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Takeuchi, Hiroyuki
Department of Administration Engineering, Keio University

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A LOCATION SHIFT PROBLEM IN NONPARAMETRIC DENSITY ESTIMATION

By

Hiroyuki Takeuchi*

Abstract

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having a probability density function $f(x)$ and $f_n(x)$ be a nonparametric density estimator of $f(x)$. We investigate the property of a location shift random variable a_n which minimizes integrated squared error $\mathrm{ISE}_n(a)$:

$$\mathrm{ISE}_n(a) = \int_{-\infty}^{\infty} |f_n(x) - f(x - a)|^2 \, dx.$$

The asymptotic normality and the order of strong convergence of the r.v. a_n and those of $\mathrm{ISE}_n(a_n)$ are studied. We also give some numerical examples and some simulations which show the effectiveness of using the a_n when one estimates $f(x)$ by $f_n(x)$.

1. Introduction

Let

$$X_1, X_2, \ldots, X_n \tag{1.1}$$

be independently and identically distributed random variables with a common distribution function $F(x)$ whose density is $f(x)$. We define $f_n(x)$:

$$f_n(x) = \frac{1}{n} \sum_{j=1}^{n} K_n(x - X_j),$$

as a nonparametric density estimator for $f(x)$. The estimator $f_n(x)$ has been so widely studied by many authors, see, for example Izenman [11] or Prakasa Rao [14]. In this paper we shall investigate the asymptotic property of the location shift r.v. a_n which minimizes the integrated squared error $\mathrm{ISE}_n(a)$:

$$\mathrm{ISE}_n(a) = \int_{-\infty}^{\infty} |f_n(x) - f(x - a)|^2 dx \tag{1.2}$$

Blackman [2] considered this problem for the empirical distribution function and he got the asymptotic distribution of $\sqrt{n}\, a_n$. In Härdle [9], some estimators for the shift parameter are cited for the robust estimation of nonparametric regression function.

* Department of Administration Engineering, Keio University, 3−14−1 Hiyoshi, Kohoku-ku, Yokohama 223, Japan.

Scott [16] defined averaged shifted histograms but it seems that there is little connection with our work.

In Section 2, we shall show the weak convergence of the a_n using Heathcote's idea [10]. Since our situation is nonparametric, so the proof is slightly more complicated than his. We show that the asymptotic variance of $\sqrt{n}\ a_n$ is inherent in the underlying distribution. The asymptotic normality of $\mathrm{ISE}_n(a) - \mathrm{ISE}_n(0)$ is also proved to show the range of the interval $\{a{:}\mathrm{ISE}_n(a) - \mathrm{ISE}_n(0) < 0\}$. In Section 3, the order of strong convergence of the a_n is considered for the kernel-type density estimator. In that section, the empirical characteristic function based on (1.1) reveals us a powerful tool to evaluate the order. Especially it is proved that if $\hat{a}_n$ is any estimator of the location shift r.v. a_n and h_n is the window width, then $\mathrm{ISE}_n(\hat{a}_n) - \mathrm{ISE}_n(0)$ can not converge to 0 slower than $O\left(\left(\frac{\log\log n}{n}\right)^{\frac{1}{4}\left(1-\frac{1}{\mu-1}\right)}\right)$ or $O(h_n^{1/2})$, for some $\mu > 2$, according to certain conditions. A simulation was conducted in Section 4 to estimate the relative efficiency of the estimation with respect to the location shift r.v. a_n.

2. Weak Convergence of the Location Shift Random Variable a_n and $\mathrm{ISE}_n(a)$

In this section we shall consider a class of the density estimator $\{f_n(x),\ n \in \boldsymbol{N}\}$ that can be written in the next form

$$f_n(x) = \frac{1}{n}\sum_{j=1}^{n} K_n(x - X_j), \tag{2.1}$$

where $\{K_n,\ n \in \boldsymbol{N}\}$ is a sequence of "kernel" functions. Many types of density estimators are contained in this class, for example, kernel estimators, trigonometric series estimators, orthogonal polynomial estimators, Fourier transform estimators and histogram estimators (Hall [7]). We can write the Fourier transform of (2.1) as,

$$\varphi_{f_n}(t) = \varphi_{K_n}(t)\cdot c_n(t)$$

where $c_n(t)$ is called the empirical characteristic function (e.c.f.), based on (1.1), defined by

$$c_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{iX_jt}.$$

It is obvious that $c_n(t)$ converges to $\varphi_f(t)$, the characteristic function of the $f(x)$, with probability one for each $t \in \boldsymbol{R}$ (Lukacs [12]). The property of the e.c.f. has been investigated by many authors, for example Csörgö [5], Marcus [13] and Feuerverger & Mureika [6].

We assume that the true probability density function (p.d.f.) $f(x)$ satisfies (2.2).

$$\int_{-\infty}^{\infty} t^2\ |\varphi_f(t)|\ dt < \infty. \tag{2.2}$$

And we suppose following conditions for the kernel.

$$K_n(y) = K_n(-y),\ \text{for all } n \in \boldsymbol{N}, \tag{2.3}$$

$$\sup_{n\in N}\int_{-\infty}^{\infty}|K_n(y)|\,dy<\infty, \tag{2.4}$$

$$K_n(y)\in L^2\,(\boldsymbol{R}). \tag{2.5}$$

$$\lim_{n\to\infty}\varphi_{K_n}(t)=1,\ \text{for each } t\in\boldsymbol{R}. \tag{2.6}$$

We use the symbol Re$[z]$ and Im$[z]$ as the real and the imaginary part of $z\in\boldsymbol{C}$, respectively. The complex number $\bar{z}$ denotes the conjugate of z. And the symbols E and *Var* denote the expectation and the variance, respectively, with corresponding probability measure P or the measure generated by $F(\cdot)$. Furthermore *a.s.*, $\xrightarrow{p}$ and $\xrightarrow{\mathscr{D}}$ denotes the convergence with probability one, in probability and in distribution, respectively.

To state Theorem 2.2 and 2.3, the main results of this section, we need the following lemmas.

LEMMA 2.1.
If

$$\int_{-\infty}^{\infty}|t\varphi_f(t)|\,|\varphi_{K_n}(t)-1|\,dt=o(n^{-1/2}) \tag{2.7}$$

for large n, then we have

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathrm{Im}\left[\int_{-\infty}^{\infty}te^{iX_jt}\cdot\varphi_{K_n}(t)\cdot\overline{\varphi_f(t)}\mathrm{dt}\right]$$
$$\xrightarrow{\mathscr{D}}N\left(0,\ E\left|\mathrm{Im}\left[\int_{-\infty}^{\infty}te^{iX_1t}\cdot\overline{\varphi_f(t)}\mathrm{dt}\right]\right|^2\right),\quad as\ n\to\infty. \tag{2.8}$$

PROOF. From theorem 4.1 in Billingsley [1], we may prove (2.9) and (2.10) to show (2.8).

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\lim_{n\to\infty}\Psi_n(X_j)\xrightarrow{\mathscr{D}}N\left(0,\ E\left|\mathrm{Im}\left[\int_{-\infty}^{\infty}te^{iX_1t}\cdot\overline{\varphi_f(t)}dt\right]\right|^2\right), \tag{2.9}$$

$$\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\Psi_n(X_j)-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\lim_{n\to\infty}\Psi_n(X_j)\right|\xrightarrow{p}0. \tag{2.10}$$

as $n\to\infty$, where $\Psi_n(X)$ is defined by

$$\Psi_n(X)=\mathrm{Im}\left[\int_{-\infty}^{\infty}te^{iXt}\cdot\varphi_{K_n}(t)\cdot\overline{\varphi_f(t)}dt\right]. \tag{2.11}$$

By (2.2), (2.4) and (2.6), we can use dominated convergence theorem such as

$$\lim_{n\to\infty}\Psi_n(X_j)=\mathrm{Im}\left[\int_{-\infty}^{\infty}te^{iX_jt}\cdot\overline{\varphi_f(t)}dt\right]. \tag{2.12}$$

The sequence of the right hand side of (2.12), suffixed by j, is i.i.d. random variables which have finite second moment by (2.2). The expectation of (2.12) is 0 for all j from

(2.3), so by virtue of the central limit theorem, we have

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \lim_{n\to\infty} \Psi_n(X_j) \xrightarrow{\mathscr{D}} N\left(0,\ Var\left[\lim_{n\to\infty} \Psi_n(X_1)\right]\right),$$

as $n \to \infty$. Thus we have (2.9) immediately. Then by (2.7), (2.10) is proved in the following way.

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left| \Psi_n(X_j) - \lim_{n\to\infty} \Psi_n(X_j) \right| \\
&\leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left| \int_{-\infty}^{\infty} t e^{iX_j t} (\varphi_{K_n}(t) - 1) \cdot \overline{\varphi_f(t)} dt \right| \\
&\leq \sqrt{n} \int_{-\infty}^{\infty} |t\varphi_f(t)|\, |\varphi_{K_n}(t) - 1| \mathrm{dt} \\
&\to 0,\ a.s.
\end{aligned}$$

Hence we get (2.8).

The condition (2.7) may seem to be technical, but it is not so strong. In fact next proposition can be stated in the kernel-type density estimation case.

PROPOSITION 2.1.
Suppose that the density estimator is a kernel-type;

$$f_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right). \tag{2.13}$$

where h_n is called the window width satisfying $\lim_{n\to\infty} h_n = 0$. And assume that the kernel is standard normal and that the characteristic function of true p.d.f. satisfies

$$|\varphi_f(t)| \leq A\ e^{-\rho|t|} \tag{2.14}$$

for some A, $\rho > 0$ that are independent of t. If the order of the window width is

$$h_n = o\left(n^{-\frac{1}{4}-\varepsilon}\right), \tag{2.15}$$

then (2.7) is satisfied for any $\varepsilon > 0$.

PROOF. Since by assumption, for any fixed $\varepsilon > 0$ we can choose $m > \frac{1}{2\varepsilon}$ such that

$$\begin{aligned}
\int_{n^\varepsilon}^{\infty} |t\varphi_f(t)|\ dt &\leq \frac{A}{\rho^2} (1 + \rho n^\varepsilon) e^{-\rho n^\varepsilon} \\
&= o(n^{-m\varepsilon}) \\
&= o(1).
\end{aligned}$$

And then we have

$$\begin{aligned}
&\sqrt{n}\int_{-\infty}^{\infty}|t\varphi_f(t)|\,|\varphi_K(h_nt)-1|dt\\
&\le \sqrt{n}\sup_{|t|\le n^\varepsilon}\left|\exp\left(-\frac{1}{2}h_n^2t^2\right)-1\right|\int_{-\infty}^{\infty}|t\varphi_f(t)|\,dt\\
&\quad+2\sqrt{n}\left(\int_{n^\varepsilon}^{\infty}|t\varphi_f(t)|\,dt+\int_{-\infty}^{-n^\varepsilon}|t\varphi_f(t)|\,dt\right)\\
&=O\left(h_n^2\cdot n^{\frac{1}{2}+2\varepsilon}\right)+o\left(n^{\frac{1}{2}-m\varepsilon}\right)\\
&=o(1),
\end{aligned}$$

by (2.15).

The class of the characteristic function satisfying (2.14) contains wide variety of p.d.f. The form of (2.14), which was used in Watson and Leadbetter [18], will also appear in Section 3.

We shall evaluate the order of convergence of a_n to 0 in Section 3 under some conditions, but we can prove following theorem without any technical conditions.

THEOREM 2.1. *The location shift random variable a_n converges to 0 with probability one.*

PROOF. By dominated convergence theorem, we have

$$\begin{aligned}
&\pi\{\mathrm{ISE}_n(a)-\mathrm{ISE}_n(0)\}\\
&=\mathrm{Re}\left[\int_{-\infty}^{\infty}(1-e^{-iat})\varphi_{K_n}(t)\cdot c_n(t)\cdot\overline{\varphi_f(t)}dt\right]\\
&\to\int_{-\infty}^{\infty}\{1-\cos(at)\}\,|\varphi_f(t)|^2\,dt,\ a.s. \qquad (2.16)
\end{aligned}$$

as $n\to\infty$, for each $a\in\boldsymbol{R}$. This means for any $a\in\boldsymbol{R}$,

$$\mathrm{ISE}_n(a)-\mathrm{ISE}_n(0)\ge 0,\ a.s.$$

for sufficiently large n. $\mathrm{ISE}_n(a)$ is differentiable about $a\in\boldsymbol{R}$, so there exists a solution a_n of the equation

$$\frac{d}{da}\mathrm{ISE}_n(a)=0$$

that converges to 0 with probability one. To show that the limit of the a_n is 0 only, we shall give a necessary and sufficient conditions for an identifiability condition with respect to $f(x)$ when we consider a location shift parameter for $f(x)$. The following (i), (ii) and (iii) are equivalent

(i) $a=0$

(ii) $\displaystyle\int_{-\infty}^{\infty}(1-e^{iat})\,|\varphi_f(t)|^2\,dt=0,$

(iii) $f(x)=f(x-a)$, almost everywhere in $x\in\boldsymbol{R}$.

(i) to (iii) is obvious. (iii) to (ii) is as follows.

$$\int_{-\infty}^{\infty} (1 - e^{iat}) |\varphi_f(t)|^2 \, dt = 2\pi \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx - \int_{-\infty}^{\infty} f(x - a) f(x) dx \right)$$

by Parseval's relation. Finally we shall show (ii) to (i). If $a \neq 0$ there exists a $\delta > 0$ such that if $t \in (0, \delta]$ then both of $1 - \cos(\text{at})$ and $|\varphi_f(t)|$ are positive by their continuity. So we have

$$\begin{aligned} \mathrm{Re}\left[\int_{-\infty}^{\infty} (1 - e^{iat}) |\varphi_f(t)|^2 \, dt \right] &\geq \int_{t \in (0,\delta]} \{1 - \cos(at)\} \, |\varphi_f(t)|^2 \, dt \\ &> 0 \end{aligned}$$

This contradicts (ii), and hence we have (ii) to (i). By the assertion above, the right hand side of (2.16) is 0 if and only if $a = 0$. This completes the proof of the theorem.

We need following lemma to prove Lemma 2.3.

LEMMA 2.2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_l, X_{m,n} : l, m, n \in N\}$ be a sequence of random variables defined on the $(\Omega, \mathcal{F}, P)$. Suppose that*

(i) $X_1, X_2, \ldots$ *are independently and identically distributed random variables,*

(ii) $E|X_1| < \infty$,

(iii) $\lim_{n \to \infty} \max_{1 \leq m \leq n} |X_{m,n} - X_m| = 0, \quad a.s.$

Then we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} X_{m,n} = EX_1, \qquad a.s.$$

PROOF. Write

$$\begin{aligned} &\left| \frac{1}{n} \sum_{m=1}^{n} X_{m,n} - EX_1 \right| \\ &\leq \left| \frac{1}{n} \sum_{m=1}^{n} X_{m,n} - \frac{1}{n} \sum_{m=1}^{n} X_m \right| + \left| \frac{1}{n} \sum_{m=1}^{n} X_m - EX_1 \right| \\ &= I_{1,n} + I_{2,n}, \end{aligned}$$

say. From (i) and (ii), we have

$$\lim_{n \to \infty} I_{2,n} = 0, \qquad a.s.$$

by Kolmogorov's strong law of large numbers. From (iii), there exists an $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 1$ and that

$$\forall \omega \in \Omega_0, \forall \varepsilon > 0, \exists N(\omega) : n \geq N(\omega) \Rightarrow \max_{1 \leq m \leq n} |X_{m,n}(\omega) - X_m(\omega)| < \varepsilon.$$

Hence, then we have

$$\begin{aligned} I_{1,n} &\leq \frac{1}{n} \sum_{m=1}^{n} |X_{m,n}(\omega) - X_m(\omega)| \\ &< \varepsilon. \end{aligned}$$

This completes the proof.

LEMMA 2.3.

$$\frac{1}{n}\sum_{j=1}^{n} \mathrm{Re}\left[\int_{-\infty}^{\infty} t^2 e^{i(X_j-\theta a_n)t}\cdot \varphi_{K_n}(t)\cdot \overline{\varphi_f(t)}dt\right] \rightarrow \int_{-\infty}^{\infty} t^2|\varphi_f(t)|^2\, dt, \qquad a.s. \tag{2.17}$$

as $n \rightarrow \infty$, *uniformly for* $\theta \in (0, 1)$, *where* θ *is depending on* f, K_n *and* a_n.

PROOF. We define r.v. Y_j as

$$Y_j = \mathrm{Re}\left[\int_{-\infty}^{\infty} t^2 e^{iX_jt}\cdot \overline{\varphi_f(t)}dt\right],$$

and $Y_{j,n}$ as

$$Y_{j,n} = \mathrm{Re}\left[\int_{-\infty}^{\infty} t^2 e^{i(X_j-\theta a_n)t}\cdot \varphi_{K_n}(t)\cdot \overline{\varphi_f(t)}dt\right].$$

Note that $Y_{j,n}$ is equivalent to

$$-\frac{d}{du}\Psi_n(X_j-u)\Big|_{u=\theta a_n}$$

where $\Psi_n(X)$ is defined by (2.11). We shall show the sequence $\{Y_j, Y_{j,n}: j, n \in \boldsymbol{N}\}$ satisfies the condition (i), (ii) and (iii) of Lemma 2.2. (i) is obvious. (ii) is also clear from (2.2). Next we show (iii).

$$\begin{aligned}&\max_{1\le j\le n} |Y_{j,n} - Y_j|\\ &\le \int_{-\infty}^{\infty} t^2|\varphi_f(t)|\, |e^{-i\theta a_n t}\cdot \varphi_{K_n}(t) - 1|dt\\ &\rightarrow 0, \qquad a.s.\end{aligned}$$

as $n \rightarrow \infty$ by (2.4). (2.6) and Theorem 2.1 with applying dominated convergence theorem. Finally we get

$$EY_1 = \int_{-\infty}^{\infty} t^2|\varphi_f(t)|^2 dt$$

by Fubini's theorem.

Now we can state next theorem.

THEOREM 2.2. *Under the condition of Lemma* 2.1, *we have*

$$\sqrt{n}\, a_n \xrightarrow{\mathscr{D}} N\left(0, \frac{E\left|\mathrm{Im}\left[\int_{-\infty}^{\infty} te^{iX_1t}\cdot\overline{\varphi_f(t)}dt\right]\right|^2}{\left(\int_{-\infty}^{\infty} t^2|\varphi_f(t)|^2dt\right)^2}\right) \tag{2.18}$$

as $n \rightarrow \infty$.

PROOF. From (2.2) and (2.5) we have $f_n, f \in L^2(\boldsymbol{R})$, so (1.2) is equivalent to

$$\mathrm{ISE}_n(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\varphi_{K_n}(t)\cdot c_n(t) - e^{iat}\cdot\varphi_f(t)|^2 dt$$

by Parseval's relation. (2.2) affords us interchange of differentiation and integration such that

$$\begin{aligned}&\frac{d}{da}\,\mathrm{ISE}_n(a)\\ &= -\frac{1}{\pi}\cdot\mathrm{Im}\left[\int_{-\infty}^{\infty} te^{-iat}\cdot\varphi_{K_n}(t)\cdot c_n(t)\cdot\overline{\varphi_f(t)}dt\right].\end{aligned}$$

So, with Ψ_n defined in (2.11), the problem becomes to find an a_n which satisfies the equation:

$$\sum_{j=1}^{n}\Psi_n(X_j - a) = 0.$$

Characteristic of a_n which satisfies $\sum_{j=1}^{n}\rho(X_j - a) = 0$ has been discussed by Huber [11], where $\rho(\cdot)$ is a continuous function not depending on n. Let a_n be a consistant solution of this equation. Then

$$\begin{aligned}0 &= \sum_{j=1}^{n}\Psi_n(X_j - a_n)\\ &= \sum_{j=1}^{n}\Psi_n(X_j) + a_n\sum_{j=1}^{n}\frac{d}{du}\Psi_n(X_j - u)\Big|_{u=\theta a_n}\end{aligned}$$

for some $\theta \in (0, 1)$ by Taylor's expansion. So we get

$$\sqrt{n}\,a_n = \frac{-\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\Psi_n(X_j)}{\frac{1}{n}\sum_{j=1}^{n}\frac{d}{du}\Psi_n(X_j - u)\Big|_{u=\theta a_n}}. \tag{2.19}$$

Hence from Lemma 2.1 and Lemma 2.3, we obtain the conclusion by Slutsky's theorem (Serfling [17]).

We remark that the asymptotic variance of $\sqrt{n}\,a_n$ does not depend on the kernel and it only depends on the true p.d.f., in other words, the asymptotic variance is inherent in the underlying distribution F.

EXAMPLE 1. We exhibit the asymptotic variance in Theorem 2.2 for Cauchy and normal distributions;

$$\begin{aligned}\varphi_f(t) &= e^{-s|t|},\ s > 0,\\ \varphi_f(t) &= e^{-\sigma^2t^2/2},\ \sigma > 0.\end{aligned}$$

The asymptotic variances are $\frac{5}{2}s^2$ and $\frac{8}{3\sqrt{3}}\sigma^2$, respectively.

Hereafter we define $\mathrm{DISE}_n(a)$ as

$$\mathrm{DISE}_n(a) = \mathrm{ISE}_n(a) - \mathrm{ISE}_n(0)$$

for simplification.

THEOREM 2.3. *Under the condition of Lemma* 2.1, *we have*

$$\lim_{n\to\infty} \mathrm{P}\left[\sqrt{n}\,\pi\cdot \mathrm{DISE}_n(a) < x\right] = \Phi_{\mu_a,\sigma_a^2}(x) \tag{2.20}$$

for each $a \neq 0$, *where* $\Phi_{\mu_a,\sigma_a^2}(x)$ *denotes a normal distribution function with mean* μ_a *and variance* σ_a^2;

$$\mu_a = \int_{-\infty}^{\infty} \{1 - \cos(at)\}\, |\varphi_f(t)|^2 dt \tag{2.21}$$

$$\sigma_a^2 = E\left|\mathrm{Re}\left[\int_{-\infty}^{\infty} (1 - e^{-iat})e^{iX_1t}\cdot \overline{\varphi_f(t)}\,dt\right]\right|^2 - \mu_a^2 \tag{2.22}$$

respectively.

PROOF. The proof is almost the same as that of Lemma 2.1, so we shall only show the outline here. We define $\Psi_n(X_j, a)$ as

$$\Psi_n(X_j, a) = \mathrm{Re}\left[\int_{-\infty}^{\infty} (1 - e^{-iat})\, e^{iX_jt}\cdot \varphi_{K_n}(t)\cdot \overline{\varphi_f(t)}\,dt\right],$$

then from (2.2) and (2.6), we have

$$\lim_{n\to\infty} \Psi_n(X_j, a) = \mathrm{Re}\left[\int_{-\infty}^{\infty} (1 - e^{-iat})\, e^{iX_jt}\cdot \overline{\varphi_f(t)}\,dt\right],$$

by dominated convergence theorem. We can show (2.23) and (2.24),

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \lim_{n\to\infty} \Psi_n(X_j, a) \xrightarrow{\mathcal{L}} N(\mu_a, \sigma_a^2), \tag{2.23}$$

$$\left|\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \Psi_n(X_j, a) - \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \lim_{n\to\infty} \Psi_n(X_j, a)\right| \xrightarrow{p} 0, \tag{2.24}$$

as $n \to \infty$. By the definition of $\mathrm{ISE}_n(a)$, we have thus proved

$$\sqrt{n}\,\pi\cdot \mathrm{DISE}_n(a) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \Psi_n(X_j, a) \xrightarrow{\mathcal{L}} N(\mu_a, \sigma_a^2), \quad \text{as } n \to \infty$$

for each $a \neq 0$.

By the Theorem 2.3 we can find the range of the interval of which the parameter a satisfies $\mathrm{DISE}_n(a) < 0$. $\mathrm{ISE}_n(0)$ is thought of as the loss when we do not take the parameter into consideration. Following example illustrates the contents above.

EXAMPLE 2. If $\varphi_f(t) = \exp(-\frac{1}{2}t^2)$ then (2.2) is satisfied. In this case we have the asymptotic mean μ_a and the asymptotic variance σ_a^2 as follows.

$$\mu_a = \sqrt{\pi}(1 - e^{-a^2/4}),$$
$$\sigma_a^2 = \pi(1 - e^{-a^2/4})\,(e^{-a^2/4} + 4/\sqrt{3} - 1),$$

Obviously the probability $p(a) = \lim_{n\to\infty} P\{\mathrm{DISE}_n(a) < 0\}$ satisfies $p(a) = p(-a)$. Furthermore we remark that $p(a)$ is a continuous function about the parameter a, except for the point $a = 0$, and $\lim_{a\to 0} p(a) = 0.5$. Table 1 shows an example of the values of $p(a)$.

Table 1. Values of $p(a)$

a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$p(a)$	0	0.49	0.47	0.46	0.45	0.43	0.42	0.41	0.40

3. Strong Consistency of the Location Shift Random Variable a_n and $\mathrm{ISE}_n(a_n)$

In this section we shall evaluate the order of strong convergence of the a_n, and confine ourselves to consider the kernel-type density estimator which is given by (2.13), where the kernel $K(y)$ satisfies following three conditions,

$$^\forall y \in \boldsymbol{R},\ K(y) \geq 0 \text{ and } \int_{-\infty}^{\infty} K(y)\, dy = 1, \tag{3.1}$$

$$K(y) \in L^2(\boldsymbol{R}), \tag{3.2}$$

$$\int_{-\infty}^{\infty} |yK(y)|\, dy < \infty. \tag{3.3}$$

Hereafter we define the sequence $\{T_n,\ n \in \boldsymbol{N}\}$ as

$$T_n = \frac{\log\log n}{n}.$$

To evaluate the convergence order of the a_n to 0, we need following lemma that shows the uniform strong convergence of the e.c.f. process. Note that the lemma is slightly different from Csörgö [5] who proved this first.

LEMMA 3.1. *For any p.d.f. $f(x)$, we have*

$$\sup_{|t|\leq U_n} |c_n(t) - \varphi_f(t)| = O(U_n \cdot T_n^{1/2}), \quad a.s. \tag{3.4}$$

for sufficiently large n, where $\{U_n,\ n \in \boldsymbol{N}\}$ is a positive monotone increasing sequence diverging to infinity.

PROOF. Using integration by parts, we have for any fixed $K > 0$,

$$
\begin{aligned}
&\sup_{|t|\le U_n} |c_n(t) - \varphi_f(t)| \\
&= \sup_{|t|\le U_n} \left| \int_{-\infty}^{\infty} e^{itx}\, d(F_n(x) - F(x)) \right| \\
&\le \sup_{|t|\le U_n} \left| [e^{itx}(F_n(x) - F(x))]_{-K}^{K} - it \int_{-K}^{K} e^{itx}(F_n(x) - F(x))dx \right| \\
&\quad + \sup_{|t|\le U_n} \left| \int_{x\in[-K,K]^c} e^{itx} d(F_n(x) - F(x)) \right| \\
&\le (4 + 2KU_n) \sup_{-\infty<x<\infty} |F_n(x) - F(x)|, \qquad (3.5)
\end{aligned}
$$

where $F_n(x)$ denotes the empirical distribution function which based on (1.1). By Chung [4], we have

$$\limsup_{n\to\infty} T_n^{-1/2} \sup_{-\infty<x<\infty} |F_n(x) - F(x)| = \frac{1}{\sqrt{2}},\ a.s.$$

so then

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| = O(T_n^{1/2}),\ a.s.$$

From above and (3.5), we have (3.4) for sufficiently large n.

LEMMA 3.2. *Suppose that the true p.d.f. $f(x)$ satisfies the following.*

(i) $$\int_{-\infty}^{\infty} |t|^{m+1} |\varphi_f(t)|\, dt < \infty,$$

for some nonnegative integer m.

(ii) *There exists a $\mu > m$ such that*

$$\int_{t\in S_n^c} |t|^m\, |\varphi_f(t)|\, dt = o\left(T_n^{\frac{\mu-m}{2(\mu-m+1)}}\right),$$

where the set S_n is given by

$$S_n = \left\{t : t \in \left[-T_n^{\frac{-1}{2(\mu-m+1)}},\ T_n^{\frac{-1}{2(\mu-m+1)}}\right]\right\}.$$

Then we have

$$
\begin{aligned}
&\sup_{-\infty<a<\infty} \left| \frac{d^\nu}{da^\nu} \mathrm{ISE}_n(a) - \lim_{n\to\infty} \frac{d^\nu}{da^\nu} \mathrm{ISE}_n(a) \right| \\
&= O\left(T_n^{\frac{\mu-m}{2(\mu-m+1)}}\right),\ a.s.\ \text{for } h_n = o\left(T_n^{\frac{\mu-m}{2(\mu-m+1)}}\right) \\
&= O(h_n), \qquad a.s.\ \text{for } T_n = o\left(h_n^{\frac{2(\mu-m+1)}{\mu-m}}\right)
\end{aligned}
\qquad (3.6)
$$

as $n \to \infty$, for $\nu = 0, 1, \ldots, m$.

PROOF. It is easy to check that

$$\int_{-\infty}^{\infty} |t|^{m_1} |\varphi_f(t)| \, dt < \infty \text{ implies } \int_{-\infty}^{\infty} |t|^{m_2} |\varphi_f(t)| \, dt < \infty,$$

for any positive integer such that $m_2 \leq m_1$. This with the dominated convergence theorem for the interchange of differentiation and integration, we have from (3.1), (3.2) and (3.3)

$$\begin{aligned}
&\pi \left| \frac{d^\nu}{da^\nu} \mathrm{ISE}_n(a) - \lim_{n\to\infty} \frac{d^\nu}{da^\nu} \mathrm{ISE}_n(a) \right| \\
&\leq \left| \int_{-\infty}^{\infty} (-it)^\nu e^{-iat} \cdot \varphi_K(h_n t) \, (c_n(t) - \varphi_f(t)) \, \overline{\varphi_f(t)} dt \right. \\
&\quad \left. + \int_{-\infty}^{\infty} (-it)^\nu e^{-iat} (\varphi_K(h_n t) - 1) \, |\varphi_f(t)|^2 dt \right| \\
&\leq \sup_{t\in S_n} |c_n(t) - \varphi_f(t)| \cdot \int_{-\infty}^{\infty} |t|^\nu \, |\varphi_f(t)| \, dt + 2 \int_{t\in S_n^c} |t|^\nu \, |\varphi_f(t)| \, dt \\
&\quad + h_n \int_{-\infty}^{\infty} |t|^{\nu+1} \, |\varphi_f(t)|^2 dt \cdot \int_{-\infty}^{\infty} |yK(y)| dy.
\end{aligned}$$

From Lemma 3.1

$$\sup_{t\in S_n} |c_n(t) - \varphi_f(t)| = O\left(T_n^{\frac{\mu-m}{2(\mu-m+1)}}\right), \ a.s. \tag{3.7}$$

By the condition (ii), we have

$$\begin{aligned}
\int_{t\in S_n^c} |t|^\nu \, |\varphi_f(t)| \, dt &\leq T_n^{\frac{m-\nu}{2(\mu-m+1)}} \int_{t\in S_n^c} |t|^m \, |\varphi_f(t)| dt \\
&= o\left(T_n^{\frac{\mu-\nu}{2(\mu-m+1)}}\right), \ a.s.
\end{aligned} \tag{3.8}$$

Together with (3.7), (3.8) and the order relation of h_n and T_n, we get (3.6).

Note that the right hand side of (3.6) may be written as $O\left(T_n^{\frac{\mu-m}{2(\mu-m+1)}} + h_n\right)$. We can remark about the condition (i) and (ii) of Lemma 3.2 as follows.

PROPOSITION 3.1. *If* $\varphi_f(t)$ satisfies (2.14) *then the condition* (i) *of Lemma* 3.2 *holds for any nonnegative integer m. And there exists a* $\mu > m$ *such that* (ii) *of Lemma* 3.2 *also holds.*

PROOF. It is easy to check that

$$\begin{aligned}
\int_{-\infty}^{\infty} |t|^{m+1} \, |\varphi_f(t)| \, dt &\leq \frac{2A}{\rho^{m+2}} \int_{-\infty}^{\infty} x^{m+1} e^{-x} dx \\
&= \frac{2A}{\rho^{m+2}} (m+1)! \\
&< \infty.
\end{aligned}$$

Then we shall show the second assertion of this proposition. Let μ_1 be a real positive number such that $\mu_1 > m$. We define the set V_n as

$$V_n = \left\{t : t \in \left[-T_n^{\frac{-1}{2(\mu_1 - m + 1)}},\ T_n^{\frac{-1}{2(\mu_1 - m + 1)}}\right]\right\}.$$

Then we have

$$\begin{aligned}
&\int_{t \in V_n^c} |t|^m \, |\varphi_f(t)| dt \\
&\leq \frac{2A}{\rho^{m+1}} \cdot \exp\left(-\rho \cdot T_n^{\frac{-1}{2(\mu_1 - m+1)}}\right) \cdot \sum_{k=0}^{m} \frac{m!}{k!} \left(\rho \cdot T_n^{\frac{-1}{2(\mu_1 - m+1)}}\right)^k \\
&= o\left(T_n^{\frac{\mu - m}{2(\mu_1 - m+1)}}\right),
\end{aligned}$$

for any $\mu > m$ by the following fact.

$$\lim_{x \to \infty} \frac{x^\mu}{e^x} = 0, \qquad \text{for any } \mu \in \boldsymbol{R}.$$

so if we set $\mu = \mu_1$, we get the conclusion.

LEMMA 3.3. *Under the conditions of Lemma* 3.2, *we have*

$$\left| \frac{d^\nu}{du^\nu} \mathrm{ISE}_n(u) \right|_{u=a} - \lim_{n \to \infty} \frac{d^\nu}{du^\nu} \mathrm{ISE}_n(u) \bigg|_{u=b} \Bigg|$$

$$\leq \frac{|a| + |b|}{\pi} \int_{-\infty}^{\infty} |t|^{\nu+1} \, |\varphi_f(t)|^2 \, dt + \begin{cases} O\left(T_n^{\frac{\mu - m}{2(\mu - m+1)}}\right), & a.s. \\ O(h_n), & a.s. \end{cases}$$

as $n \to \infty$, *according as* $h_n = o\left(T_n^{\frac{\mu - m}{2(\mu - m+1)}}\right)$ *or* $T_n = o\left(h_n^{\frac{2(\mu - m+1)}{\mu - m}}\right)$, *for* $\nu = 0, 1, \ldots, m$.

PROOF. We may use similar approach to the proof of Lemma 3.2.

$$\begin{aligned}
&\pi \left| \frac{d^\nu}{du^\nu} \mathrm{ISE}_n(u) \right|_{u=a} - \lim_{n \to \infty} \frac{d^\nu}{du^\nu} \mathrm{ISE}_n(u) \bigg|_{u=b} \Bigg| \\
&\leq \sup_{t \in S_n} |c_n(t) - \varphi_f(t)| \cdot \int_{-\infty}^{\infty} |t|^\nu \, |\varphi_f(t)| \, dt + 2 \int_{t \in S_n^c} |t|^\nu \, |\varphi_f(t)| \, dt \\
&\quad + h_n \int_{-\infty}^{\infty} |t|^{\nu+1} \, |\varphi_f(t)|^2 \, dt \cdot \int_{-\infty}^{\infty} |yK(y)| \, dy \\
&\quad + \int_{-\infty}^{\infty} |t|^\nu \, |e^{-iat} - e^{-ibt}| \, |\varphi_f(t)|^2 \, dt \\
&\leq (|a| + |b|) \int_{-\infty}^{\infty} |t|^{\nu+1} \, |\varphi_f(t)|^2 \, dt + O\left(T_n^{\frac{\mu - m}{2(\mu - m+1)}}\right), \ a.s.
\end{aligned}$$

if $h_n = o\left(T_n^{\frac{\mu - m}{2(\mu - m+1)}}\right)$, and otherwise is clear. Hence we get the conclusion.
Now we can state our main theorems.

THEOREM 3.1. *If the conditions of Lemma* 3.2 *are satisfied for* $m = 2$, *then we have*

$$a_n = O\left(T_n^{\frac{1}{4}\left(1-\frac{1}{\mu-1}\right)}\right), \ a.s. \ for \ h_n = o\left(T_n^{\frac{\mu-2}{2(\mu-1)}}\right)$$
$$= O(h_n^{1/2}), \qquad a.s. \ for \ T_n = o\left(h_n^{\frac{2(\mu-1)}{\mu-2}}\right) \tag{3.9}$$

for sufficiently large n.

PROOF. We shall only show for the case $h_n = o\left(T_n^{\frac{\mu-2}{2(\mu-1)}}\right)$ as the otherwise can be shown in the same way. By expanding $\frac{d}{du}$ $\mathrm{ISE}_n(u)$ and $\lim_{n\to\infty} \frac{d}{du}$ $\mathrm{ISE}_n(u)$ about $u = 0$, we have

$$\frac{d}{du}\mathrm{ISE}_n(u)\Big|_{u=a} - \lim_{n\to\infty}\frac{d}{du}\mathrm{ISE}_n(u)\Big|_{u=a} - \left[\frac{d}{du}\mathrm{ISE}_n(u)\Big|_{u=0} - \lim_{n\to\infty}\frac{d}{du}\mathrm{ISE}_n(u)\Big|_{u=0}\right]$$
$$= a\cdot\left[\frac{d^2}{du^2}\mathrm{ISE}_n(u)\Big|_{u=\theta a} - \frac{d}{du}\left(\lim_{n\to\infty}\frac{d}{du}\mathrm{ISE}_n(u)\right)\Big|_{u=\theta' a}\right], \tag{3.10}$$

where θ, $\theta' \in (0, 1)$. It is obvious that

$$\frac{d^{\nu-1}}{du^{\nu-1}}\left(\lim_{n\to\infty}\frac{d}{du}\mathrm{ISE}_n(u)\right) = \lim_{n\to\infty}\frac{d^\nu}{du^\nu}\mathrm{ISE}_n(u)$$

for any positive integer $\nu \le m$. And so by Lemma 3.3, we have

$$\frac{d^2}{du^2}\mathrm{ISE}_n(u)\Big|_{u=\theta_n a_n} - \frac{d}{du}\left(\lim_{n\to\infty}\frac{d}{du}\mathrm{ISE}_n(u)\right)\Big|_{u=\theta'_n a_n}$$
$$\le \frac{1}{\pi}\left(|\theta_n| + |\theta'_n|\right)|a_n|\int_{-\infty}^{\infty}|t|^3\,|\varphi_f(t)|^2\,dt + O\left(T_n^{\frac{\mu-2}{2(\mu-1)}}\right)$$
$$= O\left(a_n + T_n^{\frac{\mu-2}{2(\mu-1)}}\right), \ a.s.$$

since θ_n, $\theta'_n \in (0, 1)$. Therefore we can rewrite (3.10) as

$$O\left(T_n^{\frac{\mu-2}{2(\mu-1)}}\right) = a_n\cdot O\left(a_n + T_n^{\frac{\mu-2}{2(\mu-1)}}\right), \ a.s.$$

by Lemma 3.2. Hence we have (3.9).

It is well known that the asymptotically optimal convergence order of the window width h_n is $n^{-1/5}$, i.e. $n^{1/5}h_n \to$ const. as $n \to \infty$, see Prakasa Rao [14]. In this case it is easy to check if $2 < \mu \le \frac{8}{3}$ then $h_n = o\left(T_n^{\frac{\mu-2}{2(\mu-1)}}\right)$, and if $\mu \ge \frac{8}{3} + 0.1$ then we have $T_n = o\left(h_n^{\frac{2(\mu-1)}{\mu-2}}\right)$.

THEOREM 3.2. *Under the conditions of Lemma 3.2 for $m = 2$, we have*

$$\mathrm{DISE}_n(a_n) = O\left(T_n^{\frac{1}{4}\left(1-\frac{1}{\mu-1}\right)}\right), \ a.s. \ for \ h_n = o\left(T_n^{\frac{\mu-2}{2(\mu-1)}}\right)$$
$$= O(h_n^{1/2}), \qquad a.s. \ for \ T_n = o\left(h_n^{\frac{2(\mu-1)}{\mu-2}}\right) \tag{3.11}$$

for sufficiently large n.

PROOF. We shall also show for $h_n = o\left(T_n^{\frac{u-2}{2(u-1)}}\right)$ case. Since from the proof of Lemma 3.2

$$\frac{1}{\pi}\left|\pi\cdot \mathrm{DISE}_n(a) - \int_{-\infty}^{\infty}\{1 - \cos(at)\}\ |\varphi_f(t)|^2 dt\right|$$
$$\leq 2\sup_{t\in S_n}|c_n(t) - \varphi_f(t)|\cdot\int_{-\infty}^{\infty}|\varphi_f(t)|\ dt + 4\int_{t\in S_n^c}|\varphi_f(t)|dt$$
$$+ 2h_n\int_{-\infty}^{\infty}|t|\ |\varphi_f(t)|^2\ dt\cdot\int_{-\infty}^{\infty}|yK(y)|\ dy.$$
$$= O\left(T_n^{\frac{u-2}{2(u-1)}}\right),\ a.s.$$

therefore we have (3.12) which is more accurate than (2.16)

$$\mathrm{DISE}_n(a) = \frac{1}{\pi}\int_{-\infty}^{\infty}\{1 - \cos(at)\}\ |\varphi_f(t)|^2\ dt + O\left(T_n^{\frac{u-2}{2(u-1)}}\right),\ a.s. \tag{3.12}$$

uniformly for $a \in \boldsymbol{R}$. By Theorem 3.1

$$\int_{-\infty}^{\infty}\{1 - \cos(a_n t)\}\ |\varphi_f(t)|^2\ dt \leq |a_n|\int_{-\infty}^{\infty}|t|\ |\varphi_f(t)|^2 dt$$
$$= O\left(T_n^{\frac{1}{4}\left(1-\frac{1}{u-1}\right)}\right),\ a.s.$$

Thus we get the conclusion.

Let $\hat{a}_n$ be any estimator of the a_n then by the definition of the a_n, we have

$$\mathrm{DISE}_n(a_n) \leq \mathrm{DISE}_n(\hat{a}_n),\quad a.s.$$

for all $n \in \boldsymbol{N}$. And also we have

$$\mathrm{DISE}_n(a_n) \leq 0,\ a.s.$$

for all $n \in \boldsymbol{N}$. We define the estimator $\hat{a}_n$ is *asymptotically efficient* if $n \geq N$ then $\mathrm{DISE}_n(\hat{a}_n) \leq 0$ for some $N > 0$, with probability one. The asymptotic property of $\mathrm{ISE}_n(0)$ has been studied by Hall [8], it is obvious that the order of convergence of $\mathrm{ISE}_n(a_n)$ is the same as $\mathrm{ISE}_n(0)$. So we can say $\mathrm{DISE}_n(\hat{a}_n)$ is desired that it converges to 0 as slower as could. Theorem 3.2 gives the lower bound of the convergence discussed above. Namely, if the estimator $\hat{a}_n$ is *asymptotically efficient* then $\mathrm{DISE}_n(\hat{a}_n)$ can not converge to 0 slower than the order $O\left(T_n^{\frac{1}{4}\left(1-\frac{1}{u-1}\right)}\right)$ *or* $O(h_n^{1/2})$ with probability one.

4. Simulation Study

In this section we shall show the examples for the following case.

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$$
$$K(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)$$

We have,

$$\mathrm{ISE}_n(a, h_n) = \frac{1}{2\sqrt{\pi}nh_n}\left[1 + \frac{2}{n}\sum_{j<l}^{n}\exp\left\{-\frac{1}{4}\left(\frac{X_j - X_l}{h_n}\right)^2\right\}\right]$$
$$-\frac{1}{n}\left(\frac{2}{\pi(1 + h_n^2)}\right)^{\frac{1}{2}}\cdot\sum_{j=1}^{n}\exp\left\{-\frac{(X_j - a)^2}{2(1 + h_n^2)}\right\} + \frac{1}{2\sqrt{\pi}}.$$

We define $\mathrm{DISE}_n(a, h_n^*)$ as

$$\mathrm{DISE}_n(a, h_n^*) = \mathrm{ISE}_n(a, h_n^*) - \mathrm{ISE}_n(0, h_n^*),$$

where h_n^* is given by

$$\mathrm{ISE}_n(0, h_n^*) = \min_{0<h_n<\infty} \mathrm{ISE}_n(0, h_n).$$

Secondly, we conduct a simulation to estimate

$$E\left[\frac{\min_{-\infty<a<\infty} \mathrm{ISE}_n(a, h_n^*)}{\mathrm{ISE}_n(0, h_n^*)}\right], \tag{4.1}$$

by calculating

$$\frac{1}{N}\sum_{s=1}^{N}\frac{\min_{-\infty<a<\infty} \mathrm{ISE}_n(a, h_n^*)}{\mathrm{ISE}_n(0, h_n^*)} \tag{4.2}$$

for sample size $n = 50$ ($N = 100$) and $n = 100$ ($N = 50$). (4.1) may be defined as a relative efficiency of the estimation with respect to the location shift r.v. a_n. As a result we have 58% for $n = 50$, and 62% for $n = 100$ respectively. These percentages motivate us to construct an estimator of the a_n.

The author has been considering the estimator of a_n. For example, it may be worth while to study the following two types of the estimators.

I. From (2.19) we may construct a natural estimator,

$$\hat{a}_n = \frac{-\sum_{j=1}^{n}\widehat{\Psi}_n(X_j)}{\sum_{j=1}^{n}\frac{d}{du}\widehat{\Psi}_n(x_j - u)\Big|_{u=0}}.$$

where $\widehat{\Psi}_n(X)$ is given by

$$\widehat{\Psi}_n(X) = \mathrm{Im}\left[\int_{-\infty}^{\infty} te^{iXt}\cdot\varphi_K(h_n t)\cdot\overline{\widehat{\varphi}_f(t)}dt\right],$$

and $\widehat{\varphi}_f(t)$ is an estimator of the characteristic function $\widehat{\varphi}_f(t)$.

II. The corss-validation method. Define the equation as

$$CV(h, a) = \int_{-\infty}^{\infty} f_n^2(x)dx - \frac{2}{n}\sum_{j=1}^{n} f_{n-1,-j}(X_j - a),$$

where

$$f_{n-1,-j}(x) = \frac{1}{(n - 1)h_n}\sum_{i\neq j}^{n} K\left(\frac{x - X_i}{h_n}\right).$$

And find the $(\widehat{h}_n, \hat{a}_n)$ such that

$$CV(\widehat{h}_n, \hat{a}_n) = \min_{-\infty<a<\infty} \min_{h>0} CV(h, a).$$

The cross-validation method to find the asymptotically optimal window width h_n has been studied by many authors, see, for example Bowman [3] and Rudemo [15].

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References

[1] Billingsley, P.: *Convergence of Probability Measures*, Wiley, New York, (1968).

[2] Blackman, J.: *On the approximation of a distribution function by an empiric distribution*, Ann. Math. Statist. **26**, (1955), 256–267.

[3] Bowman, A.: *An alternative method of cross-validation for the smoothing of density estimates*, Biometrika **71**, (1984), 353–360.

[4] Chung, K. L.: *An estimate concerning the kolmogoroff limit distribution*, Trans. Amer. Math. Soc. **67**, (1949), 36–56.

[5] Csörgö, S.: *Limit behaviour of the empirical characteristic function*, Ann. Prob. **9**, (1981), 130–140.

[6] Feuerverger, A. and Mureika, R. A.: *The empirical characteristic function and its applications*, Ann. Statist. **5**, (1977), 88–97.

[7] Hall, P.: *Laws of iterated logarithm for nonparametric density estimators*, Zeit. Wahrscheinl.-theorie **56**, (1981), 47–61.

[8] Hall, P.: *Limit theorems for stochastic measures of the accuracy of density estimators*, Stoch. Processes Appl. **13**, (1982), 11–25.

[9] Härdle, W.: *Applied Nonparametric Regression*, Cambridge Univ. Press, (1990).

[10] Heathcote, C. R.: *The integrated squared error estimation of parameters*, Biometrika **64**, (1977), 255–264.

[11] Huber, P. J.: *Robust estimation of a location shift parameter*, Ann. Math. Statist. **35**, (1964), 73–101.

[12] Izenman, A. J.: *Recent developments in nonparametric density estimation*, J. Amer. Statist. Ass. **86**, (1991), 205–224.

[13] Lukacs, E.: *Characteristic Functions*, 2nd. ed. Griffin, London, (1970).

[14] Marcus, M. B.: *Weak convergence of the empirical characteristic function*, Ann. Prob. **9**, (1981), 194–201.

[15] Prakasa Rao, B. L. S.: *Nonparametric Fuctional Estimation*, Academic Press. (1983).

[16] Rudemo, M.: *Empirical choice of histograms and kernel density estimators*, Scand. J. Statist. **9**, (1982), 65–78.

[17] Scott, D. W.: *Averaged shifted histograms: effective nonparametric density estimators in several dimensions*, Ann. Statist. **13**, (1985), 1024–1040.

[18] Serfling, R. J.: *Approximation Theorems of Mathematical Statistics*, Wiley, New York, (1980)

[19] Watson, G. R. and Leadbetter, M. R.: *On the estimation of the probability density, I*, Ann. Math. Statist. 34, (1963), 480–491.

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