}$ respectively. If $q \geq 3$, i.e. $n > 2^p$, there exist no permutations satisfying (A.2) for $\pi = \pi^{n-2^p+1}$ from Lemma A4 (*ii*). Hence we have $q = 1$, i.e. $n = 2^p$. When $q = 1$, the permutations $(\pi^1, \cdots, \pi^n)$ determined above is P_n^*. Hence the proof is completed. □

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