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BAHADUR REPRESENTATION OF SAMPLE CONDITIONAL QUANTILES BASED ON SMOOTHED CONDITIONAL EMPIRICAL DISTRIBUTION FUNCTION*

By

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Abstract

Let $\{(X_i, Y_i): i = 1, 2, \ldots\}$ be a sequence of stationary independent random vectors in $\mathcal{R}^{(2)}$ with a continuous distribution, and let $G_x(\cdot)$ denote the conditional distribution function of Y_1 given $X_1 = x$. In this paper, Bahadur's almost sure representation for the sample conditional quantile $\widehat{G}_{nx}^{-1}(\lambda)$, $0 < \lambda < 1$, is established, where $\widehat{G}_{nx}$ is a smoothed (rank nearest neighbor or the Nadaraya-Watson type) estimator of G_x. Such representations are useful in the study of asymptotics of functionals of conditional quantiles.

1. Introduction

Let $\{(X_i, Y_i), i = 1, 2, \ldots\}$ be a sequence of independent identically distributed random vectors with common continuous distribution function (d.f.) H and marginal d.f.'s F and G, respectively. Further, let G_x denote the conditional d.f. of Y given $X = x \in \mathcal{R} = \mathcal{R}^{(1)}$ = (real line) and, for each λ, $0 < \lambda < 1$, let $q_x(\lambda) = G_x^{-1}(\lambda) = \inf\{y \in \mathcal{R} : G_x(y) \geq \lambda\}$, the λth quantile of G_x. Consider the smoothed conditional empirical d.f. defined by

$$\begin{aligned}\widehat{G}_{nx}(y) &= (na_n)^{-1}(t_n(x))^{-1} \sum_{i=1}^{n} W_n(a_n^{-1}(F_n(x) - F_n(X_i)), y - Y_i) \qquad (1.1)\\ &= (a_n t_n(x))^{-1} \iint W_n((F_n(x) - F_n(u))/a_n, y - v) dH_n(u, v),\end{aligned}$$

$-\infty < y < \infty$, where $H_n(x, y) = n^{-1} \Sigma_{i=1}^{n} I_{[X_i \leq x, Y_i \leq y]}$, $F_n(x) = n^{-1} \Sigma_{i=1}^{n} I_{[X_i \leq x]}$, $\{W_n\}$ is a "*Heaviside*" sequence as defined in Section 3 of Mehra, Rama and Rao [5] (hereafter abbreviated to MRR[5]) and $\{a_n\}$ is a bandwidth sequence ($a_n \downarrow 0$ but $na_n \to \infty$, as $n \to \infty$). In this paper we shall establish a pointwise Bahadur [1] type almost sure representation for the sample λ^{th} conditional quantile

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$$\widehat{G}_{nx}^{-1}(\lambda) = \inf\{y : \widehat{G}_{nx}(y) \geq \lambda\},\ 0 < \lambda < 1, \tag{1.2}$$

corresponding to the "*smoothed*" conditional empirical distribution function (s.c.e.d.f.) $\widehat{G}_{nx}$ defined by (1.1) and its Nadarya-Watson counterpart $\widehat{G}_{nx}^{*-1}(\lambda)$, $0 < \lambda < 1$ (see MRR [5] Section 1).

For unconditional unsmoothed sample quantile $G_n^{-1}(\lambda)$, $0 < \lambda < 1$, corresponding to the e.d.f. $G_n(y)$, Bahadur [1] showed that if $g(G^{-1}(\lambda)) > 0$ and g' exists and is bounded in a neighbourhood of $G^{-1}(\lambda)$, then as $n \to \infty$, for each fixed λ, $0 < \lambda < 1$, with probability one

$$\begin{aligned} G_n^{-1}(\lambda) - G^{-1}(\lambda) &= -[g(G^{-1}(\lambda))]^{-1}[G_n(G^{-1}(\lambda)) - \lambda)] \\ &\quad + O(n^{-\frac{3}{4}}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}}) \end{aligned} \tag{1.3}$$

(see also Kiefer [3]). Mack [4] extended the above a.s. representation result to sample quantiles $\widehat{G}_n^{-1}(\lambda)$ based on smoothed e.d.f. $\widehat{G}_n$(see [4] Theorem pp. 187–188), but with the order term $O(n^{-\frac{3}{4}}(\log n)^{\frac{3}{4}})$ converging to zero at a slightly slower rate as $n \to \infty$. In order to establish a similar a.s. representation for the conditional quantile (1.2) (Theorem 3.1. below), we need, in addition to Theorem 3.1 of MRR[5], some results regarding the oscillations of the $\widehat{G}_{nx}$. These results are given in Section 2. The main result along with some remarks are given in Section 3. In order to avoid repetition, we employ the same notation and assumptions as used in MRR[5] and refer to the arguments therein whenever they help to shorten the proofs. Unlike as in MRR[5], the results of this paper are established explicitly only for the case $m = 1$; the general case $m > 1$ can be dealt with using similar arguments and the results of MRR[5] for the case $m > 1$.

2. Preliminary Results

In this section, we shall prove two lemmas which are essential for the proof of the main result Theorem 3.1: Lemma 2.1 gives the a.s. rate of convergence of the smoothed conditional quantile $\widehat{G}_{nx}^{-1}(\lambda)$ to $G_x^{-1}(\lambda)$ and Lemma 2.2 pertains to the oscillation behaviour of the conditional empirical process. $\widehat{G}_{nx}(\lambda)$, $0 < \lambda < 1$. Let $\Lambda\ (F)$ denote the support of F.

LEMMA 2.1. *Suppose that, in addition to the assumptions of Theorem 3.1(a) of MRR[5], $b_n^2 = o(\tau_n)$, where $\{b_n\}$ is defined as in assumption A.III(iv) of MRR[5]. Then for each λ, $0 < \lambda < 1$ and each fixed $x \in \Lambda\ (F)$,*

$$[\widehat{G}_{nx}^{-1}(\lambda) - G_x^{-1}(\lambda)] = O(\tau_n) \tag{2.1}$$

with probability one, as $n \to \infty$, where $\widehat{G}_{nx}^{-1}(\lambda)$ is defined by (1.2) and $\tau_n = n^{-\frac{1}{2}}a_n^{-\frac{1}{2}}$ $(\log a_n^{-1})^{\frac{1}{2}}$.

PROOF. First note that for each λ, $0 < \lambda < 1$, and $\varepsilon > 0$

$$P\left[\bigcup_{k \geq n}\{\widehat{G}_{kx}^{-1}(\lambda) - G_x^{-1}(\lambda) \geq \varepsilon\tau_k\}\right]$$

$$= P\left[\bigcup_{k\geq n}\{\widehat{G}_{kx}(q_x(\lambda)+\varepsilon\tau_k)\leq\lambda\}\right]+P\left[\bigcup_{k\geq n}\{\widehat{G}_{kx}(q_x(\lambda)-\varepsilon\tau_k)\geq\lambda\}\right]$$

$$= P\left[\bigcup_{k\geq n}\{\widehat{G}_{kx}(u_k^+)\leq\lambda\}\right]+P\left[\bigcup_{k\geq n}\{\widehat{G}_{kx}(u_k^-)\geq\lambda\}\right], \tag{2.2}$$

where we have set $q_x(\lambda) = G_x^{-1}(\lambda)$, $u_n^+ = q_x(\lambda) + \varepsilon\tau_n$ and $u_n^- = q_x(\lambda) - \varepsilon\tau_n$. Now for the second term in (2.2), we obtain from (3.2) that *WOLG* (see (3.12a) of MRR[5]),

$$P\left[\bigcup_{k\geq n}\{\widehat{G}_{kx}(u_k^-)\geq\lambda\}\right] = P\left[\bigcup_{k\geq n}\{t_k(x)(\widehat{G}_{kx}(u_k^-)-G_k(u_k^-))\geq t_k(x)(\lambda-G_x(u_k^-))\}\right]$$

$$\leq P\left[\bigcup_{k\geq n}\{|\bar{J}_{k1}(u_k^-)-E(\bar{J}_{k1}(u_k^-))-(t_{k1}(x)-1)|\geq(\lambda-G_x(u_k^-))/2\}\right]$$

$$+P\left[\bigcup_{k\geq n}\left\{\left|\sum_{j=2}^{3}(J_{kj}(y)+t_{kj}(x))\right|\geq((\lambda-G_x(u_k^-))/2)-E\bar{J}_{k1}(u_k^-)\right\}\right], \tag{2.3}$$

where $\tau_n = n^{-\frac{1}{2}}a_n^{-\frac{1}{2}}(\log a_n^{-1})^{\frac{1}{2}}$ and $t_n(x) = t_{n1}(x) + t_{n2}(x) + t_{n3}(x)$ stands for the Taylor's expansion similar to that for $v_{nx}(y)$ in (3.2) of MRR[5]. Now note that, for the RHS expressions in the preceding probabilities in (2.3), we have by the mean value theorem, for some Δ_{nx} lying between $G_x^{-1}(\lambda)$ and $(G_x^{-1}(\lambda) - \varepsilon\tau_n)$,

$$\begin{aligned}\tau_n^{-1}[\lambda-G_x(u_n^-)] &= \tau_n^{-1}[G_x(G_x^{-1}(\lambda))-G_x(G_x^{-1}(\lambda)-\varepsilon\tau_n)]\\ &= \varepsilon g_x(\Delta_{nx})\\ &\to \varepsilon g_x(G_x^{-1}(\lambda))>0,\end{aligned} \tag{2.4}$$

as $n \to \infty$, and further from (3.13) and (3.15) of MRR[5] and the additional assumption $b_n^2 = \sigma(\tau_n)$, that

$$E[\bar{J}_{n1}(u_n^-)] = \sigma(\tau_n), \tag{2.5}$$

as $n \to \infty$. From (2.4) and (2.5), it follows by the arguments used in the proof of Theorem 3.1, from (3.16) to (3.19), of MRR[5] that the first probability in (2.3) is dominated by

$$\sum_{k\geq n}P[|\bar{J}_{k1}(u_k^-)-E\bar{J}_{k1}(u_k^-)-(t_{k1}(x)-1)|\geq(\varepsilon/2)\tau_k g_x(G_x^{-1}(\lambda))]$$

$$\leq c_1\sum_{k\geq n}[k^{-2}]\to 0, \tag{2.6}$$

as $n \to \infty$ (c_1 above is a positive constant). Also the second probability in (2.3) goes to zero by (3.24) of MRR[5](cf. Lemma 3.1). Thus, in view of (2.3) and (2.6), the second probability $P[\cup_{k\geq n}\{\widehat{G}_{kx}(u_k^-)\geq\lambda\}]$ on the right of (2.2) $\to 0$, as $n \to \infty$. By following parallel arguments, one can similarly show that the first term in (2.2)

$$P\left[\bigcup_{k\geq n}\{\widehat{G}_{kx}(u_k^+)\leq\lambda\}\right]\to 0, \tag{2.7}$$

as $n \to \infty$. The proof of the Lemma accordingly follows from (2.2) and the standard a.s.

convergence criterion. □

COROLLARY 2.1. *Under the conditions of Lemma 2.1, for each* λ, $0 < \lambda < 1$, $\widehat{G}_{nx}^{-1}(\lambda)$ *is a strongly consistent estimator of* $q_x(\lambda) = G_x^{-1}(\lambda)$, *as* $n \to \infty$.

The following lemma gives the order of magnitude of oscillations of the conditional empirical process based on $\widehat{G}_{nx}(y)$, $y \in \mathfrak{R}$.

LEMMA 2.2. *Suppose the conditions of lemma 2.1 hold. Then, for each* λ, $0 < \lambda < 1$, *and any fixed* $x \in \Lambda(F)$ *and constant* $c > 0$, *we have with probability one*

$$\sup_{|y-q_x(\lambda)|<c\tau_n} |\widehat{G}_{nx}(y) - \widehat{G}_{nx}(q_x(\lambda)) - G_x(y) + \lambda| = O(\tau_n^{3/2}), \tag{2.8}$$

as $n \to \infty$, *where* τ_n *is as defined in Lemma 2.1.*

PROOF. For each fixed λ, $0 < \lambda < 1$, and $x \in \Lambda(F)$, consider the real numbers η_{rn} defined by

$$\eta_{rn} = q_x(\lambda) + r\tau_n^*, \; r = 0, \pm 1, \ldots \pm d_n, \tag{2.9}$$

where $c > 0$ is a constant, $\tau_n^* = c\tau_n^{3/2}$, $d_n = [\tau_n^{-\frac{1}{2}}] + 1$, and $[\cdot]$ stands for the integral part. Then by the monotonicity of $\widehat{G}_{nx}$ and G_x, we have for any $y \in I_r = [\eta_{rn}, \eta_{(r+1)n}]$,

$$\begin{aligned} \widehat{G}_{nx}(\eta_{rn}) - G_x(\eta_{(r+1)n}) &\le \widehat{G}_{nx}(y) - G_x(y) \\ &\le \widehat{G}_{nx}(\eta_{(r+1)n}) - G_x(\eta_{rn}), \end{aligned}$$

which yields

$$\begin{aligned} &\sup_{|y-q_x(\lambda)|<c\tau_n} |\widehat{G}_{nx}(y) - G_x(y) - \widehat{G}_{nx}(q_x(\lambda)) + \lambda| \\ &\le \max_{|r|\le d_n} |\widehat{G}_{nx}(\eta_{rn}) - G_x(\eta_{rn}) - \widehat{G}_{nx}(q_x(\lambda)) + \lambda| \\ &+ \max_{-d_n\le r\le d_n-1} |G_x(\eta_{(r+1)n}) - G_x(\eta_{rn})|. \end{aligned} \tag{2.10}$$

By assumption A.I(ii), the density g_x is bounded (in some neighborhood of $q_x(\lambda)$) so that for sufficiently large n the second term on the R.H.S. in (2.10) is bounded by τ_n^*. It thus suffices to show that the first term on the R.H.S. of (2.10) is dominated by an order term given in the assertion of the lemma. For this, note that from (3.2) of MRR[5] we have

$$\begin{aligned} &t_n(x)[\widehat{G}_{nx}(\eta_{rn}) - \widehat{G}_{nx}(q_x(\lambda)) - G_x(\eta_{rn}) + \lambda] \\ &= [J_{n1}(\eta_{rn}) - J_{n1}(q_x(\lambda))] + [J_{n2}(\eta_{rn}) - J_{n2}(q_x(\lambda))] + [J_{n3}(\eta_{rn}) - J_{n3}(q_x(\lambda))] \\ &= [\varepsilon_{n1} + \varepsilon_{n2} + \varepsilon_{n3}] \text{ (say)}. \end{aligned} \tag{2.11}$$

We first deal with the terms ε_{n2} and ε_{n3}. Now as in Lemma 3.1 of MRR[5], we have from (3.4), (3.5) and (3.6) of MRR[5], with probability one and uniformly in r, for some constants C_5 and C_6

$$\begin{aligned}|\varepsilon_{n3}| &\le \varepsilon_{n31} + \varepsilon_{n32}\\ &\le C_5 n^{-\frac{3}{2}} a_n^{-2} \log a_n^{-1} (\log\log n)^{\frac{1}{2}} + C_6 n^{-1} a_n^{-1} \log a_n^{-1}\\ &= (n^{-1} a_n^{-1} \log a_n^{-1})^{\frac{3}{4}} [C_5 n^{-\frac{3}{4}} a_n^{-\frac{5}{4}} (\log a_n^{-1})^{\frac{1}{4}} (\log\log n)^{\frac{1}{2}} + C_6 n^{-\frac{1}{4}} a_n^{-\frac{1}{4}} (\log a_n^{-1})^{\frac{1}{4}}]\\ &= \sigma(\tau_n^{\frac{3}{2}}) \end{aligned} \tag{2.12}$$

as $n \to \infty$, under the assumption A.IV(ii). Now splitting ε_{n2}, on exactly the same lines as for J_{n2} in (3.8) of MRR[5], we obtain

$$\varepsilon_{n2} = \varepsilon_{n21} + \varepsilon_{n22} \qquad \text{(say)}, \tag{2.13}$$

where, in view of boundedness of W_n and W_n^* (cf (3.9) of MRR[5]), we have uniformly in r

$$\begin{aligned}|\varepsilon_{n22}| &\underset{\text{a.s.}}{\le} C_7 (n^{-1} a_n^{-1} \log a_n^{-1})^{3/4} n^{-\frac{1}{4}} a_n^{-3/4} (\log\log n)^{\frac{1}{2}} (\log a_n^{-1})^{-\frac{1}{4}}\\ &= C_7 \tau_n^{3/2} (n^{-1} a_n^{-3})^{1/4} [(\log\log n)^2 / \log a_n^{-1})]^{1/4}\\ &\underset{\text{a.s.}}{=} \sigma(\tau_n^{3/2}), \end{aligned} \tag{2.14}$$

as $n \to \infty$, in view of assumptions A.IV(ii); further on the lines of (3.10) of MRR[5], we have

$$\begin{aligned}|\varepsilon_{n21}| \underset{\text{a.s.}}{\le} \tau_n \Bigg[&\int_{A_n^*} |W_n^{*(1)}(t)| \left| \int [V_n^{(1,0)}(t, \eta_{rn} - v) - V_n^{(1,0)}(t, q_x(\lambda) - v)] dG_x(v) \right| dt\\ &+ \left(\int |W_n^{*(1)}(t)| dt \right) |G_x(\eta_{rn}) - \lambda)|\\ &+ \iint_{A_n^*} |W_n^{(1,0)}(t, \eta_{rn} - v) - W_n^{(1,0)}(t, q_x(\lambda) - v) - W_n^{(1)*}(t)(G_x(\eta_{rn}) - \lambda)|\\ &\quad d|G_{x_n(t)}(v) - G_x(v)| dt \Bigg], \end{aligned} \tag{2.15}$$

where $x_n(t) = F^{-1}(F(x) - a_n t)$ and $V_n^{(j,0)}(t, s) = W_n^{(j,0)}(t, s)/W_n^{*(j)}(t)$. Now by using integration by parts for the inside integral in the first term of (2.15), the assumption A.III(v) and the boundedness of W_n, W_n^* and g_x, it follows easily that the first two terms in (2.15) are, uniformly in r, $O(\tau_n^2)$; also by using Taylor's expansion and the preceding boundedness assumptions, it follows that the integral in the last term in (2.15) is $O(a_n)$ uniformly in r, as $n \to \infty$. In view of the preceding considerations and the assumption A.III(iv), we obtain from (2.15) that, as $n \to \infty$, $\varepsilon_{n21} = O(\tau_n^{3/2})$ with probability one. This coupled with (2.14) yields

$$\varepsilon_{n2} \underset{\text{a.s.}}{=} O(\tau_n^{3/2}), \tag{2.16}$$

uniformly in r, as $n \to \infty$. Now to deal with the first term, we follow steps similar to those in the proof of Theorem 3.1 (a) of MRR[5]:

Now $\varepsilon_{n1} = n^{-1} \Sigma_{i=1}^n Z_{ni}^*$, with

$$Z_{ni}^* = a_n^{-1} [W_n((F(x) - F(X_i))/a_n, \eta_{rn} - Y_i) - W_n((F(x) - F(X_i))/a_n, q_x(\lambda) - Y_i)$$

$$- W_n^*((F(x) - F(X_i))/a_n)(G_x(\eta_{rn}) - \lambda)], \tag{2.17}$$

so that using transformation $F(x) - F(u) = a_n t$ and integration by parts in below, we obtain (for notational convenience, we have set $q_x = q_x(\lambda)$)

$$\begin{aligned}
E[Z_{n1}^*] &= \int\int [W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v) - W_n^*(t)(G_x(q_x + r\tau_n^*) \\
&\quad - G_x(q_x))]dG_{x_n(t)}(v)dt \\
&= \int\left[\int (W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v))dG_{x_n(t)}(v)\right]dt \\
&\quad - [G_x(q_x + r\tau_n^*) - G_x(q_x)] \\
&= \int\left[\int [G_{x_n(t)}(q_x - v + r\tau_n^*) - G_{x_n(t)}(q_x - v)]dW_n(t, v)\right]dt \\
&\quad - [G_x(q_x + r\tau_n^*) - G_x(q_x)] \\
&= r\tau_n^*\int\left[\int \{g_{x_n(t)}(q_x - v) - g_x(q_x)\}dW_n^*(t, v)\right]dt + O(r^2\tau_n^{*2}),
\end{aligned} \tag{2.18}$$

where in (2.18) we have used the boundedness of g_x' and the proximity of $x_n(t) = F^{-1}(F(x) - a_n t)$ to x for $-1 \le t \le 1$ and n sufficiently large. Further, employing Taylor's expansion for $x_n(t)$ for n sufficiently large and using assumptions A.I, we obtain from (2.18) that for some constant $C_8 > 0$

$$\begin{aligned}
|E(Z_{n1}^*)| &\le C_8\tau_n \left|\int [g_x(q_x - v) - g_x(q_x)]dm_n^*(v)\right| + O(\tau_n a_n) + O(\tau_n^2) \\
&= o(\tau_n^{3/2})
\end{aligned} \tag{2.19}$$

as $n \to \infty$, in view of again the boundedness of g_x', arguments similar to those of (3.14) to (3.16) of MRR[5] and the assumption A.III(iv), which implies that $a_n = \tau_n^{\frac{1}{2}}[na_n^5(\log a_n^{-1})^{-1}]^{1/4} = o(\tau_n^{\frac{1}{2}})$, provided na_n^5 stays bounded as $n \to \infty$. Also similar calculations yield

$$\begin{aligned}
a_n EZ_{n1}^{*2} &= \int\int [W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v) \\
&\quad - W_n^*(t)(G_x(q_x + r\tau_n^*) - G_x(q_x)]^2 dG_{x_n(t)}(v)dt \\
&= \int\Big\{\int [(W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v))^2 \\
&\quad + W_n^{*2}(t)q_x^2(q_x + \delta_n r\tau_n^*)r^2\tau_n^{*2} \\
&\quad -2W_n^*(t)g_x(q_x + \delta_n r\tau_n^*)r\tau_n^*(W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v))] \\
&\qquad dG_{x_n(t)}(v)\Big\}dt \\
&\le \int\left[\int (W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v))^2 dG_{x_n(t)}(v)\right]dt
\end{aligned}$$

$$+ \left(\int W_n^{*2}(t)dt\right) r^2 \tau_n^{*2} g_x^2(q_x + \delta_n r \tau_n^*)$$

$$+ 2|r|\tau_n^* g_x(q_x + \delta_n r \tau_n^*) \int W_n^*(t) \left| \int [W_n(t, q_x - v + r\tau_n^*) \right.$$

$$\left. - W_n(t, q_x - v)] dG_{x_n(t)}(v) \right| dt$$

$$\leq \int \left[\int (W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v))^2 dG_{x_n(t)}(v) \right] dt$$

$$+ O(\tau_n), \tag{2.20}$$

where the order term holds uniformly in r, as $n \to \infty$. Now assume momentarily that $r \geq 0$; then the integral within the square bracket in the last R.H.S. expression in (2.20) does not exceed a constant times

$$\int [W_n(t, q_x - v + r\tau_n^*) - W_n(t, q_x - v)] dG_{x_n(t)}(v)$$

$$= \int [G_{x_n(t)}(q_x - v + r\tau_n^*) - G_{x_n(t)}(q_x - v)] dW_n(t, v)$$

$$= r\tau_n^* \int g_{x_n(t)}(q_x - v + \delta_n r \tau_n^*) dW_n(t, v), \tag{2.21}$$

where in (2.21), we have used integration by parts and the standard mean value theorem. From (2.20) and (2.21), one immediately obtains for some constant $C_9 > 0$

$$a_n E Z_{n1}^{*2} \leq C_9 \tau_n \int \left[\int dW_n(t, v) \right] dt + O(\tau_n)$$

$$= O(\tau_n) \tag{2.22}$$

uniformly in r, as $n \to \infty$. The case $r < 0$ can be dealt with in a similar manner. Now using the Bernstein inequality as in Theorem 3.1 of MRR[5], we obtain from (2.22)

$$P\left[n^{-1} \left| \sum_{i=1}^{n} [Z_{ni}^* - EZ_{ni}^*] \right| \geq \varepsilon \tau_n^{3/2} \right] \leq 2 \, exp\left\{ \frac{-\varepsilon^2 n \tau_n^3}{C_1 a_n^{-1} \tau_n + (2/3) a_n^{-1} \varepsilon \tau_n^{3/2}} \right\}$$

$$= 2 \, exp\left\{ \frac{-\varepsilon^2 \log a_n^{-1}}{C_1 + (2/3)\varepsilon \tau_n^{1/2}} \right\} \tag{2.23}$$

for $\varepsilon > 0$ and some positive constant C_1. Using (2.19) and arguing again as in the paragraph immediately after (3.18) of MRR[5], we obtain from (2.23) that, for some positive constants C_2,

$$P[\max_{|r| \leq d_n} |J_{n1}(\eta_{rn}) - J_{n1}(q_x(\lambda))| \geq \varepsilon \tau_n^{3/2}]$$

$$\leq 2 d_n n^{-2} \leq C_2 n^{-7/4},$$

which in view of Borel-Cantelli lemma implies that

$$\max_{|r|\le d_n}|J_{n1}(\eta_{rn}) - J_{n1}(q_x(\lambda))| = O(\tau_n^{3/2}) \tag{2.24}$$

with probability one, as $n \to \infty$. Since clearly, using mean value theorem, $\max\{|G_x(\eta_{(r+1)n}) - G_x(\eta_{rn})| : -d_n \le r \le d_n\} = O(\tau_n^{3/2})$ as $n \to \infty$, (2.24) coupled with (2.10) to (2.12), (2.16) and the fact that $t_n(x) \to 1$, with probability one, as $n \to \infty$, establishes (2.8). The proof is complete. □

3. The Main Result

We shall now state the main result of this paper concerning the almost sure representation of the λth conditional quantile $\widehat{G}_{nx}^{-1}(\lambda)$ (for fixed $0 < \lambda < 1$ and $x \in \Lambda(F)$) defined by (1.2).

THEOREM 3.1. *Under the assumptions of Lemmas 2.1 and 2.2, for each fixed λ, $0 < \lambda < 1$ and $x \in \Lambda(F)$, we have with probability one*

$$\widehat{G}_{nx}^{-1}(\lambda) = G_x^{-1}(\lambda) + [\lambda - \widehat{G}_{nx}(G_x^{-1}(\lambda))]g_x^{-1}(\lambda)) + O(\tau_n^{3/2}), \tag{3.1}$$

as $n \to \infty$, where $\tau_n = n^{-\frac{1}{2}}a_n^{-\frac{1}{2}}(\log a_n^{-1})^{\frac{1}{2}}$.

PROOF. From Lemmas 2.1 and 2.2, we have at once

$$\widehat{G}_{nx}(\widehat{G}_{nx}^{-1}(\lambda)) - \widehat{G}_{nx}(G_x^{-1}(\lambda)) \underset{\text{a.s.}}{=} (\widehat{G}_{nx}^{-1}(\lambda) - G_x^{-1}(\lambda))g_x(G_x^{-1}(\lambda)) + O(\tau_n^2) + O(\tau_n^{3/2})$$

where $\tau_n = n^{-\frac{1}{2}}a_n^{-\frac{1}{2}}(\log a_n^{-1})^{\frac{1}{2}}$, as $n \to \infty$, which yields (3.1) by rearrangement of the equality. The proof is complete □

The following corollary is an immediate consequence of Theorem 3.1 above and Theorem 3.1 of MRR[5].

COROLLARY 3.1. *Under the conditions of Theorem 3.1 above, $\widehat{G}_{nx}^{-1}(\lambda)$ is asymptotically normal i.e., for each λ, $0 < \lambda < 1$, $(na_n)^{1/2}[\widehat{G}_{nx}^{-1}(\lambda) - G_x^{-1}(\lambda)] \xrightarrow[\mathcal{L}]{} N(0, \sigma_x^2(\lambda))$ (or, $N(b_x(G_x^{-1}(\lambda)), \sigma_x^2(\lambda))$ if $na_n^{2m+3} \to \theta$) as $n \to \infty$, where $\sigma_x^2(\lambda) = \lambda(1-\lambda)(\int k_1^2(t)dt)/g_x^2(G_x^{-1}(\lambda))$ with the kernel function k_1, $b_x(\cdot)$ and θ as given in Corollary 3.1 of MRR[5].*

REMARK 3.1. It is worth noting that in our proof we have made no assumption regarding continuity or differentiability of the function $W_n(t, s)$ in the second argument. Consequently the assertions of Theorems 3.1 and Corollary 3.1 also cover the "*unsmoothed*" conditional empirical distribution and quantile functions considered by Stute [6] and Härdle et al. [2]. Further, the considerations of Remark 3.3 of MRR[5] will also apply here.

REMARK 3.2. The results of Lemmas 2.1 and 2.2, Theorem 3.1 and Corollary 3.1 concerning the RNN estimator $\widehat{G}_{nx}^{-1}(\lambda)$ also remain valid, under the same conditions, for the Nadaraya-Watson type estimator $\widehat{G}_{nx}^{*-1}(\lambda)$ of $G_x^{-1}(\lambda)$, $0 < \lambda < 1$ defined in Section 1 of MRR[5]. The proofs of these results for the Nadaraya-Watson type estimators – both smoothed and unsmoothed – are in fact contained within the

corresponding ones for the RNN estimators. This remark covers the assertions of Remark 3.1 as well.

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