UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS OF VARIANCES AND COVARIANCES OF MULTIVARIATE NORMAL DISTRIBUTIONS WITH CYCLIC COVARIANCE MATRICES

Yamato, Hajime
Department of Mathematics, Faculty of Science, Kagoshima University

Kondo, Masao
Department of Statistics, College of Liberal and Arts, Kagoshima University

https://doi.org/10.5109/13431

出版情報：Bulletin of informatics and cybernetics. 25 (3/4), pp.189-194, 1993-03. Research Association of Statistical Sciences

Bulletin of Informatics and Cybernetics, Vol. 25, No. 3~4, 1993

UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS OF VARIANCES AND COVARIANCES OF MULTIVARIATE NORMAL DISTRIBUTIONS WITH CYCLIC COVARIANCE MATRICES

By

Hajime YAMATO* and Masao KONDO**

Abstract

For a multivariate normal distribution having a cyclic covariance matrix and equal means, we give uniformly minimum variance unbiased estimators of a common variance and covariances.

1. Introduction and Summary

A signal is transmitted from a point source located at the geocenter of a regular polygon of d sides. Let $V_1, \ldots, V_d$ be vertices of the polygon and taken sequentially. The signal received at the vertex V_i is denoted by X_i $(i = 1, \ldots, d)$. Then it may be reasonable to assume that $\mathrm{Var}(X_1) = \ldots = \mathrm{Var}(X_d)$ and $\mathrm{Cov}(X_j, X_{j+1})$ does not depend on $j (j = 1, \ldots, d)$, where $X_{d+1} = X_1$; $\mathrm{Cov}(X_j, X_{j+2})$ does not depend on $j (j = 1, \ldots, d)$, where $X_{d+2} = X_2$ and so on. In other words, for a covariance matrix $\Sigma = (\sigma_{ij})$ of $(X_1, \ldots X_d)$, its elements can be written as $\sigma_{ii} = \sigma^2$ (>0) $(i = 1, \ldots, d)$ and $\sigma_{ij} = \theta_k$ $(i \neq j,\ i, j = 1, \ldots, d$ and $k = \min(|i - j|, d - |i - j|))$. A covariance matrix having this structure is said to be cyclic.

Equivalently, a $d \times d$ covariance matrix Σ is cyclic if $g\Sigma g' = \Sigma$ for all $g \in G_0 = \{I_d, C, C^2, \ldots, C^{d-1}\}$ where I_d is the $d \times d$ identity matrix, and $C = (c_{ij})$ is a $d \times d$ matrix with $c_{d1} = c_{j,j+1} = 1$ $(j = 1, \ldots, d - 1)$ and the remaining elements equal to zero. For a vector $\mathbf{x}' = (x_1, \ldots, x_d)$, $(\mathbf{Cx})' = (x_2, \ldots, x_d, x_1)$. (See for example Eaton [1] and Olkin and Press [3]).)

For a multivariate normal distribution with a cyclic covariance matrix it is generally difficult to express the density function explicitly with σ^2 and θ_k.

By transforming a sample, Olkin and Press [3] gives maximum likelihood estimators and likelihood tests for the above multivariate normal distribution.

The multivariate normal distribution with cyclic covariance matrix and equal means is equivalent to multivariate normal distribution invariant under the group of transformations $G_0 = \{I_d, C, C^2, \ldots, C^{d-1}\}$. Noting this invariance of distributions,

* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima 890, Japan
** Department of Statistics, College of Liberal Arts, Kagoshima University, Kagoshima 890, Japan

we present uniformly minimum variance unbiased (UMVU) estimators of means, variance and covariances by the method stated in Yamato [4] and Yamato and Maesono [5].

Let $(X_{1p}, \ldots, X_{dp})$, $p = 1, \ldots, n$ be a sample from a nonsingular d-variate normal distribution with cyclic covariance matrix $\Sigma = (\sigma_{ij})$ and equal means μ, where $\sigma_{ii} = \sigma^2\ (>0)$ $(i = 1, \ldots, d)$ and $\sigma_{ij} = \theta_k$ $(i \neq j, i, j = 1, \ldots, d$ and $k = \min(|i-j|, d - |i-j|))$.

The UMVU estimators of a common mean μ, variance σ^2 and covariances θ_k are presented by

$$\widehat{\mu} = \bar{X}\left(\bar{X} = \sum_{i,p} X_{ip}/(nd)\right),$$

$$\widehat{\sigma^2} = \sum_{i,j=1}^{d} \sum_{p \neq q} (X_{ip} - X_{jp})^2/[2n(n-1)d^2]$$

$$\widehat{\theta}_k = \sum_{i,j} \sum_{p \neq q} (X_{ip} - X_{jq})(X_{i+k,p} - X_{j+k,q})/[2n(n-1)d^2]$$

(for $k = 1, \ldots, r$ with $d = 2r + 1$ or $k = 1, \ldots, r$ with $d = 2r$), where $X_{d+i,p} = X_{ip}$ for $i = 1, \ldots, d$ and $p = 1, \ldots, n$. These are shown in Section 2.

In Section 3, by the method different from Section 2 we make sure that the estimators obtained in Section 2 are UMVU estimators of mean, variance and covariances.

2. UMVU Estimators

Since the multivariate normal distribution with cyclic covariance matrix and equal means is invariant under the group of transformations $G_0 = \{I_d, C, C^2, \ldots, C^{d-1}\}$, we derive the UMVU estimators by the method stated in Yamato [4] and Yamato and Maesono [5] which is quoted below.

LEMMA 2.1. $\boldsymbol{P} = \{P_\eta, \eta \in \Omega\}$ denotes a family of some distributions on the d-dimensional Euclidean space. $\boldsymbol{P}^* = \{P_\eta, \eta = \Omega^*\}$ denotes its subfamily whose distribution is invariant under a finite group of measurable transformations $G = \{g_1, \ldots, g_k\}$. Let $X_1, \ldots, X_n$ be a sample of size n from $\mathbf{P} \in \boldsymbol{P}$ and $T(X_1, \ldots, X_n)$ be a UMVU estimator of its expectation $E(T)$ for $\boldsymbol{P}$. Then the G-invariant version of T,

$$T^*(X_1, \ldots, X_n) = \frac{1}{k^n} \sum_{j_1, \ldots, j_n = 1}^{k} T(g_{j_1}X_1, \ldots, g_{j_n}X_n)$$

is a UMVU estimator of $E(T^*)\ (= E(T))$ for $\boldsymbol{P}^*$.

PROPOSITION 2.1. For a multivariate normal distribution with unknown cyclic covariance matrix and unknown but equal means, UMVU estimators of mean μ, variance σ^2 and covariances $\theta_1, \ldots, \theta_r$ are presented by

$$\widehat{\mu} = \bar{X}\left(\bar{X} = \sum_{i,p} X_{ip}/(nd) = \sum_{i=1}^{d} X_{i}\cdot/d\right), \tag{2.1}$$

$$\widehat{\sigma}^2 = \sum_{i,j=1}^{d} \sum_{p\neq q} (X_{ip} - X_{jp})^2/[2n(n-1)d^2] \tag{2.2}$$

$$\widehat{\theta}_k = \sum_{i,j}\sum_{p\neq q} (X_{ip} - X_{jq})\,(X_{i+k,p} - X_{j+k,q})/[2n(n-1)d^2] \tag{2.3}$$

(for $k = 1, \ldots, r$ with $d = 2r + 1$ or $k = 1, \ldots, r$ with $d = 2r$)

where $X_{d+i,p} = X_{ip}$ for $i = 1, \ldots, d$ and $p = 1, \ldots, n$.

PROOF. Let $\mathbf{X}_p = (X_{1p}, \ldots, X_{dp})$, $p = 1, \ldots, n$ be a sample from a nonsingular d-variate normal distribution with $E(X_{ip}) = \mu_i$ and $\mathrm{Cov}(X_{ip}, X_{jp}) = \sigma_{ij}$ ($i, j = 1, \ldots, d$, $p = 1, \ldots, n$). Then UMVU estimators of μ_i and σ_{ij} are given by

$$\widehat{\mu}_i = X_{i}.\ (i = 1, \ldots, d) \tag{2.4}$$

$$\widehat{\sigma}_{ij} = S_{ij}/(n-1) \quad (i, j = 1, \ldots, d) \tag{2.5}$$

where $X_{i}. = \sum_{p=1}^{n} X_{ip}/n$ and $S_{ij} = \sum_{p=1}^{n} (X_{ip} - X_{i}.)\,(X_{jp} - X_{j}.)$ for $i, j = 1, \ldots, d$ (see for example Lehmann [2]).

We make an UMVU estimator $\widehat{\mu}$ invariant under the group of transformations $G_0 = \{I_d, C, C^2, \ldots, C^{d-1}\}$ from the UMVU estimator $\widehat{\mu}_1$ given by (2.4) with $i = 1$, using Lemm 2.1.

Since we can write $\widehat{\mu}_1 = \sum_{p=1}^{n} u(\mathbf{X}_p)/n$ with $u(\mathbf{X}_p) = X_{1p}$, we have

$$\widehat{\mu} = \frac{1}{n}\sum_{p=1}^{n}\frac{1}{d}\sum_{i=0}^{d-1} u(C^i\mathbf{X}_p) = \bar{X},$$

where $\bar{X} = \Sigma_{j,p}X_{jp}/(nd)$.

We make an UMVU estimator $\widehat{\sigma}^2$ invariant under G_0 from the UMVU estimator $\widehat{\sigma}_{11}$ given by (2.5) with $i = j = 1$, using Lemma 2.1.

Since we can write $\widehat{\sigma}_{11} = \sum_{p\neq q} v(\mathbf{X}_p, \mathbf{X}_q)/[2n(n-1)]$ with $v(\mathbf{X}_p, \mathbf{X}_q) = (X_{1p} - X_{1q})^2$, we have

$$\widehat{\sigma}^2 = \frac{1}{2n(n-1)}\sum_{p\neq q}\frac{1}{d^2}\sum_{i,j=0}^{d-1} v(C^i\mathbf{X}_p, C^j\mathbf{X}_q)$$
$$= \frac{1}{2n(n-1)d^2}\sum_{p\neq q}\sum_{i,j}(X_{ip} - X_{jq})^2.$$

We make an UMVU estimator $\widehat{\theta}_k$ invariant under the group of transformations G_0 from the UMVU estimator $\widehat{\sigma}_{1,1+k}$ given by (2.5) with $i = 1$ and $j = 1 + k$ for $k = 1, \ldots, r$ with $d = 2r + 1$ or $d = 2r$.

Since we can write $\widehat{\sigma}_{1,1+k} = \sum_{p\neq q} w(\mathbf{X}_p, \mathbf{X}_q)/[2n(n-1)]$ with $w(\mathbf{X}_p, \mathbf{X}_q) = (X_{1p} - X_{1q})\,(X_{1+k,p} - X_{1+k,q})$ ($k = 1, \ldots, r$ with $d = 2r + 1$ or $d = 2r$) we have

$$\widehat{\theta}_k = \frac{1}{2n(n-1)} \sum_{p \neq q} \frac{1}{d^2} \sum_{i,j=0}^{d-1} w(C^i \mathbf{X}_p, C^j \mathbf{X}_q)$$
$$= \frac{1}{2n(n-1)d^2} \sum_{p \neq q} \sum_{i,j} (X_{ip} - X_{jq})(X_{i+k,p} - X_{j+k,q}),$$

where $X_{d+j,p} = X_{jp}$ ($j = 1, \ldots, d$, $p = 1, \ldots, n$). Thus we get (2.1), (2.2) and (2.3).

3. Normal Distribution with Cyclic Covariance Matrix

In Section 1 we gave the definition that a covariance matrix is cyclic. Its inverse is also cyclic as follows.

LEMMA 3.1. If a covariance matrix Σ is cyclic, then its inverse Σ^{-1} is also a cyclic covariance matrix.

PROOF. Since Σ is positive definite, Σ^{-1} is also positive definite. Because of $C^d = I_d$ and $C'C = I_d$, we have $(C^{d-j})^{-1} = C^j$ and $(C^j)^{-1} = (C^j)'(j = 1, \ldots, d-1)$.

From the assumption we have $C^j \Sigma C^{d-j} = \Sigma$ ($j = 1, \ldots, d-1$) and therefore

$$(C^{d-j})^{-1}\Sigma^{-1}(C^j)^{-1} = \Sigma^{-1} \quad \text{for } j = 1, \ldots, d-1.$$

This is equivalent to

$$C^j\Sigma^{-1}(C^j)' = \Sigma^{-1} \quad \text{for } j = 1, \ldots, d-1,$$

which shows that Σ^{-1} is cyclic.

We put $u_{kj} = d^{-1/2}\{\cos\xi_{kj} + \sin\xi_{kj}\}$ with $\xi_{kj} = 2\pi(j-1)(k-1)/d$ for $k, j = 1, \ldots, d$ and $\mathbf{u}_k' = (u_{k1}, \ldots, u_{kd})$ for $k = 1, \ldots, d$. Let Γ' be the matrix $(u_1, \ldots, u_d)$.

LEMMA 3.2. (Eaton [1]). Γ is a $d \times d$ symmetric orthogonal matrix. For any cyclic covariance matrix Σ we have

$$\Gamma\Sigma\Gamma = \Lambda,$$

where Λ is a diagonal matrix. For $d = 2r + 1$, the diagonal elements of Λ are

$$\lambda_k = \alpha_k \text{ for } k = 1, \ldots, r+1; \ \lambda_{d-k+2} = \alpha_k \text{ for } k = 2, \ldots, r+1,$$

and for $d = 2r$, the diagonal elements of Λ are

$$\lambda_k = \alpha_k \text{ for } k = 1, \ldots, r+1; \ \lambda_{d-k+2} = \alpha_k \text{ for } k = 2, \ldots, r,$$

where $\alpha_1, \ldots, \alpha_{r+1}$ denotes the eigenvalues of Σ and are positive.

LEMMA 3.3. For any cyclic covariance matrix Σ, we denote the elements of its inverse $\Sigma^{-1} = (\sigma^{ij})$ by $\sigma^{ii} = \tau_0$ ($i = 1, \ldots, d$) and $\sigma^{ij} = \tau_k(i \neq j, i, j = 1, \ldots, d$ and $k = \min(|i-j|, d-|i-j|)$. Then the range of $(\tau_0, \ldots, \tau_r)$ contains a $(r+1)$-dimensional rectangle.

PROOF. Since Σ^{-1} is cyclic by Lemma 3.1, its elements (σ^{ij}) can be described as $\sigma^{ii} = \tau_0$ ($i = 1, \ldots, d$) and $\sigma^{ij} = \tau_k$ ($i \neq j, i, j = 1, \ldots, d$ and $k = \min(|i-j|, d-|i-j|)$.

From Lemma 3.2 we have $\Gamma\Sigma^{-1}\Gamma = \Lambda^{-1}$ and $\Sigma^{-1} = \Gamma\Lambda^{-1}\Gamma$. Therefore there exists the one to one and linear correspondence between $\tau = (\tau_0, \ldots, \tau_r)'$ and $\beta = (\beta_1, \ldots, \beta_{r+1})'$, where $\beta_j = 1/\alpha_j$ ($j = 1, \ldots, r+1$). We denote this correspondence by $\tau = A\beta$ with the nonsingular matrix A. Since $\beta_1, \ldots, \beta_{r+1} > 0$, the range of $(\tau_0, \ldots, \tau_r)$ is the image of $\beta_1, \ldots, \beta_{r+1} > 0$ by the nonsingular transformation A and contains a $(r + 1)$-dimensional rectangle.

Proposition 3.1. For a multivariate normal distribution with cyclic covariance matrix and equal means, a sufficient and complete statistic is given by

$$\left\{\sum_{i=1}^{d}\sum_{p=1}^{n} X_{ip},\ \sum_{i=1}^{d}\sum_{p=1}^{n} X_{ip}^2,\ \sum_{|i-j|=k,d-k}\ \sum_{p=1}^{n} X_{ip}X_{jp}\ (k = 1, \ldots, r)\right\} \tag{3.1}$$

for $d = 2r + 1$ or $d = 2r$.

Proof. Let $\mathbf{X}_p = (X_{1p}, \ldots, X_{dp})$, $p = 1, \ldots, n$ be a sample from a nonsingular d-variate normal distribution with $\mathrm{E}(X_{ip}) = \mu$ ($i = 1, \ldots, d$) and a cyclic covariance matrix Σ. Since its inverse is also cyclic as stated in Lemma 2.1, the elements of the inverse matrix Σ^{-1} can be written as $\sigma^{ii} = \tau_0$ ($i = 1, \ldots, d$) and $\sigma^{ij} = \tau_k$ [$i \neq j$, $i, j = 1, \ldots, d$ and $k = \min(|i - j|, d - |i - j|)$). We put $v = \mu[\tau_0 + 2(\tau_1 + \ldots + \tau_r)]$ for $d = 2r + 1$ and $v = \mu[\tau_0 + 2(\tau_1 + \ldots + \tau_{r-1}) + \tau_r]$ for $d = 2r$.

The joint density function of $\mathbf{X}_1, \ldots, \mathbf{X}_n$ forms a $(r + 2)$-parameter exponential family given by $f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = c\exp(\eta_0 S_0 + \ldots + \eta_r S_r - 2v\Sigma_{i,p}x_{ip})$, where c is a constant depending upon parameters,

$$\eta_i = -\tau_i/2\ (i = 0, \ldots, r) \text{ and } S_0 = \sum_{i=1}^{d}\sum_{p=1}^{n} x_{ip}^2,$$

$$S_k = \sum_{|i-j|=k,d-k}\ \sum_{p=1}^{n} x_{ip}x_{jp}\ (k = 1, \ldots, r \text{ with } d = 2r + 1 \text{ or } d = 2r).$$

The range of $(\tau_0, \ldots, \tau_r)$ contains a $(r + 1)$-dimensional rectangle by Lemma 3.3 and therefore $(\eta_0, \ldots, \eta_r)$ contains a $(r + 1)$-dimensional rectangle. Thus a sufficient and complete statistic is $(\Sigma_{i,p}x_{ip}, S_0, \ldots, S_r)$ (see for example Lehmann [2]).

Proposition 3.2. For a multivariate normal distribution with a cyclic covariance matrix and equal means, the UMVU estimators of mean μ, variance σ^2 and covariances $\theta_1, \ldots, \theta_r$ given by (2.1), (2.2) and (2.3), respectively, are unbiased functions of the sufficient and complete statistic presented by (3.1).

Proof. It is easy to see that the estimators given by (2.1), (2.2) and (2.3) are unbiased. The estimator given by (2.1) is obviously a function of the sufficient and complete statistic given by (3.1).

Using the relation:

$$\sum_{i,j}\sum_{p} X_{ip}X_{jp} = \sum_{m=1}^{r}\ \sum_{|i-j|=m,d-m}\ \sum_{p} X_{ip}X_{jp} + \sum_{i,p} X_{ip}^2$$

we have

$$2n(n-1)d^2\widehat{\sigma}^2 = \sum_{p,q}\sum_{i,j}(X_{ip}-X_{jq})^2 - \sum_{p}\sum_{i\neq j}(X_{ip}-X_{jp})^2$$
$$= 2(nd-d+1)\sum_{i,p}X_{ip}^2 - 2(nd\bar{X})^2 + 2\sum_{m=1}^{r}\sum_{|i-j|=m,d-m}\sum_{p}X_{ip}X_{jp},$$

where the summation $\sum_{|i-j|=r,d-r}$ is taken as $\sum_{|i-j|=r}$ for $d = 2r$.

Thus $\widehat{\sigma}^2$ can be written as a function of the sufficient and complete statistic given by (3.1):

$$\widehat{\sigma}^2 = \left\{[(n-1)d+1]\sum_{i,p}X_{ip}^2 - (nd\bar{X})^2 + \sum_{m=1}^{r}\sum_{|i-j|=m,d-m}\sum_{p}X_{ip}X_{jp}\right\}\Big/[n(n-1)d^2].$$

Since $2\sum_{i=1}^{d}X_{ip}X_{i+k,p} = \sum_{|i-j|=k,d-k}X_{ip}X_{i+k,p}$ for $k = 1, \ldots, r$ with $d = 2r+1$ or $k = 1, \ldots, r-1$ with $d = 2r$, by the similar computation to the above we have for $k = 1, \ldots, r$ with $d = 2r+1$ or $k = 1, \ldots, r-1$ with $d = 2r$

$$\widehat{\theta}_k = (2nd)^{-1}\sum_{|i-j|=k,d-k}\sum_{p}X_{ip}X_{jp}$$
$$-[n(n-1)d^2]^{-1}\left[(nd\bar{X})^2 - \sum_{m=1}^{r}\sum_{|i-j|=m,d-m}\sum_{p}X_{ip}X_{jp} - \sum_{i,p}X_{ip}^2\right],$$

which is a function of the statistic given by (3.1).

Since $\sum_{i=1}^{d}X_{ip}X_{i+r,p} = \sum_{|i-j|=r}X_{ip}X_{i+r,p}$ $(d = 2r)$, we have for $d = 2r$

$$\widehat{\theta}_r = (nd)^{-1}\sum_{|i-j|=r}\sum_{p}X_{ip}X_{jp}$$
$$-[n(n-1)d^2]^{-1}\left[(nd\bar{X})^2 - \sum_{m=1}^{r}\sum_{|i-j|=m,d-m}\sum_{p}X_{ip}X_{jp} - \sum_{i,p}X_{ip}^2\right],$$

which is a function of the statistic given by (3.1).

References

[1] EATON, M. L.: *Multivariate Statistics*, Wiley & Sons Inc., New York, (1983).

[2] LEHMANN, E. L.: *Theory of Point Estimation*, Wiley & Sons Inc., New York, (1983).

[3] OLKIN, I. and PRESS, S. J.: *Testing and estimation for a circular stationary model*, Ann. Math. Statist., **40** (1969), 1358–1373.

[4] YAMATO, H.: *Uniformly minimum variance unbiased estimation for symmetric normal distributions*, J. Multivariate Anal., **34** (1990), 227–237.

[5] YAMATO, H. and MAESONO, Y.: *Estimation under invariant distributions*. Jour. Statist. Plan. Infer, 22, 55–61.

Received August 18, 1992