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HAMONIZABILITY OF A p–ADIC STATIONARY PROCESS

By

Yasushi ENDOW*

Abstract

This paper deals with a p-adic stationary process, which is an extension of a dyadic stationary process. The definition of this process is based on the p-adic group operation. A necessary and sufficient condition will be stated in order that the process (and its covariance function) is harmonizable, namely it assumes a spectral representation.

The paper discusses moreover the relation between a harmonizable p-adic stationary process with a spectral density and a linear p-adic process.

1. Introduction

It is well-known that a measureble stationary process $\{X(t); t \in \mathbf{R}\}$ in wide sense is harmonizable, i.e., it is representable as

$$X(t) = \int_{-\infty}^{\infty} e^{itx}\zeta(dx),$$

where ζ is an orthogonal random measure, and its covariance function is also expressed by

$$\operatorname{cov}(t, s) = \int_{-\infty}^{\infty} e^{itx}\overline{e^{isx}}F(dx),$$

where $F(dx) = \mathbf{E}|\zeta(dx)|^2$ is the spectral distribution function. Since

$$\operatorname{cov}(t, s) = \int_{-\infty}^{\infty} e^{i(t-s)x}F(dx) = \operatorname{cov}(t - s, 0)$$

the covariance function depends only on the difference of $t - s$.

Similar theorems for the dyadic stationary processes are shown by Endow[3][4]. In the discrete parameter case Nagai[8] first prove it, and Endow[3] completes the proof later. Under some conditions a dyadic stationary process $\{X(t); t \in \mathbf{R}^+\}$ is harmonizable, i.e., it is representable as

$$X(t) = \int_{0}^{\infty} \psi_t(x)\zeta(dx)$$

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where $\psi_t(x)$ is the Walsh function and ζ is an orthogonal random measure[4]. Its covariance function is represented by

$$\mathrm{cov}(t,s) = \int_0^\infty \psi_t(x)\overline{\psi_s(x)}F(dx),$$

where F is the spectral distribution function with $F(dx) = \mathbf{E}|\zeta(dx)|^2$. Since for any $t \in \mathbf{R}^+$ the equality

$$\psi_t(x)\overline{\psi_s(x)} = \psi_{t\ominus s}(x)$$

holds for a.a.$s \in \mathbf{R}^+$, in the case of DSP

$$\mathrm{cov}(t,s) = \mathrm{cov}(t \ominus s, 0) \quad (t, \text{a.a.}s \in \mathbf{R}^+)$$

holds of every $t \in \mathbf{R}^+$ and for a.a.$s \in \mathbf{R}^+$.

In this paper we deal with a p-adic stationary process(p-SP), and state its harmonizability. In 2 we review briefly the Selfridge functions, that are the generalizations of the Walsh functions. In 3 we prove a harmonizability theorem of the p-SP. In 4 we show that a p-SP with a spectral density is a linear p-adic process, and *vice versa.*

2. The p-adic group and the Selfridge functions

Throughout this paper p is taken to be a fixed, but arbitrary integer greater than one. Let $G = \{0, 1, ..., p-1\}$ be a topological group with the discrete topology and the addition;

$$x \oplus y \equiv x + y \pmod p \quad (x, y, \in G). \tag{1}$$

Let us define a sequence of product groups of G by

$$\mathbf{G}_k = \prod_{i=k}^{\infty} G_i \quad (k \in \mathbf{Z}), \tag{2}$$

where $G_i \equiv G$ $(i \in \mathbf{Z})$. On $\mathbf{G}_k$'s group operations and topologies are induced naturally. Let us put $\mathbf{G} = \bigcup_{k\in\mathbf{Z}} \mathbf{G}_k$, then $\mathbf{G}$ is a locally compact Abelian group with the addition $\mathbf{x} \oplus \mathbf{y}$ defined by

$$(\mathbf{x} \oplus \mathbf{y})_i = x_i \oplus y_i \quad (i \geq \min\{M, N\}), \tag{3}$$

where $\mathbf{x} \equiv (x_i)_{i\geq M} \in \mathbf{G}$ and $\mathbf{y} \equiv (y_i)_{i\geq N} \in \mathbf{G}$. Similary, we define the subtraction by

$$(\mathbf{x} \ominus \mathbf{y})_i = x_i \ominus y_i \quad (i \geq \min\{M, N\}), \tag{4}$$

where $x \ominus y \equiv x - y \pmod p$ $(x, y \in G)$. Let $\mathbf{\Gamma}$ be the dual group or character group of $\mathbf{G}$. To each $\gamma \in \mathbf{\Gamma}$ there exists a $\mathbf{y} \in \mathbf{G}$ such that

$$\gamma(\mathbf{x}) = \alpha^{\mathbf{x}\odot\mathbf{y}} \quad (\mathbf{x} \in \mathbf{G}) \tag{5}$$

where

$$\mathbf{x} \odot \mathbf{y} = \sum_{i=-M}^{N+1} x_i y_{1-i}$$

and $\alpha = \exp(2\pi i/p)$. If $\mathbf{y} \in \mathbf{G}$ is given then $\gamma(\mathbf{x})$ defined by (5) is a character on $\mathbf{G}$. The 1-1 correspondence thus established between $\mathbf{G}$ and $\mathbf{\Gamma}$ is easily seen to be an isomorphism. For further discussion see Fine[5].

Let $\mathbf{R}^+ = [0, \infty)$ and $\mathbf{D}$ be the set of p-adic rationals on $\mathbf{R}^+ \backslash \{0\}$. A mapping λ from $\mathbf{G}$ onto $\mathbf{R}^+$ defined by

$$\lambda(\mathbf{x}) = \sum_{n=M}^{\infty} x_n p^{-n} \quad (\mathbf{x} \equiv (x_n)_{n \geq M} \in \mathbf{G}) \tag{6}$$

represents a p-adic expression of an element in $\mathbf{R}^+$. Since every p-adic rational has two expressions, in which we will take a finite one to make the discussion clear, the mapping λ is not injective. Define $\mathbf{F} = \{\mathbf{x} \in \mathbf{G} : \mathbf{x} \equiv (x_i),\ x_i = p-1\ (i \geq M) \text{ for some } M\}$. The restriction of λ on $\mathbf{F}^c \equiv \mathbf{G} \backslash \mathbf{F}$ is 1-1 and onto $\mathbf{R}^+$. The inverse mapping μ satisfies $\lambda(\mu(x)) = x \ \ (x \in \mathbf{R}^+)$ and $\mu(\lambda(\mathbf{x})) = \mathbf{x} \ \ (\mathbf{x} \in \mathbf{F}^c)$. Let us introduce the addition $x \oplus y$ on $\mathbf{R}^+$ by

$$x \oplus y = \lambda(\mu(x) \oplus \mu(y)) \quad (x, y, \in \mathbf{R}^+). \tag{7}$$

The subtraction $x \ominus y$ is similarly defined on $\mathbf{R}^+$ by

$$x \ominus y = \lambda(\mu(x) \ominus \mu(y)) \quad (x, y \in \mathbf{R}^+). \tag{8}$$

If $x \oplus y = 0$ then y is said to be the *inverse* of x and denoted by x^-. Then

$$x \ominus y = x \oplus y^- \quad (x, y \in \mathbf{R}^+).$$

For each complex-valued function f defined on $\mathbf{R}^+$ there corresponds a function $\mathbf{f}$ defined on $\mathbf{G}$ such that

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= f(\lambda(\mathbf{x})) \qquad (\mathbf{x} \in \mathbf{F}^c) \\ &= \limsup_{\mathbf{y} \to \mathbf{x}} \mathbf{f}(\mathbf{y}) \quad (\mathbf{x} \in \mathbf{F}) \end{aligned} \tag{9}$$

where the approach is over those $\mathbf{y} \in \mathbf{F}^c$. The function $\mathbf{f}$ will be called the **G**-*extension of* f, and this relation will be expressed as $\mathbf{f} \sim f$. If f is continuous so is $\mathbf{f}$, but not conversely. A function f on $\mathbf{R}^+$ is said to belong to $\mathbf{C}^W$, if it is continuous at every p-adic irrational, right-continuous at every point, and has a finite left-limit at every point. If $f \in \mathbf{C}^W$ then its $\mathbf{G}$-extension is continuous on $\mathbf{G}$(c.f.[6]).

A measure $\mathbf{m}$ defined on $\mathcal{B}(\mathbf{G})$, that is the Borel field on $\mathbf{G}$, is decomposed uniquely into a usual measure, vanishing on all Borel subsets of $\mathbf{F}$, and an unsusual measure, vanishing on all Borel subsets of $\mathbf{F}^c$. There is a 1-1 correspondence between usual measures on $\mathbf{G}$ and measures on $\mathbf{R}^+$ such that

$$\mathbf{m}(A) = m(\lambda(A \cap \mathbf{F}) \quad (A \in \mathcal{B}(\mathbf{G}))$$

or

$$m(A) = \mathbf{m}(\mu(A)) \quad (A \in \mathcal{B}(\mathbf{R}^+)).$$

We denote this correspondence as $\mathbf{m} \sim m$. If $\mathbf{f} \sim f$ and $\mathbf{m} \sim m$, then

$$\int_{\mathbf{G}} \mathbf{f}(\mathbf{x})\mathbf{m}(d\mathbf{x}) = \int_0^\infty f(x)m(dx) \tag{10}$$

in particular, a Haar measure $d\mathbf{x}$ is adjustable such that

$$\int_{\mathbf{G}} \mathbf{f}(\mathbf{x})d\mathbf{x} = \int_0^\infty f(x)dx, \tag{11}$$

where dx is a Lebesque measure[5]. We assume in the sequel that the Haar measure on $\mathbf{G}$ is always adjusted as above. A function f on $\mathbf{R}^+$ is said to be *positive definite*, if it satisfies that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j f(t_i \ominus t_j) \geq 0$$

for any sequence $\{t_1, ..., t_n\}$ of non-negative numbers with $\mu(t_i) \ominus \mu(t_j) \in \mathbf{F}^c$ $(i, j = 1, ..., n)$ and for any sequence $\{c_1, ...c_n\}$ of complex numbers.

Now we shall define the Selfridge functions[10], that are the extensions of the Walsh functions[1], [5],[11]. For $t \in \mathbf{R}^+$ put

$$\psi_t(x) = \alpha^{\mu(x) \odot \mu(t)} \quad (x \in \mathbf{R}^+) \tag{12}$$

and call it the *Selfridge function*. When $p = 2$, $\psi_t(x)$ is the Walsh functions defined by Fine[5]. Putting

$$\langle x, t \rangle = \sum_{i=-M}^{T+1} x_i t_{1-i} \quad (t, x \in \mathbf{R}^+), \tag{13}$$

where $t = \sum_{i=-T}^{\infty} t_i p^{-i}$ and $x = \sum_{i=-M}^{\infty} x_i p^{-i}$, we may write

$$\psi_t(x) = \omega^{\langle x, t \rangle}. \tag{14}$$

It follows from definition that

$$\begin{aligned}
\psi_t(0) &= 1 \quad (t \in \mathbf{R}^+) && (15)\\
\psi_y(x) &= \psi_{[y]}(x)\psi_{[x]}(y) = \psi_x(y) \quad (x, y \in \mathbf{R}^+) && (16)\\
\psi_t(x)\psi_s(x) &= \psi_{t\oplus s}(x) \quad (x \in \mathbf{R}^+,\ \mu(t) \oplus \mu(s) \in \mathbf{F}^c) && (17)\\
\psi_t(x)\overline{\psi_s(x)} &= \psi_{t\ominus s}(x) \quad (x \in \mathbf{R}^+,\ \mu(t) \ominus \mu(s) \in \mathbf{F}^c) && (18)\\
\psi_t(x)\psi_t(y) &= \psi_t(x \oplus y) \quad (t \in \mathbf{R}^+,\ \mu(x) \oplus \mu(y) \in \mathbf{F}^c) && (19)\\
\psi_t(x)\overline{\psi_t(y)} &= \psi_t(x \ominus y) \quad (t \in \mathbf{R}^+,\ \mu(x) \ominus \mu(y) \in \mathbf{F}^c) && (20)
\end{aligned}$$

[10]. Since the both sets $\mathbf{F}_t \equiv \{s \in \mathbf{R}^+ : \mu(t) \oplus \mu(s) \in \mathbf{F}\}$ and $\mathbf{F}_s \equiv \{t \in \mathbf{R}^+ : \mu(t) \oplus \mu(s) \in \mathbf{F}\}$ are countable, the equality (17) holds for each $t \in \mathbf{R}^+$ and for a.a.$s \in \mathbf{R}^+$, or for each $s \in \mathbf{R}^+$ and for a.a.$t \in \mathbf{R}^+$. The equations (18)-(20) hold in the similar sense. It is clear that the limit

$$\lim_{y \uparrow x} \psi_t(y) = \psi_t(x-) \quad (x \in \mathbf{D}) \tag{21}$$

exists and is finite, and hence $\psi_t \in \mathbf{C}^W$ $(t \in \mathbf{R}^+)$. It is not difficult that

$$\lim_{s\to\infty} s^{-1}\int_0^s \psi_x(t)dt = \delta(x), \tag{22}$$

where δ is Kroneker's delta function.

For $f \in L^1(\mathbf{R}^+)$ we define its *Selfridge-Fourier (S-F) transform* by

$$\widehat{f}(t) = \int_0^\infty f(x)\overline{\psi_t(x)}dx. \tag{23}$$

The *S-F transform* of $f \in L^2(\mathbf{R}^+)$ is also defined by

$$\widehat{f}(t) = \underset{y\to\infty}{\text{l.i.m.}} \int_0^y f(x)\overline{\psi_t(x)}dx. \tag{24}$$

Since $\widehat{f} \in L^2(\mathbf{R}^+)$ for $f \in L^2(\mathbf{R}^+)$, the *inverse S-F transform* can be defined, and

$$f(x) = \underset{s\to\infty}{\text{l.i.m.}} \int_0^s \widehat{f}(t)\psi_x(t)dt. \tag{25}$$

It is easy to see that for $f \in L^1(\mathbf{R}^+)$ or for $f \in L^2(\mathbf{R}^+)$

$$\int_0^\infty f(x)\psi_s(x)\overline{\psi_t(x)}dx = \int_0^\infty f(x)\overline{\psi_{t\ominus s}(x)}dx$$

we obtain that

$$(\widehat{f\psi_s})(t) = \widehat{f}(t\ominus s) \quad (s,t \in \mathbf{R}^+). \tag{26}$$

3. The p-adic stationary porcesses and their harmonizability

Let $\{X(t,\omega); t \in \mathbf{R}^+\}$ be a second order process with a constant mean. If its covariance function

$$\text{cov}(t,s) = \mathbf{E}(X(t) - \mathbf{E}X(t))\overline{(X(s) - \mathbf{E}X(s))}$$

satisfies

$$\text{cov}(t,s) = \text{cov}(t\ominus s, 0) \quad (\mu(t)\ominus\mu(s) \in \mathbf{F}^c), \tag{27}$$

then it is calld a *p-adic stationary process (p-SP)*. For simplisity we assume throughout that $\mathbf{E}X(t) = 0$ $(t \in \mathbf{R}^+)$. We remark that the covariance function $\text{cov}(t,s)$ of a p-SP depends only on the difference $t\ominus s$ for any $t \in \mathbf{R}^+$ and a.a.$s \in \mathbf{R}^+$, or for any $s \in \mathbf{R}^+$ and a.a.$t \in \mathbf{R}^+$. We shall define a function r on $\mathbf{R}^+$ by

$$r(t) = \text{cov}(t,0) \quad (t \in \mathbf{R}^+). \tag{28}$$

Then

$$\text{cov}(t,s) = r(t\ominus s) \quad (\mu(t)\ominus\mu(s) \in \mathbf{F}^c). \tag{29}$$

It is also clear that

$$|r(t)| \leq r(0), \quad \overline{r(t)} = r(t^-) \quad (t \in \mathbf{R}^+), \tag{30}$$

$$\mathbf{E}|X(t) - X(s)|^2 = 2(r(0) - \mathrm{Re}\{r(t \ominus s)\}) \quad (\mu(t) \ominus \mu(s) \in \mathbf{F}^c). \tag{31}$$

If for each $\epsilon > 0$ there exists a $\delta > 0$ such that for h with $0 \leq h < \delta$, $\mathbf{E}|X(t \oplus h) - X(t)|^2 < \epsilon$ holds, then the process $X(t)$ is said to be *p-adic mean continuous* at t. If the process is p-adic mean continuous at every $t \in \mathbf{R}^+$, then it is simply called *p-adic mean continuous.*

LEMMA 3.1. *If a p-SP is measurable in the mean, then $r(t)$ defined by (28) is continuous from right at $t = 0$.*

This is essentially due to Crum[2]. Hence, we have the following

LEMMA 3.2. *A measureble p-SP is p-adic mean continuous, if and only if*

$$\mathbf{E}|X(t \oplus h) - X(t)|^2 \to 0, \tag{32}$$

as $h \to 0$ with $\mu(t) \oplus \mu(h) \in \mathbf{F}$.

Here we give an example.

EXAMPLE 3.1. *Let Y be a random variable with zero mean and unit variance. For any fixed $x \in \mathbf{R}^+$ put*

$$X(t, \omega) = \psi_x(t) Y(\omega) \quad (t \in \mathbf{R}^+). \tag{33}$$

Then it is a p-adic mean continuous p-SP. Actually, it is easy to see that

$$\mathbf{E}X(t) = 0,$$

$$\mathbf{E}X(t)\overline{X(s)} = \psi_x(t)\overline{\psi_x(s)} = \psi_x(t \ominus s) \quad (\mu(t) \ominus \mu(s) \in \mathbf{F}^c)$$

and

$$\mathbf{E}|X(t \oplus h) - X(t)|^2 = 2(1 - \mathrm{Re}\{\psi_x(t \oplus h)\overline{\psi_x(t)}\}) \to 0,$$

as $h \to 0$, since $\psi_x(t) \in \mathbf{C}^W$.

Now we consider the harmonizability of the covarince function of a p-SP.

THEOREM 3.3. *In order that the covariance function of measurable p-SP is harmonizable, i.e., it is representable as*

$$\mathrm{cov}(t, s) = \int_0^\infty \psi_t(x)\overline{\psi_s(x)} F(dx), \tag{34}$$

where F is a unique non-negative finite measure on $\mathbf{R}^+$, it is necessary and sufficient that it satisfies the following conditions;

(a)

$$\lim_{h \to 0} \mathrm{cov}(t, s \oplus h) = \mathrm{cov}(t, s) \quad (s, t \in \mathbf{R}^+).$$

(b) *The limit* $\lim_{s\uparrow t}\mathrm{cov}(s,0)$ *exist and is finite at* $t\in\mathbf{D}$.

(c)

$$\lim_{n\to\infty} p^{-n}\int_0^{p^n} r(t)\psi_{x-}(t)dt = 0 \quad (x\in\mathbf{D}).$$

PROOF. (Sufficiency) It is easy that the $\mathbf{G}$-extensions ρ of r defined by (28) is continuous, since $r\in\mathbf{C}^W$ by (a) and (b). For any sequence $\{t_1,...t_n\}$ of non-negetive numbers with $\mu(t_i)\ominus\mu(t_j)\in\mathbf{F}$ $(i,j=1,...,n)$ and for any sequence $\{c_1,...c_n\}$ of complex numbers we have that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{c}_j r(t_i\ominus t_j) = \mathbf{E}\Big|\sum_{i=1}^{n} c_i X(t_i)\Big|^2 \geq 0,$$

thus its $\mathbf{G}$-extension ρ is positive definite[6]. Hence it follows from Bochner's theorem[9] that

$$\rho(\mathbf{t}) = \int_{\mathbf{G}} \theta^{\mathbf{x}\odot\mathbf{t}}\mathbf{m}(d\mathbf{x}),$$

where $\mathbf{m}$ is a unique non-negative finite measure on $\mathbf{G}$. By definition

$$\begin{aligned} r(t) &= x\rho(\mu(t)) = \Big(\int_{\mathbf{F}^c} + \int_{\mathbf{F}}\Big)\theta^{\mathbf{x}\odot\mu(t)}\mathbf{m}(d\mathbf{x}) \\ &= J_1 + J_2, \end{aligned}$$

say. Let us evaluate J_2 first. Since the Haar measure of the set $\mathbf{F}$ is zero,

$$\int_{I(n)} \rho(\mathbf{t})\theta^{\mathbf{x}\odot\mathbf{t}}d\mathbf{t} = \int_0^{p^n} r(t)\psi_{x-}(t)dt \quad (n\in\mathbf{N}),$$

where $\mathbf{x}\in\mathbf{F}$ with $\lambda(\mathbf{x}) = x$ and $I(n)\equiv\{\mathbf{t}:\lambda(\mathbf{t})<\rho^n\}$. By (c) the last integral multiplied by p^{-n} tends to zero as $n\to\infty$. This means that $\mathbf{m}(\{\mathbf{x}\}) = 0$ $(\mathbf{x}\in\mathbf{F})$ by Lemma 2.5 in [4] and (c), hence this shows $J_2 = 0$. Next put $F(A) = \mathbf{m}(\mu(A))$ for $A\in\mathcal{B}(\mathbf{R}^+)$, then

$$r(t) = \int_0^\infty \theta^{\mu(x)\odot\mu(t)}\mathbf{m}(d\mu(x)) = \int_0^\infty \psi_x(t)F(dx).$$

Thus

$$\mathrm{cov}(t,s) = \int_0^\infty \psi_{t\ominus s}(x)F(dx) = \int_0^\infty \psi_t(x)\overline{\psi_s(x)}F(dx),$$

whenever $\mu(t)\ominus\mu(s)\in\mathbf{F}^c$. In the case of $\mu(t)\ominus\mu(s)\in\mathbf{F}$, by (a)

$$\begin{aligned} \mathrm{cov}(t,s) &= \lim_{h\to 0}\mathrm{cov}(t,s\oplus h) = \lim_{h\to 0}\int_0^\infty \psi_t(x)\overline{\psi_{s\oplus h}(x)}F(dx) \\ &= \int_0^\infty \psi_t(x)\overline{\psi_s(x)}F(dx), \end{aligned}$$

where the approach is over those h with $\mu(s)\ominus\mu(h)\in\mathbf{F}^c$. The last equality is justified, since $\psi_t\in\mathbf{C}^W$.

(Necessity) The condition (a) is obvious, and (b) is also clear by $\psi_s \in \mathbf{C}^W$. As for (c), let us define r as (28), and put $\rho \sim r$ and $\mathbf{m} \sim F$, respectively, then

$$\rho(\mu(t)) = \int_{\mathbf{F}^c} \alpha^{\mathbf{x}\odot\mu(t)}\mathbf{m}(d\mathbf{x}).$$

Note that $\mathbf{y} \neq \mathbf{x}$ in the last integral. Hence the inner integral multiplied by p^{-n} converges to zero boundedly as $n \longrightarrow \infty$.

The measure F is called the *sprectral measure* of the p-SP. Since $F(\mathbf{R}^+) = \mathbf{E}\big|X(t)\big|^2 < \infty$ $(t \in \mathbf{R}^+)$, it is, as was stated above, a finite measure. If F is absolutely continuous with respect to Lebesque mesure, then its Radon-Nykodym derivative f, which satisfies $F(A) = \int_A f(x)dx$ $(A \in \mathcal{B}(\mathbf{R}^+))$, is called the *spectral density function* of the p-SP. Let F be a non-negative finite measure on $\mathbf{R}^+$. Then there exists a p-SP which has F as its spectral measure. We shall illustrate this by an example.

EXAMPLE 3.2. *Let Y be a non-negative random variable with its probability $\Pr\{Y \in A\} = F(A)/F(\mathbf{R}^+)$ for $A \in \mathcal{B}(\mathbf{R}^+)$. Let Z be a random variable independent of Y with $\mathbf{E}Z = 0$, $\mathbf{E}\big|Z\big|^2 = F(\mathbf{R}^+)$. Define $X(t) = Z\psi_Y(t)$ $(t \in \mathbf{R}^+)$. Then*

$$\begin{aligned} \mathbf{E}X(t) &= 0, \\ \operatorname{cov}(t,s) &= \mathbf{E}\big|Z\big|^2\psi_Y(t)\overline{\psi_Y(s)} = \int_0^\infty \psi_y(t)\overline{\psi_y(s)}F(dy). \end{aligned}$$

Next we consider the harmonizability of the p-SP.

THEOREM 3.4. *Let $\{X(t); t \in \mathbf{R}^+\}$ be a p-SP. If its covarance function assumes a representation (34), then it is harmonizable, i.e., it is representable as*

$$X(t) = \int_0^\infty \psi_t(x)\zeta(dx) \quad \text{a.s.,} \tag{35}$$

where ζ is an orthogonal random measure with

$$\begin{aligned} \mathbf{E}\zeta(A) &= 0, \\ \mathbf{E}\zeta(A)\overline{\zeta(B)} &= F(A \cap B) \quad (A, B \in \mathcal{B}(\mathbf{R}^+)). \end{aligned}$$

Conversely, if a p-SP is harmonizable, then its covariance function is also harmonizable, i.e., it is representable in the form of (34).

We ommit the proof since the first part is a corollary of the result(c.f.[7, p.201]), and the second part is straightforward.

REMARK. The integral in (35) is defined in the limit in the mean sense, i.e.,

$$\lim_{s\to\infty} \mathbf{E}\left|\int_0^s \psi_t(x)\zeta(dx) - X(t)\right|^2 = 0.$$

The orthogonal random measure ζ is called the *spectral random measure* of the p-SP. The conditions (a) and (b) in Theorem 1 assure that $r \in \mathbf{C}^W$, and (c) means that the spectral measure on $\mathbf{G}$ is usual.

4. Linear p-adic processes and p-SP's

Let η be a random measure on $\mathcal{B}(\mathbf{R}^+)$ with

$$\begin{aligned} \mathbf{E}\eta(A) &= 0, \\ \mathbf{E}\eta(A)\overline{\eta(B)} &= \sigma^2 \int_{A\cap B} dx \quad (A, B \in \mathcal{B}(\mathbf{R}^+)). \end{aligned}$$

For $\Phi \in L^2(\mathbf{R}^+)$, define a process $\{X(t); t \in \mathbf{R}^+\}$ by

$$X(t) = \int_0^\infty \Phi(s \ominus t)\eta(ds) \quad \text{a.s.} \quad (t \in \mathbf{R}^+). \tag{37}$$

and it will be called a *linear p-adic process (Lp-P)*.

THEOREM 4.1. *A p-SP $\{X(t); t \in \mathbf{R}^+\}$ has a spectral density function, if and only if it is a Lp-P.*

PROOF. (Sufficiency) Let $X(t)$ be represented by (37). For the characteristic function $1_A(x)$ of a bounded set $A \in \mathcal{B}(\mathbf{R}^+)$, define its S-F transform

$$J_A(t) = \int_0^\infty 1_A(x)\overline{\psi_t(x)}dx = \int_A \overline{\psi_t(x)}dx.$$

Putting

$$\xi(A) = \int_0^\infty J_A(t)\eta(dt) \quad (A \in \mathcal{B}(\mathbf{R}^+)),$$

we obtain by Parseval's relation

$$\mathbf{E}\xi(A)\overline{\xi(B)} = \sigma^2 \int_0^\infty J_A(t)\overline{J_B(t)}dt = \sigma^2 \int_0^\infty 1_A(x)1_B(x)dx = \sigma^2 \int_{A\cap B} dx,$$

Hence ξ is an orthogonal random measure.

It will be shown that

$$\int_0^\infty \widehat{f}(t)\eta(dt) = \int_0^\infty f(x)\xi(dx), \tag{38}$$

where $\widehat{f}$ is the S-F transform of f in $L^2(\mathbf{R}^+)$ sense. First, consider a simple function

$$f(x) = \sum_{k=1}^n a_k 1_{A_k}(x) \quad (x \in \mathbf{R}^+)$$

where the A_k's are bounded and disjoint set belonging to $\mathcal{B}(\mathbf{R}^+)$. By linearity of the S-F transform

$$\int_0^\infty \widehat{f}(t)\eta(dt) = \sum_{k=1}^n a_k \int_0^\infty J_{A_k}(t)\eta(dt) = \sum_{k=1}^n a_k\xi(A_k) = \int_0^\infty f(x)\xi(dx).$$

It is easy to see that for every pair of simple functions f and g

$$\mathbf{E}\int_0^\infty \widehat{f}(t)\eta(dt)\overline{\int_0^\infty \widehat{g}(t)\eta(dt)} = \mathbf{E}\int_0^\infty f(x)\xi(dx)\overline{\int_0^\infty g(x)\xi(dx)},$$

where $\widehat{g}$ is the S-F transform of g in $L^2(\mathbf{R}^+)$ sense. This relation can be extended to $L^2(\mathbf{R}^+)$, since the set of all simple functions is dense in $L^2(\mathbf{R}^+)$.

Now we recall that $\Phi \in L^2(\mathbf{R}^+)$ and its S-F transform ϕ also belongs to $L^2(\mathbf{R}^+)$. The application of (38) to Φ and ϕ therefore gives with using (26)

$$\int_0^\infty \Phi(s \ominus t)\eta(ds) = \int_0^\infty \psi_t(x)\phi(x)\xi(dx) \quad \text{a.s.} \tag{39}$$

Put

$$\zeta(A) = \int_A \phi(x)\xi(dx) \quad (A \in \mathcal{B}(\mathbf{R}^+)),$$

then

$$\mathbf{E}\zeta(A) = 0,$$

and

$$\mathbf{E}\zeta(A)\overline{\zeta(B)} = \sigma^2 \int_{A\cap B} |\phi(x)|^2 dx \quad (A, B \in \mathcal{B}(\mathbf{R}^+)).$$

Hence (39) can be rewritten as

$$X(t) = \int_0^\infty \psi_t(x)\zeta(dx) \quad \text{a.s.,}$$

and

$$\text{cov}(t,s) = \sigma^2 \int_0^\infty \psi_t(x)\overline{\psi_s(x)}|\phi(x)|^2 dx.$$

(Necessity) Suppose that $\{X(t); t \in \mathbf{R}^+\}$ has a spectral density function, i.e.,

$$X(t) = \int_0^\infty \psi_t(x)\zeta(dx) \quad \text{a.s.,}$$

and

$$\mathbf{E}\zeta(A)\overline{\zeta(B)} = \int_{A\cap B} f(x)dx \quad (A, B \in \mathcal{B}(\mathbf{R}^+)),$$

where f is the spectral density function of the process. Let ϕ be a measurble function such that $|\phi(x)|^2 = f(x)$, thus $\phi \in L^2(\mathbf{R}^+)$. Let ζ_1 be a random measure which is orthogonal to ζ with

$$\mathbf{E}\zeta_1(A)\overline{\zeta_1(B)} = \int_{A\cap B} dx \quad (A, B \in \mathcal{B}(\mathbf{R}^+)).$$

Put

$$\xi(A) = \int_0^\infty 1_A(x)(1/\phi(x))\zeta(dx) + \int_0^\infty 1_A(x)\phi_1(x)\zeta_1(dx),$$

where $\phi_1(x) = 0$ when $\phi(x) \neq 0$; $= 1$ when $\phi(x) = 0$, and $1/\phi(x)$ is taken zero if $\phi(x) = 0$, then

$$\begin{aligned}\mathbf{E}\xi(A)\overline{\xi(B)} &= \int_0^\infty 1_A(x)1_B(x)\left(1/|\phi(x)|^2\right)f(x)dx + \int_0^\infty 1_A(x)1_B(x)|\phi(x)|^2 dx \\ &= \int_{A\cap B} dx.\end{aligned}$$

By defintion

$$\begin{aligned}\int_0^\infty \phi(x)\psi_t(x)\xi(dx) &= \int_{\{x:\phi(x)\neq 0\}} \phi(x)\psi_t(x)(1/\phi(x))\zeta(dx) + \int_{\{x:\phi(x)=0\}} \phi(x)\phi_1(x)\zeta_1(dx) \\ &= \int_0^\infty \psi_t(x)\zeta(dx).\end{aligned}$$

The last equality is justified since

$$\mathbf{E}\left|\int_{\{x:\phi(x)=0\}} \psi_t(x)\zeta(dx)\right|^2 = \int_{\{x:\phi(x)=0\}} f(x)dx = 0.$$

Thus

$$X(t) = \int_0^\infty \phi(x)\psi_t(x)\xi(dx) \quad \text{a.s.}$$

Putting

$$\eta(A) = \int_0^\infty J_A(x)\xi(dx) \quad (A \in \mathcal{B}(\mathbf{R}^+),$$

we have in the same way as (39) that

$$X(t) = \int_0^\infty \Phi(s \ominus t)\eta(ds) \quad \text{a.s.},$$

where Φ is the S-F transform of ϕ in the $L^2(\mathbf{R}^+)$ sense.

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