EXISTENCE OF THE CHARACTERISTIC NUMBERS ASSOCIATED WITH CELLULAR AUTOMATA WITH LOCAL TRANSITION RULE 90

Kawahara, Yasuo
Department of Mathematics, Kyushu University

https://doi.org/10.5109/13413

出版情報：Bulletin of informatics and cybernetics. 24 (3/4), pp.121-136, 1991-03. Research Association of Statistical Sciences
バージョン：

Bulletin of Informatics and Cybernetics, Vol. 24, No. 3~4, 1991

EXISTENCE OF THE CHARACTERISTIC NUMBERS ASSOCIATED WITH CELLULAR AUTOMATA WITH LOCAL TRANSITION RULE 90

By

Yasuo KAWAHARA*

Abstract

This paper concerns the fundamentals on a new type of finite linear cellular automata with so-called local transition rule 90. Some problems about a few characteristic numbers associated with these automata, initially introduced by S. Huzino, are extensively solved and consequently the author gives a new algebraic formulation of these types of cellular automata, which enables us to challenge further investigations on these types of cellular automata of higher dimensions.

1. Introduction

The notion of cellular automata was introduced as a model of self-reproducing systems by J. von Neumann in 1950's and researches in this field have been continued, via several fluctuations, from physical, biological and mathematical points of view. Many types of cellular automata and their properties were considered by many authors. For instance it is well-known that the von Neumann automaton [6, 7] has the computability of a universal Turing machine, and simpler automata [1] with the same computability of the von Neumann model were studied. T. Kitagawa [8, 9] was one of the first authors to discuss the dynamics of several bounded neural networks. He also discussed possible biological applications. Recently S. Wolfram [12] recognized importance of cellular automata and studied them from various physical viewpoints, regarding as a theoretical model of highly complex systems. In particular theoretical and experimental study on behaviors of highly complex systems and cellular automata consisting of huge numbers of cells (or artificial neurons), gives a mathematical basis of theory of neural networks and neuroscience. It seems that the importance of cellular automata might increase from theoretical and applicative features among mathematical and computer sciences. Though cellular automata with infinite cells were dealed in at least earlier works of this field, S. Huzino [2−5] pointed out the importance of cellular automata with finite cells. He analyzed a quantity of their behaviors by making use of electric computers and inductively found a lot of interesting phenomena, as experimental results, about cellular automata with finite cells. The series of his works [2−5] strongly

* Department of Mathematics, Kyushu University 33, Fukuoka 812, Japan.

suggest that a new trend of mathematics is in unification of theory and computer experiment.

First of all, we briefly recall cellular automata $CA-90(m)$ studied by Huzino [2–5]. Let m be a positive integer $\geqq 2$. The m cells of cellular automata $CA-90(m)$ are linearly sited as follows:

$$\begin{array}{ccccccccccc} \square & - & \square & - & \square & - & \cdots & - & \square & - & \square \\ 1 & & 2 & & 3 & & & & m-1 & & m \end{array}$$

Then $CA-90(m)$ is a 1-dimensional cellular automaton with m cells. Each cell of $CA-90(m)$ is in only two states 0 or 1. A *configuration* of cellular automaton $CA-90(m)$ is an m-dimensional vector $c = (c(1), c(2), c(3), \ldots, c(m))$, where $c(i) = 0$ or 1 for all $i = 1, 2, \ldots, m$. The standard global transition function τ of $CA-90(m)$ is defined by

$$\tau\,(c(1), c(2), \ldots, c(m-1), c(m)) = (c(2), c(1) + c(3), \ldots, c(m-2) + c(m), c(m-1))$$

for each configuration $(c(1), c(2), \ldots, c(m-1), c(m))$ of $CA-90(m)$, where the all additions appearing here are of modulo 2. For example,

$$\tau\,(1, 1, 1, \ldots, 1, 1, 1) = (1, 0, 0, \ldots, 0, 0, 1).$$

That is, the set of all configurations of $CA-90(m)$ is an m-dimensional vector space $V(m)$ over a finite field $F_2 = \{ 0, 1 \}$ and its standard global transition function $\tau : V(m) \rightarrow V(m)$ is a linear transformation defined above. Wolfram called transition rules like τ the rule 90, because of a natural coding of local rules of transitions. Another global transition function $\delta : V(m) \rightarrow V(m)$ of $CA-90(m)$ is easily defined by

$$\delta\,(c) = \tau\,(c) + \alpha$$

for each configuration $c \in V(m)$, where α is a fixed configuration $(1, 0, 0, \ldots, 0, 0, 1) \in V(m)$. Adding two extra cells with fixed state 0 at both sides of m cells of $CA-90(m)$, the next state of normal (inner) cell by τ is given by modulo 2 addition of states of 2 cells next to the cell. Similarly adding two extra cells with fixed state 1 at both sides of m cells of $CA-90(m)$, the next state of normal (inner) cell by δ is given by modulo 2 addition of states of 2 cells next to the cell. This is a reason why we call τ the global transition function with boundary condition $< 0, 0 >$ and δ the global transition function with boundary condition $< 1, 1 >$, respectively. One can easily consider alternative global transition functions of $CA-90(m)$ with boundary conditions $< 0, 1 >$ and $< 1, 0 >$.

Huzino [2–5] obtained some fundamental properties on behaviors of cellular automata $CA-90(m)$ and proposed several problems by a lot of observations on behaviors of cellular automata $CA-90(m)$ using electric computers. The following problem is one of them and strongly motivated this research.

PROBLEM 1. *Find the smallest positive integer n such that $\tau^n = \delta^n$ in $CA-90(m)$,* where τ^n *denotes n times iteration of τ.*

One can easily see that there exists such positive integer n satisfying the equations $\tau^n = \delta^n$ in Problem 1 for some positive integer m but there exists no such n for other m. Huzino said the smallest positive integer n satisfying $\tau^n = \delta^n$ to be the *characteristic number associated with* $CA-90(m)$, if such n exists and denoted it by $h(m)$. It is obvious that a positive integer n satisfying $\tau^n = \delta^n$ is always a multiple of $h(m)$. It immediately follows from definition of δ and linearity of τ that

$$\delta^n(c) = \tau^n(c) + \Sigma_{j=0}^{n-1} \tau^i(\alpha).$$

Therefore Problem 1 is clearly equivalent to the following

PROBLEM 2. *Find the smallest positive interger n satisfying*

$$\Sigma_{j=0}^{n-1} \tau^j (\alpha) = 0$$

in $CA-90(m)$, *where* 0 *is zero vector of* $V(m)$.

It is easier to write a program to compute characteristic number $h(m)$ from Problem 2 rather than from Problem 1 directly. The following table of the characteristic numbers $h(m)$ is a part of a large quantity of data on cellular automata $CA-90(m)$ obtained by Huzino [2–5] using electric computers.

The table of the characteristic numbers $h(10a+b)$ $(0 \leqq a \leqq 3,\ 0 \leqq b \leqq 9)$

a \ b	0	1	2	3	4	5	6	7	8	9
0	/	/	2	*	3	4	7	*	14	6
1	31	*	53	14	30	*	15	28	511	*
2	126	62	2047	*	1023	126	1022	*	16383	60
3	31	*	62	30	4095	*	87381	1022	8190	*

In the above table * denotes non existence of the characteristic number. One can easily see that all $h(m)$ appearing in the table, except for $h(36) = 87381 = (2^{18}-1)/3$, is of form 2^u-1, $2(2^u-1)$ or $4(2^u-1)$ for some integer u. Just these observations from the table provide several interesting questions on behaviors of cellular automata $CA-90(m)$. It is really very difficult to completely determine the characteristic number $h(m)$ associated with $CA-90(m)$. This paper affirmatively solves Huzino's conjecture about existence of the characteristic numbers $h(m)$ and gives partially its formula for some m of special forms. Also the paper obtains a relationship between the characteristic numbers $h(m)$ and another characteristic numbers associated with $CA-90(m)$. Thus the characteristic numbers $h(m)$ $(2 \leqq m \leqq 300)$ except for a few cases are easily computable from a table of the characteristic *numbers* $K(m)$ at the end of the paper. The basic idea to discuss systematically these cellular automata with local transition rule 90 is a new algebraic formulation of them using so-called Laurant polynomials, which enables us to challenge further investigations on these types of cellular automata of higher dimensions.

1. Fundamentals on Cellular Automata ca−90(m)

The purpose of this section is to provide fundamental properties of cellular automata *ca−90*(m) for the existence theorem of characteristic numbers $H(m)$ and $K(m)$ and for later discussions.

We define cellular automata *ca−90*(m) by a simple modification of cellular automata *CA−90*(m) because of a notational convention. Let m be a positive integer $\geqq 2$ throughout the rest of this paper. A *configuration* of cellular automaton *ca−90*(m) is an $(m-1)$ dimensional vector $c = (c(1), c(2), \ldots, c(m-1))$, where $c(i) = 0$ or 1 for all $i = 1, 2, \ldots, m-1$. The set of all configurations of *ca−90*(m) is then an $(m-1)$-dimensional vector space $V(m-1)$ over a finite field $F_2 = \{0, 1\}$. The standard global transition function $\tau: V(m-1) \rightarrow V(m-1)$ of *ca−90*(m) is defined by

$$\tau(c(1), c(2), \ldots, c(m-2), c(m-1))$$
$$= (c(2), c(1) + c(3), \ldots, c(m-3) + c(m-1), c(m-2))$$

for each configuration $(c(1), c(2), \ldots, c(m-2), c(m-1))$ of *ca−90*(m), where the all additions appearing here are of modulo 2. That is, $ca{-}90(m) = CA{-}90\ (m-1)$. (Note that $\tau(c(1)) = (0) = 0$ when $m = 2$.)

PROPOSITION 1.1. *The standard global transition function* $\tau: V(m-1) \rightarrow V(m-1)$ *of ca−90(m) is bijective if and only if m is odd.*

PROOF. At first suppose that m is odd, that is, $m = 2k + 1$ for an integer $k \geqq 1$, and $b = \tau(c)$ for $b = (b(1), b(2), \ldots, b(2k))$, $c = (c(1), c(2), \ldots, c(2k)) \in V(m-1)$. Then it easily follows from the definition of τ that

$$c(2i) = b(1) + b(3) + \ldots + b(2i-1),$$
$$c(2i-1) = b(2i) + b(2i+2) + \ldots + b(2k)$$

for all $i = 1, 2, \ldots, k$. This clearly shows that τ is injective and hence it is bijective since $V(m-1)$ is finite. Finally assume that m is even, that is, $m = 2k$ for an integer $k \geqq 1$. Then one can easily see that $\tau(1, 0, 1, 0, \ldots, 0, 1, 0, 1) = (0, 0, 0, 0, \ldots, 0, 0, 0, 0) = 0$ in $V(2k-1)$ and so τ is not bijective. □

PROPOSITION 1.2. *Let* $\tau: V(m-1) \rightarrow V(m-1)$ *be the standard global transition function of ca−90(m).*
(*a*) *When* $m = 2k$ $(k \geqq 1)$, *a configuration* $c = (c(1), c(2), \ldots, c(2k-1)$ *of ca−90(m) belongs to the image of* τ *(that is,* $c \in \mathrm{Im}\ \tau$*) if and only if* $c(1) + c(3) + \ldots + c(2k-1) = 0$.
(*b*) If $m = 4k + 2$ $(k \geqq 1)$, *then* $\mathrm{Im}\ \tau = \mathrm{Im}\ \tau^2$ *(that is,* $\mathrm{Im}\ \tau = \mathrm{Im}\ \tau^n$ *for each integer* $n \geqq 2$).

PROOF. (a) In case of $k = 1$ it is trivial since $\mathrm{Im}\ \tau = \{0\}$. Assume $m = 2k (k \geqq 2)$ and $c \in \mathrm{Im}\ \tau$. Then there exists $b = (b(1), b(2), \dots, b(2k-1)) \in V(m-1)$ such that $c = \tau(b)$ and hence we have

$$c(1) + c(3) + c(5) + \dots + c(2k-3) + c(2k-1)$$

$$= b(2) + (b(2) + b(4)) + (b(4) + b(6)) + \dots + (b(2k-4) + b(2k-2)) + b(2k-2)$$

$$= 0.$$

Conversely assume $c(1) + c(3) + c(5) + \dots + c(2k-3) + c(2k-1) = 0$. Define $b = (b(1), b(2), \dots, b(2k-1)) \in V(m-1)$ as follows.

$b(1)$ is arbitrary,

$b(2) = c(1)$,

$b(i+2) = c(i+1) + b(i)\ (1 \leqq i \leqq 2k-3)$.

Then $c = \tau(b)$ follows from $b(2k-2) = c(2k-3) + c(2k-5) + \dots + c(3) + c(1) = c(2k-1)$.

(b) Let $m = 4k + 2\ (k \geqq 1)$. Assume $c = (c(1), c(2), \dots, c(4k+1)) \in \mathrm{Im}\ \tau$. Then one can construct $b = (b(1), b(2), \dots, b(4k+1)) \in V(m-1)$ with $c = \tau(b)$ by the method shown in the proof of (a). But $b(1)$ is arbitrary so one can put $b(1) = c(4) + c(8) + \dots + c(4k)$. Then it follows that

$$b(1) + b(3) + \dots + b(4k-1) + b(4k+1) = b(1) + c(4) + c(8) + \dots + c(4k) = 0$$

and hence $b \in \mathrm{Im}\ \tau$, which proves $\mathrm{Im}\ \tau = \mathrm{Im}\ \tau^2$. □

The *inner product* $< c, b >$ of two configurations $c = (c(1), c(2), \dots, c(m-1))$, $b = (b(1), b(2), \dots, b(m-1)) \in V(m-1)$ is defined as usual by

$$< c, b > = c(1)\, b(1) + c(2)\, b(2) + \dots + c(m-1)\, b(m-1).$$

PROPOSITION 1.3. *For each configuration, $c, b \in V(m-1)$ of ca−90(m), $< \tau(c), b > = < c, \tau(b) >$ holds.*

PROOF. It is obvious since τ is represented by a symmetric matrix

$$\begin{pmatrix} 0 & 1 & & & & & 0 \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & 1 & \\ 0 & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}. \quad \square$$

Similarly to Wolfram [12] we will reformulate cellular automata *ca−90(m)* making use of so-called Laurent polynomials over a finite field. This method is essentially important for the later computations of this paper.

Let m be a positive interger $\geqq 2$. Consider the quotient ring $F_2[x]/(x^{2m}-1)$ of the polynomial ring $F_2[x]$ over $F_2 = \{0, 1\}$ by the ideal $(x^{2m}-1)$ generated by a polynomial $x^{2m}-1$. We define elements $t(k) \in F_2[x]/(x^{2m}-1)$ for each integer k by $t(k) = x^k + x^{-k}$. In particular we put $t = x + x^{-1}$ ($= t(1)$).

PROPOSITION 1.4. (*The fundamental properties of* $t(k)$) *In the quotient ring* $F_2[x]/(x^{2m}-1)$ *the following formulae are valid for each integers* i, j, k *and each nonnegative integer* p.

(1) $t(0) = t(m) = 0$

(2) $t^{2^p} = t(2^p)$, $t(k)^{2^p} = t(2^p k)$

(3) $t(i)\, t(j) = t(i+j) + t(i-j)$

(4) $t(-k) = t(k)$

(5) $t(2m+k) = t(k)$

(6) $t(m+k) = t(m-k)$

PROOF. (1) easily follows from $t(0) = 1 + 1 = 0$ and $t(m) = x^m + x^{-m} = 0$ since $x^m = x^{-m}$ by $x^{2m} = 1$. (2) follows from $t(k)^2 = (x^k + x^{-k})^2 = x^{2k} + x^{-2k} = t(2k)$ and induction. (3) follows from $t(i)\, t(j) = (x^i + x^{-i})(x^j + x^{-j}) = x^{i+j} + x^{-i-j} + x^{i-j} + x^{j-i} = t(i+j) + t(i-j)$. (4) follows from $t(i) + t(-i) = t(0)\, t(2i) = 0$ by (3) and (1). (5) follows from $t(2m+k) + t(k) = t(m)\, t(m+k) = 0$ by (3) and (1). (6) follows from $t(m+k) + t(m-k) = t(m)\, t(k) = 0$ by (3) and (1). □

Now we imbed the cellular automaton $ca-90(m)$ into the quotient ring $F_2[x]/(x^{2m}-1)$. That is, we assign an element

$$\xi(c) = \Sigma_{j=1}^{m-1} c(j)t(j)$$

of the quotient ring $F_2[x]/(x^{2m}-1)$ to each configuration $c = (c(1), c(2), \ldots, c(m-1)) \in V(m-1)$ of $ca-90(m)$. The transformation ξ is clearly an injective linear transformation and we have

$$\begin{aligned} t\,\xi(c) &= \Sigma_{j=1}^{m-1} c(j)\, t(j)\, t(1) = \Sigma_{j=1}^{m-1} c(j)\, [t(j+1) + t(j-1)] \\ &= \Sigma_{j=1}^{m-1} c(j-1)\, t(j) + \Sigma_{j=1}^{m-2} c(j+1)\, t(j) \\ &= c(2)\, t(1) + \Sigma_{j=2}^{m-2} [c(j-1) + c(j+1)]\, t(j) + c(m-2)\, t(m-1) \\ &= \xi(\tau(c)), \end{aligned}$$

which shows that $ca-90(m)$ can be simulated by ξ. Therefore one can identify the configuration space $V(m-1)$ of $ca-90(m)$ with Im ξ and a configuration $c = (c(1), c(2), \ldots, c(m-1))$ with $\xi(c)$. We denote Im ξ by $W_m[x]$, that is,

$$W_m[x] = \{\Sigma_{j=1}^{m-1} c(j)\, t(j) \mid c(1), c(2), \ldots, c(m-1) \in F_2\}.$$

Therefore we have

$$c = (c(1), c(2), \ldots, c(m-1)) = \Sigma_{j=1}^{m-1} c(j)\, t(j),\ \tau(c) = tc = (x + x^{-1})\, c$$

for each $c \in W_m[x]$.

The following proposition is concerning on the kernel and fixed points of the global transition function of $ca-90(m)$.

PROPOSITION 1.5. *Let* $\tau: W_m[x] \to W_m[x]$ *be the standard global transition function of* $ca-90(m)$.
(*a*) *If* m *is even* (*that is,* $m = 2k$ ($k \geqq 1$)), *then*

$$\mathrm{Ker}\ \tau = \{0, \Sigma_{j=1}^{k} t(2j-1)\}.$$

(*b*) τ *has a unique nonzero fixed point if and only if* $3|m$. *In the case of* $m = 3k$ ($k \geqq 1$) *the unique nonzero fixed point is*

$$\Sigma_{j=1}^{k} [t(3j-2) + t(3j-1)].$$

PROOF. (a) In the case of $m = 2$ the assertion is trivial. Let $m = 2k$ ($k \geqq 2$) and assume that $\tau(c) = 0$ for $c = (c(1), c(2), \ldots, c(m-1)) \in W_m[x]$. Then it follows that $c(2) = c(4) = \ldots = c(2k-2) = 0$ and $c(1) = c(3) = \ldots = c(2k-1)$. Therefore $c(1) = 0$ implies $c = 0$, and $c(1) = 1$ implies $c = (1, 0, 1, 0, \ldots, 0, 1)$.
(b) Let $m = 3k$ ($k \geqq 1$) and assume that $\tau(c) = c$ for $c = (c(1), c(2), \ldots, c(m-1)) \in W_m[x]$. Then it follows that if $3k - 4 \leqq m - 2$ then $c(1) = c(2) = c(4) = c(5) = \ldots = c(3k-5) = c(3k-4)$ and $c(3) = c(6) = \ldots = c(3k-3) = 0$. Hence in case of $m = 3k - 2$ ($3k - 4 = m - 2$) we have also $c(m-1) = c(3k-3) = c(3k-4)$ and so $c = 0$. In case of $m = 3k - 1$ ($3k - 4 = m - 3$) we have $c(m-2) = c(3k-3) = c(3k-4) + c(3k-2)$ and $c(m-1) = c(3k-2) = c(3k-3)$ and so $c = 0$. In case of $m = 3k$ ($3k - 4 = m - 4$) we have $c(m-3) = c(3k-3) = c(3k-4) + c(3k-2)$, $c(m-2) = c(3k-2) = c(3k-3) + c(3k-1)$ and $c(m-1) = c(3k-1) = c(3k-2)$ and so $c(3k-4) = c(3k-2) = c(3k-1)$. Hence $c(1) = 0$ implies $c = 0$ and $c(1) = 1$ implies

$$c = \Sigma_{j=1}^{k} [t(3j-2) + t(3j-1)]. \quad \square$$

2. Existence of Characteristic Numbers

This section proves the existence and non-existence theorem of characteristic numbers associated with cellular automata $ca-90(m)$. Consequently it is shown that Huzino's conjectures [2–5] was the case. We begin with a review of characteristic numbers associated with cellular automata $ca-90(m)$ introduced by Huzino (Cf. Introduction).

Let m be a integer $\geqq 2$ and let α be a fixed configuration $t(1) + t(m-1) \in W_m[x]$ of a cellular automaton $ca-90(m)$. *The characteristic number* $H(m)$ *associated with* $ca-90(m)$ is the smallest positive integer n satisfying

$$\Sigma_{j=0}^{n-1} \tau^j(\alpha) = 0$$

in $ca-90(m)$ if such n exists. (Note that $h(m) = H(m+1)$ ($m \geqq 1$), where $h(m)$ was defined in Introduction.) We now consider another characteristic number associated with $ca-90(m)$ related to $H(m)$. *The characteristic number* $K(m)$ associated with $ca-90(m)$ is defined to be the smallest positive integer n satisfying $\tau^n(\alpha) = \alpha$ (or equivalently $t^n\alpha = \alpha$) if such n exists. It is easily seen that $K(m)$ exists if and only if α

is in a limit cycle of configurations and that $K(m)$ is the length of the limit cycle. The characteristic numbers associated with $ca-90(m)$ defined above does not exist for each integer $m \geqq 2$. For example, $t\alpha = 0$ in $ca-90(4)$. We now state the basic relationship between $H(m)$ and $K(m)$.

Let n be a positive integer. Define $\chi(n) = \Sigma_{j=0}^{n-1} t^j\alpha$ in $ca-90(m)$. Then we have

$$\chi(n) - t\chi(n) = \Sigma_{j=0}^{n-1} t^j\alpha - \Sigma_{j=0}^{n-1} t^{j+1} \alpha = \alpha - t^n\alpha.$$

Thus $\chi(n) = 0$ implies $t^n\alpha = \alpha$. Conversely $t^n\alpha = \alpha$ implies $\chi(n) = t\chi(n)$, which indicates that $\chi(n)$ is a fixed point of τ but we cannot conclude that $\chi(n) = 0$ (Cf. 1.5 (b)). However, when $t^n\alpha = \alpha$ holds, it follows that

$$\chi(2n) = \Sigma_{j=0}^{2n-1} t^j\alpha = \chi(n) + \chi(n) = 0$$

since the addition is of modulo 2. Therefore the existence of a positive integer n with $\chi(n) = 0$ is equivalent to the existence of a positive integer n with $t^n\alpha = \alpha$ in $ca-90(m)$.

Let us continue the above consideration. Assume that the characteristic number $H(m)$ (or equivalently $K(m)$) does exist and put $h = H(m)$, $k = K(m)$.

1°) In the case of $\chi(k) = 0$. Then it follows that $\chi(n) \neq 0$ for each integer n with $0 < n < k$. For $\chi(n) = 0$ for some n with $0 < n < k$ implies $t^n\alpha = \alpha$ (from the previous argument), which contradicts to the minimality of $k = K(m)$. Hence $h = k$, that is, $H(m) = K(m)$.

2°) In the case of $\chi(k) \neq 0$. Then it follows that $\chi(2k) = 0$ (from the previous argument) and $\chi(n) \neq 0$ for all n with $0 < n < 2k$. For $\chi(n) = 0$ for some n with $0 < n < 2k$ implies $t^n\alpha = \alpha$, which shows that n is a multiple of k, that is, $n = k$ and this contradicts to $\chi(k) \neq 0$. Hence $h = 2k$, that is, $H(m) = 2K(m)$.

To summarize the above arguments we have the following theorem.

THEOREM 2.1. *The existence of characteristic numbers $H(m)$ associated with ca$-$90(m) is equivalent to the existence of characteristic numbers $H(m)$ associated with ca$-$90(m). Moreover if these characteristic numbers $H(m)$ and $K(m)$ exist, then $\chi(K(m)) = 0$ implies $H(m) = K(m)$ and $\chi(K(m)) \neq 0$ implies $H(m) = 2K(m)$.* □

THEOREM 2.2. *The characteristic numbers $H(m)$ and $K(m)$ associated with ca$-$90(m) exist if m is not a multiple of 4.*

PROOF. Assume that m is odd. Then it follows from 1.1 that the global tradition function τ: $W_m[x] \to W_m[x]$ of $ca-90(m)$ is bijective and hence there exists a positive integer n such that $\tau^n(\alpha) = \alpha$ because of the finiteness of $W_m[x]$. In case of $m = 2$, we clearly have $\alpha = t(1) + \mathrm{t}(1) = 0$ and $K(2) = 1$. Next assume that $m = 4k + 2$ for some integer $k \leqq 1$. Again by the finiteness there are nonnegative integer p, q with $0 \leqq p < q$ satisfying $\tau^p = \tau^q$. But $\alpha \in \mathrm{Im}\, \tau$ from 1.2(a) and so $\alpha = \tau^q(v)$ for some $v \in W_m[x]$ by the virtue of 1.2(b). Hence we have

$$\alpha = \tau^q(v) = \tau^{q-p} \cdot \tau^p(v) = \tau^{q-p} \cdot \tau^q(v) = \tau^{q-p}(\alpha). \quad \square$$

The proof of the last theorem suggesting that the existence of the characteristic numbers $H(m)$ and $K(m)$ associated with $ca-90(m)$ is closely related to the structure of images of the global tradition function τ: $W_m[x] \to W_m[x]$. Now we need the following definitions. In $ca-90(m)$ we denote the set $\cap_{n=1}^{\infty}$ Im τ^n by Im τ^{∞}, and $\cup_{n=1}^{\infty}$ Ker τ^n by Ker τ^{∞}, respectively. By the virtue of the finiteness of $W_m[x]$ there is a positive integer n such that Im τ^n = Im τ^{n+1} and Im τ^n = Im τ^{∞}. Similarly there is a positive integer n such that Ker τ^n = Ker τ^{n+1} and Ker τ^n = Ker τ^{∞}.

LEMMA 2.3. *The characteristic numbers $H(m)$ and $K(m)$ associated with $ca-90(m)$ exist if and only if $\alpha = t(1) + t(m-1) \in$ Im τ^{∞}.*

PROOF. It is obvious that $\tau^k(\alpha) = \alpha$ for some positive integer k implies $\alpha \in$ Im τ^{∞}. By the virtue of the finiteness of $W_m[x]$ there exist integers p, q $(0 \leqq p < q)$ satisfying $\tau^p = \tau^q$. Conversely assume that $\alpha \in$ Im τ^{∞}. Then there exists a configuration $v \in W_m[x]$ such that $\alpha = \tau^q(v)$ and hence $\alpha = \tau^q(v) = \tau^{q-p} \cdot \tau^p(v) = \tau^{q-p} \cdot \tau^q(v) = \tau^{q-p}(\alpha)$. □

LEMMA 2.4. *Let m be an even integer $\geqq 2$ and let $\kappa \in W_m[x]$ be a configuration of $ca-90(m)$ such that $\tau^{n-1}(\kappa) \neq 0$ and $\tau^n(\kappa) = 0$ $(n \geqq 1)$. Then $c \in$ Im τ^i for an integer i with $1 \leqq i \leqq n$ if and only if $< c, \tau^{n-j}(\kappa) > = 0$ for all integer j with $1 \leqq j \leqq i$.*

PROOF. Assume that $c \in$ Im τ^i for some integer i with $1 \leqq i \leqq n$. Then there exists $b \in W_m[x]$ such that $c = \tau^i(b)$ and so using 1.3 we have

$$< c, \tau^{n-j}(\kappa) > = < \tau^i(b), \tau^{n-j}(\kappa) > = < b, \tau^{n-j+i}(\kappa) >$$
$$= < b, 0 > = 0.$$

for all integer j with $1 \leqq j \leqq i$. The converse will be shown by induction on i. In case of $i = 1$ we have $\tau^{n-1}(\kappa) = t(1) + t(3) + t(5) + \dots + t(m-1)$ by 1.5(a) and $< c, \tau^{n-1}(\kappa) > = 0$ implies $c(1) + c(3) + c(5) + \dots + t(2m-1) = 0$ and $c \in$ Im τ from 1.2(a). Next assume that $i + 1 \leqq n$ and $< c, \tau^{n-j}(\kappa)) > = 0$ for all j with $1 \leqq j \leqq i + 1$. Then it follows from the induction hypothesis that $c \in$ Im τ^i and so $c = \tau^i(c')$ for some $c' \in W_m[x]$. But from the condition $< c, \tau^{n-1}(\kappa) > = 0$ we have

$$< c', \tau^{n-1}(\kappa) > = < \tau^i(c'), \tau^{n-i-1}(\kappa) > = < c, \tau^{n-i-t}(\kappa) > = 0$$

and hence $c' \in$ Im τ using 1.2(a) once more. This proves $c \in$ Im τ^{i+1}. □

COROLLARY 2.5. *Let m be an even integer $\geqq 2$ and let $\kappa \in W_m[x]$ be a configuration of $ca-90(m)$* such that $\tau^{n-1}(\kappa) \neq 0$ *and $\tau^n(\kappa) = 0$ $(n \geqq 1)$. Then $c \in$ Im τ^n if and only if $< c, \tau^j(\kappa) > = 0$ for all integer j with $0 \leqq j \leqq n-1$.* □

LEMMA 2.6. *Let m be an even integer $\geqq 2$ and let $\kappa \in W_m[x]$ be a configuration of $ca-90(m)$ such that $\tau^{n-1}(\kappa) \neq 0$, $\tau^n(\kappa) = 0$ $(n \geqq 1)$ and $\kappa \notin$ Im τ.
Then*

(*a*) $\tau^{n-1}(\kappa)$, $\tau^{n-2}(\kappa)$, ..., $\tau(\kappa)$, κ *is a basis of Ker τ^n over F_2.*
(*b*) *If $\tau^{n-1}(c) \neq 0$ and $\tau^n(c) = 0$, then $c \notin$ Im τ.*
(*c*) Im τ^n = Im τ^{n+1} (= Im τ^{∞})
(*d*) Ker τ^n = Ker τ^{n+1} (= Ker τ^{∞})

(*e*) Ker $\tau^n \oplus$ Im $\tau^n = W_m[x]$

PROOF. (a) Let i be an integer with $1 \leqq i \leqq n$. By induction on i we show that $\tau^{n-1}(\kappa), \tau^{n-2}(\kappa), \ldots, \tau^{n-i}(\kappa)$ is a basis of Ker τ^i over F_2. When $i = 1$ it is trivial. Assume that $c \in$ Ker τ^{j+1} $(1 \leqq i \leqq n-1)$. Then $\tau(c) \in$ Ker τ^i and so by the induction hypothesis

$$\begin{aligned}\tau(c) &= \Sigma_{j=1}^{i} c'(j)\, \tau^{n-j}(\kappa)\\ &= \tau\, (\Sigma_{j=1}^{i} c'(j)\, \tau^{n-j-1}(\kappa)).\end{aligned}$$

Hence we have

$$c = \Sigma_{j=1}^{i} c'(j)\, \tau^{n-j-1}(\kappa) \ or\ c = \tau^{n-1}(\kappa) + \Sigma_{j=1}^{i} c'(j)\, \tau^{n-j-1}(\kappa).$$

Finally we show that $\tau^{n-1}(\kappa), \tau^{n-2}(\kappa), \ldots, \tau^{n}(\kappa), \kappa$ are linearly independent. Assume

$$\Sigma_{j=1}^{n} c'(j)\, \tau^{n-j}(\kappa) = 0\ (c'(j) \in F_2\ for\ all\ j).$$

Applying τ^{n-1} to the both sides of the above equation we obtain $c'(n) = 0$ because $\tau^n(\kappa) = 0$ and $\tau^{n-1}(\kappa)) \neq 0$. Next $c'(n-1) = 0$ follows from applying τ^{n-2} to the equation. Continuing these operations $c'(1) = c'(2) = \ldots = c'(n) = 0$ are obtained.
(b) Assume $c \in$ Ker τ^n and $\tau^{n-1}(c) \neq 0$. Using (a) c can be represented as follows:

$$c = \Sigma_{j=1}^{n} c'(j)\, \tau^{n-j}(\kappa).$$

It is obvious that $c'(n) = 1$ since $c'(n)\tau^{n-1}(\kappa)) = \tau^{n-1}(c) \neq 0$.
Therefore we have

$$\begin{aligned}c &= \kappa + \Sigma_{j=1}^{n-1} c'(j)\, \tau^{n-j}(\kappa)\\ &= \kappa + \tau\, (\Sigma_{j=1}^{n-1} c'(j)\, \tau^{n-j-1}(\kappa))\end{aligned}$$

and so $c \notin$ Im τ by $\kappa \notin$ Im τ.
(c) First assume that $c \in$ Im τ^n and $c \notin$ Im τ. Then there is some positive integer s with $\tau^{s-1}(c) \notin$ Im τ, $\tau^s(c) \in$ Im τ and we can choose u and v such that $c = \tau^n(u)$, $\tau^{n+s}(v) = \tau^s(c)$ and $v \in$ Im τ. Thus it follows from $\tau^{n+s-1}(u) = \tau^{s-1}(c) \notin$ Im τ and $\tau^{n+s-1}(v) \in$ Im τ that $\tau^n(\tau^s(u) - \tau^s(v)) = 0$ and $\tau^{n-1}(\tau^s(u) - \tau^s(v)) \neq 0$. Hence by (b) we can conclude $\tau^s(u) - \tau^s(v) \notin$ Im τ, which is a contradiction.
(d) Assume $\tau^{n+1}(c) = 0$. Then $\tau^{n+1}(c) = \tau^{n-1}(\tau(c)) = 0$. Applying (b) we obtain $\tau^n(c)) = \tau^{n-1}(\tau(c)) = 0$.
(e) Let $c \in W_m[x]$. As $\tau^n(c) \in$ Im τ^∞ by (c), $\tau^n(c) = \tau^n(u)$ for some $u \in$ Im τ^n. Thus $c = u + v$ for some $v \in$ Ker τ^n. Next we show uniqueness of this decomposition. Assume that $w \in$ (Ker τ^n)$\cap$ (Im τ^n) and $w \neq 0$. Then there exists $w' \in W_m[x]$ with $w = \tau^n(w')$ and there exists a positive integer s with $\tau^{s-1}(w) \neq 0$ and $\tau^s[w] = 0$. Since $\tau^n(\tau^s(w')) = 0$ and $\tau^{n-1}(\tau^s(w')) \neq 0$ it follows from (b) that $\tau^s(w') \notin$ Im τ, which is a contradiction. □

The following lemma gives a particular choice of configurations $\kappa \in W_m[x]$ satisfying the conditions in 2.4–2.6.

Proposition 2.7. *Let* $m = 2^p(2q - 1)$ *for integers* $p \geqq 1$ *and* $q \geqq 1$ *and put* $n = 2^p - 1$. *A configuration* $\kappa = t(1) + \Sigma_{j=1}^{q-1} [t(2^{p+1} j - 1)) + t(2^{p+1} j + 1)] \in W_m[x]$ *satisfies* $\tau^{n-1}(\kappa) \neq 0$, $\tau^n(\kappa) = 0$ *and* $\kappa \notin \mathrm{Im}\, \tau$. □

Theorem 2.8. *The characteristic numbers* $H(m)$ *and* $K(m)$ *associated with ca–90(m) do not exist if* m *is a multiple of* 4.

Proof. Assume that $4|m$, that is, $m = 2^p(2q - 1)$ $(p \geqq 2,\ q \geqq 1)$. Put $n = 2^p - 1$. As $2^{p+1}(q - 1) + 1 < m - 1$ $(2^p(2q - 1) - 1 - (2^{p+1}(q - 1) + 1) = 2^p - 2)$ we have $< \alpha, \kappa > = 1 \neq 0$, where $\alpha = t(1) + t(m - 1)$ and κ is a particular configuration given in 2.7. Therefore $c \in \mathrm{Im}\, \tau^n$ by 2.5 and so $H(m)$ and $K(m)$ do not exist from 2.3. □

3. Further Properties of the Characteristic Numbers

In this section we will state further properties of the characteristic numbers $H(m)$ and $K(m)$ associated with *ca–90(m)*. The entire relationship between $H(m)$ and $K(m)$ will be given. All the results stated in the section will be useful for the further investigations of higher dimensional cellular automata with rule 90.

The reverse transformation $\# : W_m[x] \to W_m[x]$ is a linear transformation such that $\#(t(j)) = t(m - j)$ for all j $(1 \leqq j \leqq m - 1)$. In particular $\#(t(1)) = t(m - 1)$.

Theorem 3.1. *For* $m = 2k + 1$ $(k \geqq 1)$ *the following four statements are equivalent in ca–90(m)*:

(*a*) $t^{n+1} = t(m - 1)$

(*b*) $\tau^n = \#$ (*the reverse transformation*)

(*c*) n *is odd and* $t^n[t(1) + t(m - 1)] = t(1) + t(m - 1)$

(*d*) $t^n = \Sigma_{j=1}^{k} t(2j - 1)$

Proof. It is trivial that (b) ⇒ (a) ⇒ (c) and (d) ⇒ (a). It suffices to show (a) ⇒ (b) and (c) ⇒ (d). At first we prove (a) ⇒ (b). Assume that $t^{n+1} = t(m - 1)$. Then it follows that

$$\tau^{n+1}(t(j)) = t^{n+1}t(j) = t(m - 1)\, t(j) = t(m - 1 + j) + t(m - 1 - j)$$

$$= t(m + 1 - j) + t(m - 1 - j) = t(1)\, t(m - j) = \tau(t(m - j))$$

for each j $(1 \leqq j \leqq m - 1)$. Since m is odd and τ is bijective we obtain $\tau^n = \#$. Next we prove (c) ⇒ (d). Assume that

$$t^n = (x + x^{-1})^n = \Sigma_{j=2}^{m-1} \alpha(j)\, t(j) \quad (\alpha(j) = 0 \text{ or } 1).$$

Because of $t^n[t(1) + t(m - 1)] = t(1) + t(m - 1)$ we have

$$t(1) + t(m - 1) = [t(1) + t(m - 1)]\, \Sigma_{j=1}^{m-1} \alpha(j)\, t(j)$$

$$= [\alpha(2) + \alpha(2k - 1)]\, t(1) + \Sigma_{j=2}^{m-2} [\alpha(j - 1) + \alpha(j + 1) + \alpha(2k - j + 2) + \alpha(2k - j)]\, t(j) + [\alpha(2k - 1) + \alpha(2)]\, t(m - 1)$$

and so

$$\alpha(2) + \alpha(2k - 1) = 1,$$

$$\alpha(j - 1) + \alpha(j + 1) + \alpha(2k - j + 2) + \alpha(2k - j) = 0 \ (1 \leqq j \leqq m - 2).$$

But n is odd so $\alpha(2i) = 0$ $(1 \leqq i \leqq k)$. Hence $\alpha(1) = \alpha(3) = \ldots = \alpha(2k - 1) = 1$ is obtained by the last system of equations. □

The following lemma gives an upper bound of the characteristic numbers $H(m)$ and $K(m)$ associated with $ca-90(m)$. However it may be very difficult to determine exactly these characteristic numbers as stated in Introduction.

LEMMA 3.2. *Let m be an odd integer $\geqq 3$. If $2^u = \pm 1 \pmod m$, then $t^{2^u} = t(m - 1)$.*

PROOF. Since m is odd and $2^u = \pm 1 \pmod m$, there exists an integer s such that $2^u = 2ms + m \pm 1$. Hence it follows that

$$t^{2^u} = t(2^u) = t(2ms + m \pm 1) = t(m \pm 1) = t(m - 1). \quad □$$

For each odd integer m the existence of positive integer u such that $2^u = \pm 1$ (mod m) is given from Fermat-Euler theorem asserting that $(a, m) = 1$ implies $a^{\varphi(m)} = 1$ (mod m). Where $\varphi(m)$ is *Euler's function*, that is, $\varphi(m)$ denotes the number of positive integer k satisfying $1 \leqq k \leqq m$ *and* $(k, m) = 1$. *The multiplicative suborder u of* 2 *modulo m* is the smallest positive number satisfying $2^u = \pm 1$ (mod m) and it will be denoted by *sord* (2; m).

COROLLARY 3.3. *If m is an odd integer $\geqq 3$, then $K(m)|2^{sord(2;m)} - 1$ in $ca-90(m)$.*

PROOF. It is trivial from 3.1 and 3.2. □

As log as I know from computer experiments, $K(m) = 2^{\mathrm{sord}(2;m)} - 1$ holds for almost odd integer m $(3 \leqq m \leqq 299)$ (Cf. Appendix – Table of $K(m)$ $(3 \leqq m \leqq 299)$ – at the end of the paper). For example, $K(37) = (2^\wedge 18-1)/3$, $K(95) = (2^\wedge 36-1)/3$, $K(101) = (2^\wedge 50-1)/3$, $K(199) = (2^\wedge 99-1)/7$ and $K(203) = (2^\wedge 84-1)/105$ are counter-examples. However the author does not know when the characteristic number $K(m)$ associated with $ca-90(m)$ is a proper divisor of $2^{\mathrm{sord}(2;m)} - 1$ for odd m.

The following statement concerns on an lower bound of the characteristic number $K(m)$ associated with $ca-90(m)$.

PROPOSITION 3.4. *If m is an odd integer $\geqq 3$, then $K(m) \geqq m - 2$.*

PROOF. Assume that m is an odd integer $\geqq 3$. Then, by using induction, one can easily show that

$$t^n = t(n) + \Sigma_{j=1}^{n-1} \alpha(j)\, t(j) \quad (\alpha(j) = 0, 1).$$

for $1 \leqq n \leqq m - 1$ holds in $ca-90(m)$. Thus we have $K(m) \geqq m - 2$ from 3.1. □

The following corollary determines the characteristic number $K(m)$ for two particular cases as an immediate result from 3.3 and 3.4.

COROLLARY 3.5.
(a) $K(2^p - 1) = 2^p - 1$ $(p \geqq 3)$,
(b) $K(2^p + 1) = 2^p - 1$ $(p \geqq 1)$. □

The lower bound of $K(m)$ for odd integers m given in 3.4 is a best possible one in general because of 3.5(b). The next theorem indicates that it suffices to compute the characteristic number $K(m)$ only for odd integers m.

THEOREM 3.6. *If m is an odd integer $\geqq 3$, then $K(2m) = 2K(m)$.*

PROOF. Put $n = K(m)$ (which exists by 2.2). As τ ($= x + x^{-1} = t$) is bijective (since m is odd) and $\alpha = t(1) + t(m-1) = (x + x^{-1})\ [\Sigma_{j=1}^{m-1} t(i)]$ we have

$$(x + x^{-1})^{n+1}\ [\Sigma_{i=1}^{m-1} t(i)] = (x + x^{-1})\ [\Sigma_{i=1}^{m-1} t(i)]$$

in $ca{-}90(m)$. Substituting x^2 for x into the last equation

$$(x + x^{-1})^{2n}\ [\Sigma_{i=1}^{m-1} t(2i)] = [\Sigma_{i=1}^{m-1} t(2i)]$$

holds in $ca{-}90(2m)$ and multipling $t = x + x^{-1}$ on the both sides we have

$$(x + x^{-1})^{2n}\ [t(1) + t(2m-1)] = [t\ (1) + t(2m-1)]\ (= \alpha)$$

in $ca{-}90(2m)$. This proves that $K(2m) \mid 2K(m)$. Finally we must show the converse. It can be seen without difficulty that $K(2m)$ is even. So we can put $K(2m) = 2n$. Then an equation

$$(x + x^{-1})^{n+1}\ [\Sigma_{i=1}^{m-1} t(i)] = (x + x^{-1})\ [\Sigma_{i=1}^{m-1} t(2i)]$$

in $ca{-}90(2m)$ follows from $(x + x^{-1})^{2n}\ [t(1) + t(2m-1)] = [t(1) + t(2m-1)]$ in $ca{-}90(m)$. But it is easily seen that an equation

$$(x + x^{-1})^{n+1}\ [\Sigma_{j=1}^{m-1} t(i)] = (x + x^{-1})\ [\Sigma_{i=1}^{m-1} t(i)]$$

holds in $ca{-}90(m)$ from a basic property of division algorithm for polynomials asserting that if $f(x^2) = q(x)\ g(x^2) + r(x)$ and $deg\ r(x) < deg\ g(x^2)$ then $q(x)$ and $r(x)$ are also polynomials of x^2. At last we have $t^n\ \alpha = \alpha$ in $ca{-}90(m)$ by the injectibility of τ of $ca{-}90(m)$, which shows that $K(m)|K(2m)/2$. The proof is complete. □

COROLLARY 3.7.
(a) $K(2^p - 2) = 2^p - 2\ (p \geqq 4)$
(b) $K(2^p + 2) = 2^p - 2\ (p \geqq 2)$

PROOF. It is immediate from 3.5 and 3.6. □

The following three theorems show entire relationships between the characteristic numbers $H(m)$ and $K(m)$ associated with $ca{-}90(m)$ (Cf. 2.1). Hence all the characteristic numbers $H(m)$, $K(2m)$ and $H(2m)$ can be computable from the characteristic number $K(m)$ for odd integer m.

THEOREM 3.8. *If m is not a multiple of 3 and not a multiple of 4, that is, if $3 \nmid m$ and $4 \nmid m$, then $H(m) = K(m)$.*

PROOF. If m is not a multiple of 3, then τ does not have nonzero fixed points by 1.5(b) and so we have $\chi(n) = 0$ (Cf. the argument at the beginning of the section 2). Therefore $H(m) = K(m)$ follows from 2.2. □

THEOREM 3.9. *If $m = 3\ (2q + 1)\ (q \geqq 1)$, then $H(m) = 2K(m)$.*

PROOF. By the birtue of 2.1 it suffices to prove that $\chi\ (K(m)) \neq 0$. Put u = sord

(2; m). Then $2^u = 2ms + m \pm 1$ for some s and $K(m)|2^u - 1$. Since $(2^u - 1)/K(m)$ is obviously odd, one can not that $\chi(K(m)) = \chi(2^u - 1)$. Hence it suffices to prove that $\chi(2^u - 1) \neq 0$. Put $\alpha = t(1) + t(m - 1)$,

$$v = \Sigma_{i=1}^{2q} [t(3i - 1) + t(3i)] + t(6q + 2),$$

and

$$w = \Sigma_{i=1}^{2q+1} t(3i - 2).$$

Then $v = tv + \alpha$ and $v = tw$ are clearly valid. On the other hand

$$\chi(n) = \Sigma_{j=0}^{n-1} t^j \alpha$$

can be inductively defined by $\chi(1) = \alpha$ and $\chi(j + 1) = t\chi(j) + \alpha$ $(j \geqq 1)$. Noticing that $\chi(j + 1) = t(\chi(j) + v)$ it follows that

$$\chi(j) + v = t^{j-1}(\chi(1) + v) = t^{j-1}(\alpha + v) = t^j v = t^{j+1} w \ (j \geqq 1)$$

and so $\chi(2^u - 1) = t^{2^u} w + v$. Using this result we have

$$\begin{aligned} t^{2^u} w &= t(2^u) \Sigma_{i=1}^{2q+1} t(3i - 2) \\ &= \Sigma_{i=1}^{2q+1} [t(m - 1 + 3i - 2) + t(m - 1 - 3i + 2)] \qquad (\text{since } t(2^u) = t(m - 1)) \\ &= \Sigma_{i=1}^{2q+1} [t(m - 3i + 1) + t(m - 3i + 3)] \\ &= \Sigma_{i=1}^{2q+1} [t(6q - 3i + 4) + t(6q - 3i + 6)] \qquad (\text{since } m = 6q + 3) \\ &= \Sigma_{i=1}^{2q} [t(3i - 2) + t(3i)] + t(6q + 1) \end{aligned}$$

and finally

$$\begin{aligned} \chi(2^u - 1) &= \Sigma_{i=1}^{2q} [t(3i - 2) + t(3i - 1)] + t(6q + 1) + t(6q + 2) \\ &= \Sigma_{i=1}^{2q+1} [t(3i - 2) + t(3i - 1)], \end{aligned}$$

which shows that $\chi(2^u - 1)$ is a unique nonzero fixed point of τ (Cf. 1.5(b)). This completes the proof. □

THEOREM 3.10. *If* $m = 3(2q + 1)$ $(q \geqq 1)$, *then* $H(2m) = 2K(2m)$.

PROOF. Put $k = K(m)$. It suffices to show that $\chi(K(2m)) = \chi(2k) \neq 0$. In the proof of 3.9 we have proved that

$$\chi(k) = \Sigma_{j=0}^{k-1} t^j [t(1) + t(2) + \ldots + t(m - 1)] = \Sigma_{i=0}^{2q} [t(3i + 1) + t(3i + 2)]$$

(nonzero fixed point of τ of $ca-90(m)$)

holds in $W_m[x]$. Substituting x^2 for x in the last equation

$$\Sigma_{j=0}^{k-1} t^{2j} [t(2) + t(4) + \ldots + t(2m - 2)] = \Sigma_{i=0}^{2q} [t(6i + 2) + t(6i + 4)]$$

holds in $W_{2m}[x]$. On the other hand

$$\begin{aligned} \chi(2k) &= \Sigma_{j=0}^{2k-1} t^j [t(2) + t(4) + \ldots + t(2m - 2)] \\ &= (1 + t)(\Sigma_{j=0}^{k-1} t^{2j}) [t(2) + t(4) + \ldots + t(2m - 2)] \end{aligned}$$

in $W_{2m}[x]$ and using the above result we have

$$\chi(2k) = (1 + t)\ \Sigma_{j=0}^{2q}\ [t(6j + 2) + t(6j + 4)]$$
$$= \Sigma_{j=0}^{2q}\ [t(3j + 1) + t(3j + 2)] \neq 0,$$
$$(\text{nonzero fixed point of } \tau \text{ of } ca-90(2m))$$

which complete the proof. □

COROLLARY 3.11.
(*a*) *If p is an odd integer, then*

$$H(2^p - 1) = 2^p - 1 \quad (p \geqq 3), \qquad H(2^p - 2) = 2^{p+1} - 4 \quad (p \geqq 5),$$
$$H(2^p + 1) = 2^{p+1} - 2 \quad (p \geqq 1), \qquad H(2^p + 2) = 2^p - 2 \quad (p \geqq 3).$$

(*b*) *If p is an even integer, then*

$$H(2^p - 1) = 2^{p+1} - 2 \quad (p \geqq 4), \qquad H(2^p - 2) = 2^p - 2 \quad (p \geqq 4),$$
$$H(2^p + 1) = 2^p - 1 \quad (p \geqq 2), \qquad H(2^p + 2) = 2^{p+1} - 4 \quad (p \geqq 2).$$ □

Acknowledgement

The author would like to express his deep gratitude to Professor Seiiti Huzino, Kyushu University, for his very helpful suggestions throughout this work.

References

[1] CODD, E. F.: *Cellular automata*, Academic Press, (1968).
[2] HUZINO, S.: *On the behaviors of 1-dimensional cellular automata CA − 90(m)*, Research Report of Mathematics of Computation **62−05J**, Kyushu University, (1987). (in Japanese)
[3] HUZINO, S.: *On the behaviors of linearly sited bounded cellular automata CA − 90(m),* Research Report of Mathematics of Computation **63−03J**, Kyushu University, (1988). (in Japanese)
[4] HUZINO, S.: *The dynamical system associated with cellular automata CA − 90(m) (I) − Fixed Points −*, Research Report of Mathematics of Computation **63−04J**, Kyushu University, (1988). (in Japanese)
[5] HUZINO, S.: *The dynamical system associated with cellular automata CA − 90(m) (II) − Existence of characteristic numbers h(m) −*, Research Report of Mathematics of Computation **63−05J**, Kyushu University, (1988). (in Japanese)
[6] VON NEUMANN, J.: *The general and logical theory of automata*, in J. von Neumann, Collected Works, edited by A. H. Taub, **5**, 288, (1963).
[7] VON NEUMANN, J.: *Theory of self-reproducing automata*, edited by A. W. Burks, University of Illinois, Urbana, (1966).
[8] KITAGAWA, T: *Dynamical systems and operators associated with a single neuronic equation*, Math. Biosciences **18** (1973), 191−244.
[9] KITAGAWA, T.: *Cell space approaches in biomathematics*, Math. Biosciences **19** (1974), 27−71.
[10] MARTIN, O., OLDLYZKO, A. M. and WOLFRAM, S.: *Algebraic properties of cellular automata*, Comm. Math. Physics, **93** (1984), 219−258.
[11] PU-HUA GUAN and YU HE: *Exact results for deterministic cellular automata with additive rules*, J. Statist. Phys. **43** (1986), 463−478.

[12] WOLFRAM, S.: *Theory and applications of cellular automata*, World Scientific, Singapore, (1986).

Received October 4, 1990

Appendix

A table of the characteristic number $K(m)$ associated with ca−90(m) ($m = 10a+b$, $0 \leqq a \leqq 29$, $b = 1, 3, 5, 7, 9$)

a \ b	1	3	5	7	9
0		2^1−1	2^2−1	2^3−1	2^3−1
1	2^5−1	2^6−1	2^4−1	2^4−1	2^9−1
2	2^6−1	2^11−1	2^10−1	2^9−1	2^14−1
3	2^5−1	2^5−1	2^12−1	(2^18−1)/3	2^12−1
4	2^10−1	2^7−1	2^12−1	2^23−1	2^21−1
5	2^8−1	2^26−1	2^20−1	2^9−1	2^29−1
6	2^30−1	2^6−1	2^6−1	2^33−1	2^22−1
7	2^35−1	2^9−1	2^20−1	2^30−1	2^39−1
8	2^27−1	2^41−1	2^8−1	2^28−1	2^11−1
9	2^12−1	2^10−1	(2^36−1)/3	2^24−1	2^15−1
10	(2^50−1)/3	2^51−1	2^12−1	2^53−1	2^18−1
11	(2^36−1)/3	2^14−1	2^44−1	2^12−1	2^24−1
12	2^55−1	2^20−1	2^50−1	2^7−1	2^7−1
13	2^65−1	2^18−1	2^36−1	2^34−1	2^69−1
14	(2^46−1)/3	2^60−1	2^14−1	2^42−1	2^74−1
15	2^15−1	2^24−1	2^20−1	2^26−1	2^52−1
16	2^33−1	2^81−1	2^20−1	2^83−1	2^78−1
17	2^9−1	2^86−1	2^60−1	2^29−1	2^89−1
18	2^90−1	2^60−1	2^18−1	2^40−1	2^18−1
19	2^95−1	2^48−1	2^12−1	(2^98−1)/3	(2^99−1)/7
20	2^33−1	(2^84−1)/105	2^10−1	2^66−1	2^45−1
21	2^105−1	2^70−1	2^28−1	2^15−1	2^18−1
22	2^24−1	2^37−1	2^60−1	2^113−1(?)	2^38−1
23	2^30−1	2^29−1	2^92−1	2^78−1	2^119−1(?)
24	2^12−1	2^81−1	2^84−1	2^36−1	2^41−1
25	2^25−1	2^110−1	2^8−1	2^8−1	2^36−1
26	2^84−1	2^131−1(?)	2^26−1	2^22−1	(2^134−1)/3(?)
27	2^135−1(?)	2^12−1	2^20−1	2^46−1	2^30−1
28	2^35−1	2^47−1	2^36−1	2^60−1	2^68−1
29	2^48−1	2^146−1(?)	2^116−1(?)	2^45−1	2^132−1