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Lee, Hyen Yeal
Department of Computer Science, Pusan University

Kawahara, Yasuo
Research Institute of Fundamental Information Science , Kyushu University

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ON DYNAMICAL BEHAVIORS OF CELLULAR AUTOMATA CA-60

By

Hyen Yeal LEE* and Yasuo KAWAHARA**

Abstract

Dynamical behaviors of some particular cases of cellular automata $ca-60(m)$ and $ca-60(m, n)$ will be completely shown by representing their configurations as truncated polynomials.

1. Introduction

In general there are two ways in which the global transition function of configurations of a cellular automaton (or automata network) are decided for a given local transition rule. The first one is a way in which the next state (value) of each cell is computed by applying a given local transition function in parallel (or synchronously). Such global transition function is called to be parallel (or synchronous). The second one is a way in which the next state of each cell is sequentially (or asynchronously) computed by applying the given local transition function in a previously given linear order on the set of all cells, more precisely the next state of a cell is computed by evaluating the local transition function using updated states of cells if the domain of the local transition function (or the neighborhood of the cell involved) includes cells which have been already updated at present. Such global transition function is called to be of Gauss-Seidel type (sequential or asynchronous) named after a method for solving systems of equations.

The transition of the former cellular automata is clearly local but that of latter ones is not local even if given local transition functions are local. Lee and Huzino have observed that the behaviors of Gauss-Seidel type cellular automata are more complex.

The relationship between dynamical behaviors of transitions of parallel type and Gauss-Seidel type for a given local transition function might be much complicated and interested. Goles-Martinez [1] showed such simple examples.

Kumamoto, Yamamoto and Nohmi [5] studied dynamical behaviors of 2-dimensional Gauss-Seidel type cellular automata with so-called local transition rule 90 by determining representative matrices of the global transition functions and their Jordan normal forms using theory of elementary divisors of linear algebra over finite fields. Huzino [4] showed that the global transition function of the above Gauss-Seidel type cellular automaton is decomposed into two global transition functions of parallel

* Department of Computer Science, Pusan University, Pusan 605, Korea.

** Research Institute of Fundamental Information Science, Kyushu University 33, Fukuoka 812, Japan.

cellular automata $ca-60(m, n)$ and $ca-102(m, n)$.

In this paper the authors give some results on the dynamical behaviors of $ca-60(m)$ and $ca-60(m, n)$ mentioned by Huzino [4]. The main idea of the paper is based on representing configurations of cellular automata by truncated polynomials and on corresponding the global transition function with a multiplication of a simple polynomial.

Nohmi [2] showed that all fundamental invariants related to the structure of kernel trees and limit cycles of finite additive cellular automata are completely determined from Jordan normal forms of the representative matrices of the global transition functions. The representative matrix of the global transition function of $ca-60(m)$ is a Jordan normal form itself. So in this case Nohmi's result [2] can be directly applied. However the computational complexity of computing fundamental invariants associated with 2-dimensional cellular automaton $ca-60(m, n)$ according to this paper is much less than Nohmi's method [2].

2. One-Dimensional Cellular Automaton $ca-60(m)$

Let m be a positive integer. A configuration of 1-dimensional cellular automaton $ca-60(m)$ is an m -dimensional vector $(c_0, c_1, \dots, c_{m-1})$ over a finite field $F_2 (= \{0, 1\})$ and the configuration space, denoted by $ca-60(m)$, is an m -dimensional vector space F_2^m over F_2 . The global transition function $\tau: ca-60(m) \rightarrow ca-60(m)$ of $ca-60(m)$ is defined by

$$\tau(c) = (c_0, c_0 + c_1, c_1 + c_2, \dots, c_{m-2} + c_{m-1})$$

for each configuration $c = (c_0, c_1, \dots, c_{m-1}) \in ca-60(m)$. It is clear that the transition function τ is additive and bijective. Note that the canonical representative matrix of the transition function τ is of Jordan normal form. (Consequently Nohmi's result [2] can be applied to this case.)

Now we assign a polynomial $c(x) = c_0 + c_1x + \dots + c_{m-1}x^{m-1}$ to a configuration $c = (c_0, c_1, \dots, c_{m-1})$ of $ca-60(m)$. It simply follows that

$$\begin{aligned} (1+x)c(x) &= (1+x)(c_0 + c_1x + \dots + c_{m-1}x^{m-1}) \\ &= c_0 + (c_0 + c_1)x + (c_1 + c_2)x^2 + \dots \\ &\quad \dots + (c_{m-2} + c_{m-1})x^{m-1} + c_{m-1}x^m \\ &= \tau(c)(x) + c_{m-1}x^m. \end{aligned}$$

Introducing a relation $x^m = 0$, i.e. identifying a configuration c of $ca-60(m)$ with an element $c(x)$ of the quotient ring $F_2[x]/(x^m)$, we have $\tau(c)(x) = (1+x)c(x)$, where $F_2[x]$ is the polynomial ring of x over F_2 and (x^m) is the ideal generated by a monomial x^m . (An element of the quotient ring $F_2[x]/(x^m)$ is called a truncated polynomial.)

LEMMA 1. (Huzino [4]) *If $2^{k-1} < m \leq 2^k$, then $\tau^{2^k}(c) = c$ for each configuration c of cellular automaton $ca-60(m)$.*

PROOF. Since $x^{2^k} = 0$ from $m \leq 2^k$, it is clear that

$$\tau^{2^k}(c) = (1+x)^{2^k}c = (1+x^{2^k})c = c.$$

The last lemma shows that each configuration of cellular automaton $ca-60(m)$ is on a limit cycle with length a power of 2.

Let $c = c_0 + c_1x + \dots + c_{m-1}x^{m-1}$ be a configuration of $ca-60(m)$. Define $codeg(c) = \max\{i; c_{m-i} = 1 \ (0 \leq i \leq m)\}$. For example, $c = x^{m-1}$ if $codeg(c) = 1$ and $c = 0$ if $codeg(c) = 0$. Continuing the argument of Lemma 1 in detail we have the following result.

THEOREM 2. *Let c be a configuration of $ca-60(m)$. If $k \geq 0$ and $2^{k-1} < codeg(c) \leq 2^k$, then $\tau^{2^{k-1}}(c) \neq c$ and $\tau^{2^k}(c) = c$.*

PROOF. Set $w = codeg(c)$. Then c can be represented as $c = x^{m-w} + c'$, where c' is a higher term. Hence it is easy to see that

$$\tau^{2^k}(c) = (1 + x^{2^k})c = c + x^{m+2^k-w} + x^{2^k}c' = c$$

because of $m \leq m + 2^k - w$, and similarly $\tau^{2^{k-1}}(c) \neq c$.

Let p be a positive integer. For cellular automaton $ca-60(m)$ with $m \geq 2$, define nonnegative integer $\chi_m(p)$ to be the number of configurations in $ca-60(m)$ which lie on a (limit) cycle of length p . Note that the number of cycles with length p of $ca-60(m)$ is given by $\chi_m(p)/p$.

COROLLARY 3. *For cellular automaton $ca-60(m)$ with $m \geq 2$, the following holds:*

1. $\chi_m(1) = 2$,
2. $\chi_m(2^k) = 2^{2^k} - 2^{2^{k-1}}$ for $0 < k$ and $2^k < m$,
3. $\chi_m(2^k) = 2^m - 2^{2^{k-1}}$ for $2^{k-1} < m \leq 2^k$,
4. $\chi_m(p) = 0$ otherwise.

By virtue of the last corollary the fundamental invariants (Cf. Nohmi [2]) associated with a cellular automaton $ca-60(m)$ will be completely determined.

3. Two-Dimensional Cellular Automaton $ca-60(m, n)$

Let m and n be positive integers. A configuration of 2-dimensional cellular automaton $ca-60(m, n)$ is an $m \times n$ matrix

$$\begin{pmatrix} c_{00} & c_{01} & \dots & c_{0n-1} \\ c_{10} & c_{11} & \dots & c_{1n-1} \\ \dots & \dots & \dots & \dots \\ c_{m-10} & c_{m-11} & \dots & c_{m-1n-1} \end{pmatrix}$$

over F_2 , and then the configuration space $ca-60(m, n)$ is an $m \times n$ -dimensional vector space over F_2 . The global transition function of $ca-60(m, n)$ $\tau: ca-60(m, n) \rightarrow ca-60(m, n)$ is defined by

$$\tau(c) = \begin{pmatrix} c_{00} & c_{00} + c_{01} & \dots & c_{0n-2} + c_{0n-1} \\ c_{00} + c_{10} & c_{01} + c_{10} + c_{11} & \dots & c_{0n-1} + c_{1n-2} + c_{1n-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-20} + c_{m-10} & c_{m-21} + c_{m-10} + c_{m-11} & \dots & c_{m-2n-2} + c_{m-1n-2} + c_{m-1n-1} \end{pmatrix}$$

for each configuration $c = (c_{ij}) \in ca-60(m, n)$. Similarly to the one-dimensional case the transition function τ is additive and bijective.

Again one can identify a configuration $c = (c_{ij})$ of 2-dimensional cellular automaton $ca-60(m, n)$ with a (truncated) polynomial of a quotient ring $F_2[x, y]/(x^m, y^n)$, where $F_2[x, y]$ is the polynomial ring of variables x and y over F_2 , and (x^m, y^n) is the ideal of $F_2[x, y]$ generated by nomominals x^m and y^n . That is,

$$ca-60(m, n) = F_2[x, y]/(x^m, y^n),$$

$$c = \sum_{i=0, j=0}^{m-1, n-1} c_{ij} x^i y^j \quad (c_{ij} \in F_2),$$

$$\tau(c) = (1 + x + y)c.$$

LEMMA 4. (Huzino [4]) *If $2^{k-1} < \max\{m, n\} \leq 2^k$, then $\tau^{2^k}(c) = c$ for each configuration c of cellular automaton $ca-60(m, n)$.*

PROOF. Noticing that $x^{2^k} = 0$ and $y^{2^k} = 0$ by $\max\{m, n\} \leq 2^k$, it follows that

$$\tau^{2^k}(c) = (1 + x + y)^{2^k} c = (1 + x^{2^k} + y^{2^k})c = c.$$

LEMMA 5. (Huzino [4]) *If $m \leq n$, then the transition function τ of cellular automaton $ca-60(m, n)$ has 2^m fixed points, that is,*

$$\#\{c \in ca-60(m, n); \tau(c) = c\} = 2^m,$$

where $\#$ denotes the cardinality of sets.

PROOF. For each $c \in ca-60(m, n)$ we can put

$$c = \sum_{i=0, j=0}^{m-1, n-1} c_{ij} x^i y^j.$$

If $\tau(c) = (1 + x + y)c = c$, then $xc + yc = 0$ and so

$$\sum_{i=1}^{m-1} c_{i-10} x^i + \sum_{j=1}^{n-1} c_{0j-1} y^j + \sum_{i=1, j=1}^{m-1, n-1} (c_{i-1j} + c_{ij-1}) x^i y^j = 0.$$

Hence we have

$$\begin{aligned} c_{00} &= c_{10} = \dots = c_{m-20} = 0, \\ c_{00} &= c_{01} = \dots = c_{0n-2} = 0, \\ c_{i-1j} &= c_{ij-1} \quad (1 \leq i \leq m-1, 1 \leq j \leq n-1), \end{aligned}$$

which shows that a configuration c with $\tau(c) = c$ is determined by a choice of m

elements

$$c_{0n-1}, c_{1n-1}, \dots, c_{m-1n-1}$$

of F_2 . Therefore the number of such c is 2^m .

Let p be a positive integer. For cellular automaton $ca-60(m, n)$ with $m, n \geq 2$, define nonnegative integer $\lambda(p) = \lambda_{m,n}(p)$ by

$$\lambda(p) = \#\{c \in ca-60(m, n); \tau^p(c) = c\}.$$

It is obvious that $\lambda(p) = 0$ if p is not a power of 2 and the number of configurations which lie on cycles of length 2^k is $\lambda(2^k) - \lambda(2^{k-1})$.

COROLLARY 6. *If $0 \leq k$ and $1 \leq m \leq n$, then $\lambda(2^k) = 2^{2^k m}$ in $ca-60(2^k m, 2^k n)$.*

COROLLARY 7. *If $0 \leq a$ and $0 \leq k \leq a$, then $\lambda(2^k) = 2^{2^{a+k}}$ in $ca-60(2^a, 2^a)$.*

COROLLARY 8. *If $0 \leq a$ and $1 \leq n$, then the following holds in $ca-60(2^a, 2^a n)$:*

1. $\lambda(2^{a+k}) = 2^{2^{2^{a+k}}}$ if $0 \leq k$ and $2^k < n$,
2. $\lambda(2^{a+k}) = 2^{2^{2^a n}}$ if $0 \leq k$ and $2^{k-1} < n \leq 2^k$.

THEOREM 9. *If $0 \leq a$ and $1 \leq n$, then the following holds in $ca-60(2^a, 2^a n)$:*

1. $\lambda(2^k) = 2^{2^{2^{a+k}}}$ if $0 \leq k$ and $2^k < 2^a n$,
2. $\lambda(2^k) = 2^{2^{2^a n}}$ if $0 \leq k$ and $2^{k-1} < 2^a n \leq 2^k$.

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