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Basu, A. K.
University of Calcutta

Sahoo, D. K.
R. K. M. College

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RATE OF CONVERGENCE FOR NON PARAMETRIC DENSITY ESTIMATION IN LINEAR PROCESS

By

A. K. BASU* and D. K. SAHOO**

Abstract

Rate of convergence to normality for the density estimators of Kernel type is obtained when the observations are from a stationary linear processes. At first, the case of estimating the density at a fixed point is considered and latter on, it is extended for estimating joint density. Also the problem of estimating the density at several points is considered.

1. Introduction

Problem of estimating the unknown density function of a population have been considered by several authors. An excellent survey of the results is given in Rosenblatt [9] when the sample observations are independent. Attempts also have been made to extend the results to other than independent observations. Rosenblatt [8] considered the case where the sample observations are dependent in the sense that they are sampled from a stationary Markov sequence and also derived some interesting results about the Kernel type density estimators.

In this paper our aim is to extend these results for the random sequence $\{X_t\}_{t=\dots-1,0,1,\dots}$ when they form a stationary linear process. Most of the important stochastic process models, such as, Auto regressive schemes, Moving average schemes etc. are linear processes. In an unpublished paper, Kamal Chanda considered the problem of estimating the probability density function for the linear process and try to investigate the asymptotic properties like almost sure convergence and asymptotic normality. But unfortunately there are some mistakes in his calculations and also he does not consider the rate of convergence to normality. We exploit the techniques of Blume and Wittwer [2] and N. Kersten [5] and use some results of Ibragimov [4] to find the rate of convergence to normality of the estimated non parametric density function. In section 2 we consider the case of estimating the density at a fixed point and in section 3 we extend it for estimating joint density. Also in section 4 we consider the problem of estimating probability density at several points. Using different methods uniform rates of convergence to normality for non parametric density estimates have been obtained by Wertz [10], Prakasha Rao [6, 7], Basu and Sahoo [1] and their rates where at best of the order of $n^{-1/3+\gamma}$, $0<\gamma<1/3$. In this paper assuming quadratic

* University of Calcutta, India.
** R.K.M. College, Narendrapur, India.

Kernels and existence of higher moments we are able to get rate of convergence of the order of $n^{-1/2}(\log n)^{\alpha}$, $\alpha>0$. Our method can be used for polynomial Kernels and Kernels which can well approximated by polynomial Kernels.

2. Probability Density Estimate and Its Rate of Convergence to Normality

Let $\{X_t\}_{t=\cdots,-1,0,1,\cdots}$ be a stationary linear process defined by

$$X_t=\sum_{j=0}^{\infty}\alpha_j\varepsilon_{t-j} \tag{2.1}$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables such that $E(\varepsilon_t)=0$, $E(\varepsilon_t^2)=1$ and $\{\alpha_j\}_{j=0}^{\infty}$ is a sequence of real numbers with $\alpha_0\neq 0$ and $\sum_{j=0}^{\infty}|\alpha_j|<\infty$.

Suppose the process is observed at the points $t=1, 2, \cdots, N$ and X_1 has the pdf 'f'. We formally define the estimator $\hat{f}_N(x)$ of $f(x)$ as,

$$\hat{f}_N(x)=(1/N)\sum_{t=1}^{N}(1/h)K[(x-X_t)/h] \tag{2.2}$$

where $\{h=h_N\}$ is a sequence of real numbers such that $h_N\to 0$ but $Nh_N\to\infty$ as $N\to\infty$ (e.g. $h_N\propto N^{-1/5}$) and K is a Kernel function satisfying:

(i) $\sup_{-\infty<y<\infty} K(y)<\infty$

(ii) $\int_{-\infty}^{\infty}K(y)dy<\infty$, $\lim_{y\to\pm\infty} yK(y)=0$. (2.3)

We make the following additional assumptions:

(iii) if ϕ_0 denote the characteristic function of ε_1 then $\int_{-\infty}^{\infty}|u\phi_0(u)|du<\infty$

(iv) $E|\varepsilon_t|<\infty$

(v) $\sum_{j=N}^{\infty}|\alpha_j|\leqq C\cdot 2^{-\alpha N}/N^{1/2}$, where C is a generic symbol which denotes a positive finite constant, independent of N and α is a positive constant.

We set

$$\begin{aligned} f_t&=\frac{1}{\sqrt{h}}[K((x-X_{-t})/h)-EK((x-X_1)/h)]\\ &=f(\varepsilon_{-t}, \varepsilon_{-t-1}, \cdots) \end{aligned} \tag{2.4}$$

and $T_N=\sqrt{Nh}[\hat{f}_N-\bar{f}_N]$, where $\bar{f}_N=(1/h)EK((x-X_1)/h)$

$$\begin{aligned} &=(1/Nh)^{1/2}\sum_{t=1}^{N}[K((x-X_t)/h)-EK((x-X_1)/h)]\\ &=(1/Nh)^{1/2}\sum_{t=-N}^{-1}[K((x-X_{-t})/h)-EK((x-X_1)/h)]\\ &=(1/N)^{1/2}\sum_{t=-N}^{-1}f_t. \end{aligned} \tag{2.5}$$

In case the strict stationarity of $\{X_t\}$ is valid,

$$P\Big[\frac{1}{\sqrt{N}}\sum_{t=-N}^{-1} f_t<z\Big]=P\Big[(1/N)^{1/2}\sum_{t=1}^{N} f_t<z\Big].$$

We take a second degree polynomial Kernel $K(y)=a_0+a_1y+a_2y^2$. Such type of Kernels are widely used. In fact if we take $a_0=3/4\sqrt{5}$, $a_1=0$, $a_2=-3/20\sqrt{5}$ then we will get Epanechnikov [3] Kernel which is optimum in the sense that it minimizes integrated mean square error. So

$$K((x-X_{-t})/h)=A_0+A_1X_{-t}+A_2X_{-t}^2\,, \quad \text{where } A_0=a_0+a_1x/h+a_2x^2/h^2$$

$$A_1=-(a_1/h+2a_2x/h^2), \quad A_2=a_2/h^2\,.$$

Therefore,

$$\begin{aligned} f_t&=(1/h)^{1/2}[A_0+A_1X_{-t}+A_2X_{-t}^2-A_3], \quad \text{where } A_3=EK((x-X_1)/h)\\ &=(1/h)^{1/2}[A_1X_{-t}+A_2X_{-t}^2-A_4]\,, \qquad A_4=A_3-A_0\\ &=(1/h)^{1/2}[A_1\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-t-j}+A_2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-t-i}\varepsilon_{-t-j}-A_4]\,. \end{aligned} \tag{2.6}$$

Under the above representation we have the following theorem:

THEOREM 2.1. *For a stationary linear process $\{X_t\}$ with $E\varepsilon_t^6<\infty$ let α be a positive constant such that,*

$$\sum_{j=N}^{\infty}|\alpha_j|\leqq C(2^{-\alpha N})/\sqrt{N} \tag{2.7}$$

and $h_N\propto N^{-1/5}$ then,

$$\Big|P\Big(\frac{\sqrt{Nh}}{\sigma}\{\hat{f}_N-\bar{f}_N\}<z\Big)-\Phi(z)\Big|\leqq C\cdot(\log N/N)^{1/2} \tag{2.8}$$

where

$$\sigma^2=Ef_1^2+2\sum_{j=2}^{\infty}E(f_1f_j) \tag{2.9}$$

and $Ef_1^2\to f(x)\int K^2(u)du>0$ as $N\to\infty$.

To prove the above theorem we require the following result.

RESULT 2.1. *Suppose the conditions of theorem 2.1 hold. Then*

$$\sum_{t=2}^{\infty}|E(f_1f_t)|\leqq C\cdot h^{1/2} \tag{2.10}$$

PROOF. Let J_t be the distribution function of $Y=X_{t-2,t}$ and F_t be the joint distribution function of X_1 and

$$Y_2=X_{t-2,t}^* \qquad \text{where } X_{t-2,t}=\sum_{j=0}^{t-2}\alpha_j\varepsilon_{t-j}$$

$$X_{t-2,t}^*=X_t-X_{t-2,t}$$

Define

$$I=E\left[\frac{1}{h^2}K((x-X_1)/h)K((x-X_t)/h)\right]$$

$$=\frac{1}{h^2}\int K((x-X_1)/h)K((x-Y_2-y)/h)dF_t(Y_1,\ Y_2)dJ_t(y)\,. \tag{2.11}$$

Since ε_t's are iid and ϕ_0 is the characteristic function of ε_1, the characteristic function of $X_{t-2,t}$ is

$$E\left(e^{iu\sum_{j=0}^{t-2}\alpha_j\varepsilon_{t-j}}\right)=\prod_{j=0}^{t-2}E(e^{iu\alpha_j\varepsilon_1})$$

$$=\prod_{j=0}^{t-2}\phi_0(\alpha_j u)$$

and, therefore by (iii)

$$J_t'(y)\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}|\phi_0(u)|\,du<c \tag{2.12}$$

and

$$|J_t''(y)|\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}|u\phi_0(u)|\,du<c \tag{2.13}$$

Now for some constant $\delta>0$, we define $A=\{|Y_2|\leqq\delta\}$. Hence we have

$$I=I_1+I_2\,, \tag{2.14}$$

where

$$I_1=\int_A\frac{1}{h}K\Big(\frac{x-Y_1}{h}\Big)dF_t(X_1,\ Y_2)\int_{-\infty}^{\infty}K(y)J_t'(x-y_2-hy)dy$$

and

$$I_2=\int_{A^c}\frac{1}{h}K\Big(\frac{x-Y_1}{h}\Big)dF_t(X_1,\ Y_2)\int_{-\infty}^{\infty}K(y)J_t'(x-y_2-hy)dy$$

Again in A, by (2.13) we have

$$m_N\leqq\int K(y)J_t'(x-y_2-hy)dy\leqq M_N \tag{2.15}$$

where

$$m_N=\min\Big(C,\ \int_{-\infty}^{\infty}K(y)\{J_t'(x-hy)+C\delta\}\,dy\Big)$$

$$M_N=\int_{-\infty}^{\infty}K(y)\max\{0,\ J_t'(x-hy)-C\delta\}\,dy\,.$$

Obviously $m_N\geqq0$, $M_N\geqq0$ and

$$M_N-m_N\leqq\int K(y)\{J_t'(x-hy)+C\delta\}\,dy+2C\delta\int_{B^C}K(y)dy$$

where $B=\{y,\ J_t'(x-hy)\leqq C\delta\}$. Therefore

$$0<M_N-m_N\leqq C\delta \tag{2.16}$$

and

$$I_1\leqq M_N\int\frac{1}{h}K((x-Y_1)/h)dF_t(Y_1,\ Y_2)=M_N\bar{f}_N(x)\,. \tag{2.17}$$

Defining $G_t=P(|X^*_{t-2,t}|>\delta)$ we have

$$I_2\leqq(C/h)G_t\,, \tag{2.18}$$

because of the fact that $|\phi(u)|<C$ for all u and $J'_t(y)<C$ for all y. Hence, combining (2.17) and (2.18) we get

$$I=I_1+I_2\leqq M_N\bar{f}_N(x)+(C/h)G_t\,. \tag{2.19}$$

Again

$$\begin{aligned} I&\geqq m_N\int_R 1/h\,K((x-Y_1)/h)\,dF_t(Y_1,\,Y_2)\\ &\geqq m_N\bar{f}_N(x)-(C/h)\cdot G_t\,. \end{aligned}\tag{2.20}$$

Therefore, by (2.19) and (2.20), we have,

$$m_N\bar{f}_N(x)-(C/h)\cdot G_t\leqq I\leqq M_N\bar{f}_N(x)+(C/h)\cdot G_t\,. \tag{2.21}$$

Denote V_t as the distribution function of $X^*_{t-2,t}$, then

$$\begin{aligned} \bar{f}_N(x)&=\int_A+\int_{A^C}\frac{1}{h}K((x-y_2-y)/h)d\,V_t(y_2)dJ_t(y)\\ &=I_3+I_4\,,\qquad\text{say} \end{aligned}\tag{2.22}$$

where

$$\begin{aligned} I_3&=\int_A d\,V_t(Y_2)\int K(y)J'_t(x-Y_2-hy)dy\\ &\leqq M_N\int_A d\,V_t(Y_2)\,,\qquad\text{by (2.15)}\\ &\leqq C\cdot M_N \end{aligned}$$

and

$$\begin{aligned} I_4&\leqq C\cdot\int d\,V_t(Y_2)\\ &\leqq C\cdot G_t\,. \end{aligned}$$

Therefore

$$\bar{f}_N(x)=I_3+I_4\leqq C\cdot(M_N+G_t)\,. \tag{2.23}$$

Also

$$\bar{f}_N(x)\geqq I_3\geqq m_N\int_A d\,V_t(Y_2)=m_N(1-G_t)\,. \tag{2.24}$$

Combining (2.23) and (2.24) we have,

$$m_N(1-G_t)\leqq\bar{f}_N(x)\leqq C(M_N+G_t)\,.$$

This implies

$$m_N\bar{f}_N(x)(1-G_t)\leqq\bar{f}^2_N(x)\leqq C\bar{f}_N(x)(M_N+G_t)\,. \tag{2.25}$$

Hence from (2.21) and (2.25) we have,

$$\begin{aligned} &m_N\bar{f}_N(x)-(C/h)G_t-C\bar{f}_N(x)(M_N+G_t)\\ &\quad\leqq I-\bar{f}^2_N(x)\leqq M_N\bar{f}_N(x)+(C/h)G_t-m_N\bar{f}_N(x)(1-G_t)\,. \end{aligned}\tag{2.26}$$

Since M_N, m_N, $\bar{f}_N(x)<C$ and $h=h_N\to 0$ as $N\to\infty$ for all sufficiently large N, we have

$$|E(f_1f_t)|=\Big|\frac{1}{h}[E\{K((x-X_{-1})/h)$$

$$-EK((x-X_1)/h)\}\{K((x-X_{-t})/h)-EK((x-X_1)/h)\}]\Big|$$

$$=\Big|\frac{1}{h}[E\{K((x-X_1)/h)$$

$$-EK((x-X_1)/h)\}\{K((x-X_t)/h)-EK((x-X_1)/h)\}]\Big|,$$

because of sationarity.

$$=h|I-\bar{f}_N^2(x)|$$

$$\leqq Ch[\bar{f}_N(M_N-m_N)+G_t/h]$$

$$\leqq C\cdot(h\delta+G_t). \tag{2.27}$$

Also, it is easy to prove that

$$G_t=P\{|X^*_{t-2,t}|>\delta\}$$

$$\leqq E|X^*_{t-2,t}|/\delta$$

$$\leqq C\cdot\sum_{j=t-1}^{\infty}|\alpha_j|/\delta, \qquad \text{by } A_2$$

$$=C\cdot\gamma_t/\delta \qquad \text{where } \gamma_t=\sum_{j=t-1}^{\infty}|\alpha_j|.$$

Hence from (2.27), we have

$$|E(f_1f_t)|\leqq C\cdot(h\delta+\gamma_t/\delta)$$

$$=C\cdot h^{1/2}(\gamma_t)^{1/2}, \qquad \text{selecting } \delta=h^{-1/2}(\gamma_t)^{1/2}. \tag{2.28}$$

Now (2.10) is true because of the fact that

$$\sum_{t=2}^{\infty}(\gamma_t)^{1/2}=\sum_{t=2}^{\infty}\Big(\sum_{j=t-1}^{\infty}|\alpha_j|\Big)^{1/2}$$

$$\leqq C\cdot\sum_{t=2}^{\infty}(2^{-\alpha}(t-1)/(t-1)^{1/2})^{1/2}$$

since by (v),

$$\sum_{j=N}^{\infty}|\alpha_j|\leqq C\cdot 2^{-\alpha N}/(N)^{1/2}, \quad \alpha>0$$

$$=C\sum_{n=1}^{\infty}(2^{n\alpha/2}\cdot n^{1/4})^{-1}$$

$$<\infty.$$

PROOF OF THEOREM 2.1. From (2.6) we have,

$$f_1=\frac{1}{\sqrt{h}}\left[A_1\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-1-j}+A_2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}-A_4\right] \tag{2.29}$$

and

$$\begin{aligned}
&E[f_1/\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-N}]\\
&=E\Big[(A_1/\sqrt{h})\cdot\sum_{j=0}^{N-1}\alpha_j\varepsilon_{-1-j}+(A_2/\sqrt{h})\cdot\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}\\
&\qquad+(A_1/\sqrt{h})\cdot\sum_{j=N}^{\infty}\alpha_j\varepsilon_{-1-j}+(A_2/\sqrt{h})\cdot\sum_{i=N}^{\infty}\sum_{j=N}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}-A_4/\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-N}\Big]\\
&=(A_1/\sqrt{h})\cdot\sum_{j=0}^{N-1}\alpha_j\varepsilon_{-1-j}+(A_2/\sqrt{h})\cdot\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}+(A_2/\sqrt{h})\cdot\sum_{i=N}^{\infty}\alpha_i^2-A_4.
\end{aligned}$$

We set $S_{ij}=\varepsilon_{-1-i}\varepsilon_{-1-j}-E(\varepsilon_{-1-i}\varepsilon_{-1-j})$. Following Blume and Wittwer [2] and using the inequality

$$E^{1/3}[|X+Y+Z|^3]\leqq E^{1/3}|X|^3+E^{1/3}|Y|^3+E^{1/3}|Z|^3.$$

We get,

$$\begin{aligned}
&E^{1/3}(|f_1-E[f_1/\varepsilon_{-1},\cdots,\varepsilon_{-N}]|)^3\\
&\leqq E^{1/3}\left(\left|\frac{A_1}{\sqrt{h}}\sum_{j=N}^{\infty}\alpha_j\varepsilon_{-1-j}\right|^3\right)+E^{1/3}\left(\left|\frac{A_2}{\sqrt{h}}\sum_{i=N}^{\infty}\sum_{j=0}^{N-1}\alpha_i\alpha_jS_{ij}\right|^3\right)\\
&\quad+E^{1/3}\left(\left|\frac{A_2}{\sqrt{h}}\sum_{i=0}^{\infty}\sum_{j=N}^{\infty}\alpha_i\alpha_jS_{ij}\right|^3\right).
\end{aligned} \tag{2.30}$$

Further using the inequality $E|XYZ|\leqq[E|X|^3E|Y|^3E|Z|^3]^{1/3}$ we see that, the first summand is not greater than

$$\begin{aligned}
&\frac{A_1}{\sqrt{h}}E^{1/3}\Big(\sum_{j_1,j_2,j_3=N}^{\infty}|\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}\varepsilon_{-1-j_1}\varepsilon_{-1-j_2}\varepsilon_{-1-j_3}|\Big)\\
&\leqq\frac{A_1}{\sqrt{h}}\Big(\sum_{j_1,j_2,j_3=N}^{\infty}|\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}|\{E|\varepsilon_{-1-j_1}|^3E|\varepsilon_{-1-j_2}|^3E|\varepsilon_{-1-j_3}|^3\}^{1/3}\Big)^{1/3}
\end{aligned} \tag{2.31}$$

and also

$$E^{1/3}|\varepsilon_{-1-j_1}|^3=E^{1/3}|\varepsilon_0|^3. \tag{2.32}$$

Similarly, the second summand is not greater than

$$\begin{aligned}
&\frac{A_2}{\sqrt{h}}E^{1/3}\Big(\sum_{i_1,i_2,i_3=N}^{\infty}\sum_{j_1,j_2,j_3=0}^{N-1}|\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}S_{i_1j_1}S_{i_2j_2}S_{i_3j_3}|\Big)\\
&\leqq\frac{A_2}{\sqrt{h}}\Big(\sum_{i_1,i_2,i_3=N}^{\infty}\sum_{j_1,j_2,j_3=0}^{N-1}|\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}|\{E|S_{i_1j_1}|^3E|S_{i_2j_2}|^3E|S_{i_3j_3}|^3\}^{1/3}\Big)^{1/3}.
\end{aligned} \tag{2.33}$$

An analogous inequality is also valid for the 3rd summand. Further we obtain,

$$\begin{aligned}
E^{1/3}|S_{ij}|^3&\leqq E^{1/3}(|\varepsilon_{-1-i}\varepsilon_{-1-j}|^3)+1\\
&\leqq E^{1/3}(\varepsilon_0^6)+1.
\end{aligned} \tag{2.34}$$

Therefore, combining (2.31)-(2.34) we get, from (2.30), the following:

$$E^{1/3}(|f_1-E(f_1|\varepsilon_{-1},\cdots,\varepsilon_{-N})|^3)$$

$$\leqq \frac{A_1}{\sqrt{h}}\sum_{j=N}^{\infty}|\alpha_j|E^{1/3}(\varepsilon_0^3)+\frac{A_2}{\sqrt{h}}\Big(\sum_{i=N}^{\infty}|\alpha_i|\sum_{j=0}^{\infty}|\alpha_j|+\sum_{i=0}^{\infty}|\alpha_i|\sum_{j=N}^{\infty}|\alpha_j|\Big)(E^{1/3}(\varepsilon_0^6)+1)$$

$$\leqq C\cdot\frac{A_1}{\sqrt{h}}\frac{2^{-\alpha N}}{\sqrt{N}}E^{1/3}(\varepsilon_0^3)+\frac{A_2}{\sqrt{h}}(E^{1/3}(\varepsilon_0^6)+1)\Big(C\cdot\frac{2^{-\alpha N}}{\sqrt{N}}+C\cdot\frac{2^{-\alpha N}}{\sqrt{N}}\Big)\sum_{i=0}^{\infty}|\alpha_i|,$$

since by (v), $\sum_{i=N}^{\infty}|\alpha_i|\leqq C\cdot\frac{2^{-\alpha N}}{\sqrt{N}}$.

$$\leqq \frac{C}{h^{5/2}}\Big[E^{1/3}(\varepsilon_0^3)+2(E^{1/3}(\varepsilon_0^6)+1)\sum_{i=0}^{\infty}|\alpha_i|\Big]\frac{2^{-\alpha N}}{\sqrt{N}}$$

since $A_1\leqq C/h^2$ and $A_2\leqq C/h^2$

$$\leqq C\cdot 2^{-\alpha N},$$ since $h\infty N^{-1/5}$ and $E(\varepsilon_0^3)<\infty$, $E(\varepsilon_0^6)<\infty$, $\sum_{i=0}^{\infty}|\alpha_i|<\infty$.

Hence the condition (1.4) of Ibragimov [4] Theorem 2 is fulfilled for $\delta=1$.

Also

$$Ef_1^2=\frac{1}{h}EK^2(x-X_{-1})/h)-h\bar{f}_N^2(x),\qquad \text{where } \bar{f}_N(x)=(1/h)EK((x-X_1)/h)$$

$$=f(x)\int K^2(u)du-h\bar{f}_N^2(x)$$

$$\to f(x)\int K^2(u)du>0,\qquad \text{as } N\to\infty.$$

Therefore using (2.10) of Result 2.1 we have

$$\sigma^2=Ef_1^2+2\sum_{j=2}^{\infty}E(f_1f_j)\neq 0\quad\text{and also } \sigma^2<\infty.$$

Hence the condition 1.5 of Ibragimov [4], Theorem 2 is also fulfilled. So by Ibragimov's Theorem 2 we get, for $\delta=1$,

$$P\Big|\Big(\frac{1}{\sigma\sqrt{N}}\sum_{t=1}^{N}f_t<z\Big)-\Phi(z)\Big|\leqq C\cdot\Big(\frac{\log N}{N}\Big)^{1/2}.$$

This implies

$$\Big|P\Big(\frac{\sqrt{Nh}\{\hat{f}_N-\bar{f}_N\}}{\sigma}<z\Big)-\Phi(z)\Big|\leqq C\cdot\Big(\frac{\log N}{N}\Big)^{1/2},$$

which is (2.8). Hence the theorem is proved.

3. Estimation of Joint Density and Its Rate of Convergence to Normality

We consider the covariance of estimates defined by (2.2) at two distinct points x and y as follows:

$$\begin{aligned}\mathrm{Cov}(\hat f_N(x),\ \hat f_N(y)) &= (1/Nh^2)\,\mathrm{Cov}(K((x-X)/h),\ K((y-X)/h))\\ &= (1/Nh^2)\Big[\int K((x-X)/h)K((y-X)/h)f(X)dX\\ &\qquad -\Big\{\int K((x-X)/h)f(X)dX\Big\}\Big\{\int K((y-X)/h)f(X)dX\Big\}\Big]\\ &= (1/Nh)\Big[\int K(u)K((y-x)/h+u)f(x-hu)du\\ &\qquad -h\Big\{\int K(u)f(x-hu)du\Big\}\Big\{K(u)f(y-hu)du\Big\}\Big].\end{aligned}$$

From this it follows that $\sqrt{Nh}[\hat f_N(x)-E\hat f_N(x)]$ and $\sqrt{Nh}[\hat f_N(y)-E\hat f_N(y)]$ are asymptotically independent.

We assume that all joint distributions of a finite number of distinct X_t's (where X_t's are as defined in (2.1)) are absolutely continuous with uniform bounded continuous density functions. Let $f(x, y)$ be the joint density function of X_t, X_{t+1}. A natural estimate of $f(x, y)$, used by Rosenblatt [8] is given by

$$\hat f_N(x,\ y)=(1/Nh^2)\sum_{t=1}^{N-1}K((x-X_t)/h,\ (y-X_{t+1})/h) \tag{3.1}$$

where $K(x, y)$ is a bounded continuous density function. We set

$$\begin{aligned} f_t &= \frac{1}{h}[K((x-X_{-t})/h,\ (y-X_{-t+1})/h)-EK((x-X_1)/h,\ (y-X_2)/h)]\\ &= f(\varepsilon_{1-t},\ \varepsilon_{-t},\ \varepsilon_{-t-1},\ \cdots) \end{aligned}\tag{3.2}$$

and $T_N=(Nh^2)^{1/2}[\hat f_N-\bar f_N]$, where $\bar f_N=(1/h^2)E[K((x-X_1)/h,\ (y-X_2)/h]$

$$\begin{aligned} &=(Nh^2)^{-1/2}\sum_{t=1}^{N-1}\Big[K((x-X_t)/h,\ (y-X_{t+1})/h)-\frac{N}{N-1}EK((x-X_1)/h,\ (y-X_2)/h)\Big]\\ &\sim(Nh^2)^{-1/2}\sum_{t=1}^{N}[K((x-X_t)/h,\ (y-X_{t+1})/h)-EK((x-X_1)/h,\ (y-X_2)/h)]\\ &=(Nh^2)^{-1/2}\sum_{t=-N}^{-1}[K((x-X_{-t})/h,\ (y-X_{-t+1})/h)-EK((x-X_1)/h,\ (y-X_2)/h)]\\ &=N^{-1/2}\sum_{t=-N}^{-1}f_t. \end{aligned}\tag{3.3}$$

In case the strict stationarity of $\{X_t\}$ is valid then

$$P\Big[N^{-1/2}\sum_{t=-N}^{-1}f_t\leqq z\Big]=P\Big[N^{-1/2}\sum_{t=1}^{N}f_t\leqq z\Big].$$

Like in section 2 we take a second degree polynomial in two variables as Kernel $K(x, y)=a_0+a_1x+a_2y+a_3x^2+a_4y^2+a_5xy$. Such bivariate Kernels are again optimal in the sense that it minimizes integrated mean square error and widely been used in the literature. Therefore

$$K((x-X_{-t})/h, (y-X_{-t+1})/h)$$
$$=A_0+A_1X_{-t}+A_2X_{-t+1}+A_3X_{-t}^2+A_4X_{-t+1}^2+A_5X_{-t}X_{-t+1}$$

where $A_i \leqq C/h^2$ for $i=0, 1, \cdots, 5$. We write $A_6=EK((x-X_1)/h, (y-X_2)/h)$ and $A_7=A_6-A_0$.

Therefore

$$f_t=\frac{1}{h}[A_1X_{-t}+A_2X_{-t+1}+A_3X_{-t}^2+A_4X_{-t+1}^2+A_5X_{-t}X_{-t+1}-A_7]$$
$$=\frac{1}{h}\Big[A_1\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-t-j}+A_2\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-t+1-j}+A_3\sum_0^{\infty}\sum_0^{\infty}\alpha_i\alpha_j\varepsilon_{-t-i}\varepsilon_{-t-j}$$
$$+A_4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-t+1-i}\varepsilon_{-t+1-j}+A_5\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-t-i}\varepsilon_{-t+1-j}-A_7\Big]. \tag{3.4}$$

Now under this representation we have the following theorem:

THEOREM 3.1. *If the conditions of Theorem* 2.1 *together with the assumption*

$$\mathrm{Max}\Big[\sum_{i=N-1}^{\infty}|\alpha_i|, \sqrt[3]{\sum_{i=N-1}^{\infty}\alpha_i^6}, \sqrt[3]{\sum_{i=N-1}^{\infty}\alpha_i^3}\Big]\leqq C\cdot 2^{-\alpha N}/N^{3/5}$$

holds, then for normalised deviation $\sqrt{Nh^2}[\hat{f}_N(x, y)-\bar{f}_N(x, y)]$,

$$\left|P\Big(\frac{\sqrt{Nh^2}(\hat{f}_N-\bar{f}_N)}{\sigma}<z\Big)-\Phi(z)\right|\leqq C\cdot\Big(\frac{\log N}{N}\Big)^{1/2} \quad \textit{holds} \tag{3.5}$$

where

$$\sigma^2=Ef_1^2+2\sum_{j=2}^{\infty}E(f_1f_j) \quad \textit{and} \quad Ef_1^2\to f(x, y)\int K^2(u, v)dudv \quad \textit{as } N\to\infty. \tag{3.6}$$

PROOF. Proceeding exactly similar way as in the proof of Theorem 2.1 we shall verify the conditions of Ibragimov's Theorem. We note that,

$$f_1=\frac{1}{h}\Big[A_1\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-1-j}+A_2\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-j}+A_3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}$$
$$+A_4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}+A_5\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-j}-A_7\Big].$$

Therefore, for $N\geqq 2$,

$$E[f_1/\varepsilon_0, \varepsilon_{-1}, \cdots, \varepsilon_{-(N-1)}]$$
$$=E\Big[\Big\{\frac{A_1}{h}\sum_{j=0}^{N-2}\alpha_j\varepsilon_{-1-j}+\frac{A_2}{h}\sum_{j=0}^{N-2}\alpha_j\varepsilon_{-j}+\frac{A_3}{h}\sum_{i=0}^{N-2}\sum_{j=0}^{N-2}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}$$
$$+\frac{A_4}{h}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}+\frac{A_5}{h}\sum_{i=0}^{N-2}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-j}\Big\}$$
$$+\Big\{\frac{A_1}{h}\sum_{j=N-1}^{\infty}\alpha_j\varepsilon_{-1-j}+\frac{A_2}{h}\sum_{j=N}^{\infty}\alpha_j\varepsilon_{-j}+\frac{A_3}{h}\sum_{i=N-1}^{\infty}\sum_{j=N-1}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}$$
$$+\frac{A_4}{h}\sum_{N}^{\infty}\sum_{N}^{\infty}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}+\frac{A_5}{h}\sum_{i=N-1}^{\infty}\sum_{N}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-j}\Big\}-A_7\Big]$$

$$=\left\{\frac{A_1}{h}\sum_{j=0}^{N-2}\alpha_j\varepsilon_{-1-j}+\frac{A_2}{h}\sum_{j=0}^{N-1}\alpha_j\varepsilon_{-j}+\frac{A_3}{h}\sum_{i=0}^{N-2}\sum_{j=0}^{N-2}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}\right.$$

$$\left.+\frac{A_4}{h}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}+\frac{A_5}{h}\sum_{i=0}^{N-2}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-j}\right\}$$

$$+\left\{\frac{A_3}{h}\sum_{i=N-1}^{\infty}\alpha_i^2+\frac{A_4}{h}\sum_{i=N}^{\infty}\alpha_i^2+\frac{A_5}{h}\sum_{i=N-1}^{\infty}\alpha_i\alpha_{i+1}\right\}-A_7.$$

Therefore,

$$f_1-E[f_1/\varepsilon_0,\ \varepsilon_{-1},\ \cdots,\ \varepsilon_{-(N-1)}]$$

$$=\frac{A_1}{h}\sum_{j=N-1}^{\infty}\alpha_j\varepsilon_{-1-j}+\frac{A_2}{h}\sum_{j=N}^{\infty}\alpha_j\varepsilon_{-j}+\frac{A_3}{h}\sum_{i=N-1}^{\infty}\sum_{j=N-1}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}$$

$$+\frac{A_4}{h}\sum_{i=N}^{\infty}\sum_{j=N}^{\infty}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}+\frac{A_5}{h}\sum_{i=N-1}^{\infty}\sum_{j=N}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-j}$$

$$-\frac{A_3}{h}\sum_{i=N-1}^{\infty}\alpha_i^2-\frac{A_4}{h}\sum_{i=N}^{\infty}\alpha_i^2-\frac{A_5}{h}\sum_{i=N-1}^{\infty}\alpha_i\alpha_{i+1}.$$

Now following Blume and Wittwer [2] and Kersten [5] and by using the inequality

$$E^{1/3}[|X+Y+Z+\cdots|^3]\leqq E^{1/3}|X|^3+E^{1/3}|Y|^3+E^{1/3}|Z|^3+\cdots$$

we get,

$$E^{1/3}(|f_1-E[f_1/\varepsilon_0,\ \cdots,\ \varepsilon_{-(N-1)}]|^3)$$

$$\leqq E^{1/3}\left(\left|\frac{A_1}{h}\sum_{j=N-1}^{\infty}\alpha_j\varepsilon_{-1-j}\right|^3\right)+E^{1/3}\left(\left|\frac{A_2}{h}\sum_{j=N}^{\infty}\alpha_j\varepsilon_{-j}\right|^3\right)$$

$$+E^{1/3}\left(\left|\frac{A_3}{h}\sum_{i=N-1}^{\infty}\sum_{j=0}^{N-2}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}\right|^3\right)$$

$$+E^{1/3}\left(\left|\frac{A_3}{h}\sum_{i=0}^{\infty}\sum_{j=N-1}^{\infty}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}\right|^3\right)$$

$$+E^{1/3}\left(\left|\frac{A_4}{h}\sum_{i=N-1}^{\infty}\sum_{j=0}^{N-2}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}\right|^3\right)$$

$$+E^{1/3}\left(\left|\frac{A_4}{h}\sum_{i=0}^{\infty}\sum_{j=N-1}^{\infty}\alpha_i\alpha_j\varepsilon_{-i}\varepsilon_{-j}\right|^3\right)$$

$$+E^{1/3}\left(\left|\frac{A_5}{h}\left\{\sum_{i=N-1}^{\infty}\sum_{j=N}^{\infty}\alpha_i\alpha_jT_{ij}+\sum_{i=0}^{N-2}\sum_{j=N}^{\infty}\alpha_i\alpha_jT_{ij}\right.\right.\right.$$

$$\left.\left.\left.+\sum_{i=N-1}^{\infty}\sum_{j=0}^{N-1}\alpha_i\alpha_jT_{ij}-\sum_{i=N-1}^{\infty}\alpha_i\alpha_{i+1}\right\}\right|^3\right),$$

where

$$T_{ij}=\varepsilon_{-1-i}\varepsilon_{-j}=e_1+e_2+e_3+e_4+e_5+e_6+e_7,\ \text{say.} \tag{3.7}$$

Now

$$e_1\leqq\frac{A_1}{h}E^{1/3}\left(\sum_{j_1,j_2,j_3=N-1}^{\infty}|\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}\varepsilon_{-1-j_1}\varepsilon_{-1-j_2}\varepsilon_{-1-j_3}|\right)$$

$$\leqq\frac{A_1}{h}\left(\sum_{j_1,j_2,j_3=N-1}^{\infty}|\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}|\{E|\varepsilon_{-1-j_1}|^3E|\varepsilon_{-1-j_2}|^3E|\varepsilon_{-1-j_3}|^3\}^{1/3}\right)^{1/3}$$

using the inequality

$$E|XYZ| \leqq [E|X|^3 E|Y|^3 E|Z|^3]^{1/3} \leqq \frac{A_1}{h} \sum_{j=N-1}^{\infty} |\alpha_j| E^{1/3} |\varepsilon_0|^3.$$

Similarly

$$e_2 \leqq (A_2/h) \sum_{j=N}^{\infty} |\alpha_j| E^{1/3}(\varepsilon_0^3).$$

Also

$$e_3 \leqq \frac{A_3}{h} E^{1/3}\Big(\sum_{i_1, i_2, i_3=N-1}^{\infty} \sum_{j_1, j_2, j_3=0}^{N-2} |\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}\varepsilon_{-1-i_1}\varepsilon_{-1-i_2}\varepsilon_{-1-i_3}\varepsilon_{-1-j_1}\varepsilon_{-1-j_2}\varepsilon_{-1-j_3}|\Big)$$

$$\leqq \frac{A_3}{h}\Big(\sum_{i_1, i_2, i_3=N-1}^{\infty} \sum_{j_1, j_2, j_3=0}^{N-2} |\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}| \{E|\varepsilon_{-1-i_1}\varepsilon_{-1-j_1}|^3$$

$$E|\varepsilon_{-1-i_2}\varepsilon_{-1-j_2}|^3 E|\varepsilon_{-1-i_3}\varepsilon_{-1-j_3}|^3\}^{1/3}\Big)^{1/3}$$

$$= \frac{A_3}{h}\Big(\sum_{i=N-1}^{\infty} |\alpha_i| \sum_{j=0}^{\infty} |\alpha_j|\Big) E^{1/3}(\varepsilon_0^6).$$

Similarly

$$e_4 \leqq \Big(\frac{A_3}{h} \sum_{i=0}^{\infty} |\alpha_i| \sum_{j=N-1}^{\infty} |\alpha_j|\Big) E^{1/3}(\varepsilon_0^6)$$

$$e_5 \leqq \Big(\frac{A_4}{h} \sum_{i=N-1}^{\infty} |\alpha_i| \sum_{j=0}^{\infty} |\alpha_j|\Big) E^{1/3}(\varepsilon_0^6)$$

and

$$e_6 \leqq \Big(\frac{A_4}{h} \sum_{i=0}^{\infty} |\alpha_i| \sum_{j=N-1}^{\infty} |\alpha_j|\Big) E^{1/3}(\varepsilon_0^6).$$

Also following Kersten [5] we get

$$e_7^3 \leqq \frac{A_5^3}{h^3}\Big[E(\varepsilon_0^6) \sum_{i=N-1}^{\infty} \alpha_i^6 + 4\Big(\sum_{i=0}^{\infty} \alpha_i^3\Big)\Big(\sum_{i=N-1}^{\infty} \alpha_i^3\Big)\Big]$$

$$\leqq \frac{A_5^3}{h^3}\Big(E(\varepsilon_0^6) + 4 \sum_{i=0}^{\infty} \alpha_i^3\Big) \cdot M$$

where $M = \operatorname{Max}\Big\{\sum_{i=N-1}^{\infty} \alpha_i^6, \sum_{i=N-1}^{\infty} \alpha_i^3\Big\}$. This implies

$$e_7 \leqq C \cdot \frac{A_5}{h} M^{1/3}, \quad \text{because} \quad \sum_{i=0}^{\infty} |\alpha_i| < \infty \Longrightarrow \sum_{i=0}^{\infty} \alpha_i^3 < \infty.$$

Hence

$$E^{1/3}(|f_1 - E[f_1/\varepsilon_0, \cdots, \varepsilon_{-(N-1)}]|^3)$$

$$\leqq \frac{A_1}{h} \sum_{i=N-1}^{\infty} |\alpha_i| E^{1/3}(\varepsilon_0^3) + \frac{A_2}{h} \sum_{j=N}^{\infty} |\alpha_j| E^{1/3}(\varepsilon_0^3)$$

$$+ \frac{A_3}{h}\Big(\sum_{i=N-1}^{\infty} |\alpha_i|\Big)\Big(\sum_{j=0}^{\infty} |\alpha_j|\Big) E^{1/3}(\varepsilon_0^6)$$

$$+ \frac{A_3}{h}\Big(\sum_{i=0}^{\infty} |\alpha_i|\Big)\Big(\sum_{j=N-1}^{\infty} |\alpha_j|\Big) E^{1/3}(\varepsilon_0^6)$$

$$+\frac{A_4}{h}\sum_{i=N-1}^{\infty}|\alpha_i|\sum_{j=0}^{\infty}|\alpha_j|E^{1/3}(\varepsilon_0^6)$$

$$+\frac{A_4}{h}\sum_{i=0}^{\infty}|\alpha_i|\sum_{j=N-1}^{\infty}|\alpha_j|E^{1/3}(\varepsilon_0^6)+C\cdot\frac{A_5}{h}\sqrt[3]{M}$$

$$\leqq\left(\frac{A_1}{h}\sum_{i=N-1}^{\infty}|\alpha_i|+\frac{A_2}{h}\sum_{j=N}^{\infty}|\alpha_j|\right)E^{1/3}(\varepsilon_0^3)+2\left(\frac{A_3}{h}+\frac{A_4}{h}\right).$$

$$\sum_{i=N-1}^{\infty}|\alpha_i|\sum_{j=0}^{\infty}|\alpha_j|E^{1/3}(\varepsilon_0^6)+C\cdot\frac{A_5}{h}\sqrt[3]{M}$$

$$\leqq\frac{C}{h^3}\left[\sum_{i=N-1}^{\infty}|\alpha_i|(E^{1/3}(\varepsilon_0^3)+C_1E^{1/3}(\varepsilon_0^6))+\sum_{i=N}^{\infty}|\alpha_i|E^{1/3}(\varepsilon_0^3)+\sqrt[3]{M}\right]$$

$$\leqq Ch^{-3}M^* \quad \text{where} \quad M^*=\operatorname{Max}\left[\sum_{i=N-1}^{\infty}|\alpha_i|,\ \sqrt[3]{\sum_{i=N-1}^{\infty}\alpha_i^6},\ \sqrt[3]{\sum_{i=N-1}^{\infty}\alpha_i^3}\right]$$

$$\leqq C\cdot 2^{-\alpha N},\ \text{since}\ h\leqq C\cdot N^{-1/5}\leqq C\cdot 2^{-\alpha N}/N^{3/5}$$

So the condition (1.4) of Ibragimov [4] Theorem 2 is fulfilled for $\delta=1$. Proceeding exactly similar way as in Result 2.1 we get $\sum_{j=2}^{\infty}E(f_1f_j)\to 0$ as $N\to\infty$. Also $Ef_1^2\to f(x,y)\int K^2(u,v)dudv>0$. Hence $\sigma^2\neq 0$. So the condition (1.5) of Ibragimov [4] Theorem 2 is also fulfilled and hence the theorem follows immediately.

4. Estimation of Probability Density at Several Points and Its Rate of Convergence to Normality

We consider the rate of convergence of a density estimate at several points x_1, $x_2, \cdots, x_m$. Assume that K is a continuous density function with $K(u)=o(|u|)^{-1}$ as $|u|\to\infty$. The density 'f' is taken to be bounded and continuous. Rosenblatt [9] considers the asymptotic distribution of

$$\sqrt{Nh}[\hat{f}_N(x_p)-E\hat{f}_N(x_p)], \qquad p=1,2,\cdots,m$$

for any finite m-tuple of points x_p. He shows that these random variables are asymptotically jointly normal and independent with means zero and variances given by (4.1) below:

$$\lim_{n\to\infty} Nh\operatorname{Cov}[\hat{f}_N(x_p),\hat{f}_N(x_q)]=\begin{cases} f(x)\int K^2(u)du & \text{if } p=q \\ 0 & \text{if } p\neq q.\end{cases} \tag{4.1}$$

Our object is to consider the rate of convergence of the normalized deviations

$$\sqrt{Nh}[\hat{f}_N(x_p)-\bar{f}_N(x_p)], \qquad p=1,2,\cdots,m.$$

where

$$\bar{f}_N(x_p)=\frac{1}{h}EK\left(\frac{x_p-X_1}{h}\right).$$

To do this we require the lemma-1 of Kersten [5], (due to Gabasov (1977)) which is stated below:

Let $\{\varepsilon_t\}_{t=0,1,\dots}$ be a stationary sequence of random variables and let $f_p(\varepsilon_1, \varepsilon_2, \cdots)$, $p=1, 2, \cdots, m$ be random variables with $Ef_p=0$. The random vector $\underset{\sim}{f}_t$ is defined by $\underset{\sim}{f}_t=(f_{1t}, \cdots, f_{mt})^T$ for $t=0, 1, 2, \cdots$ with $f_{pt}=f_p(\varepsilon_t, \varepsilon_{t+1}, \cdots)$. By Φ_V we denote the distribution function of a m-dimensional contered Gaussian distribution Z with the covariance matrix V. With these denotations we have the following:

LEMMA. *Let the following conditions be fulfilled with the positive constant C_1 and C_2 and any positive and constant number u*

(i) $E\|\underset{\sim}{f}_1\|^u<\infty$

(ii) $E|f_{p1}-E(f_{p1}/\varepsilon_1, \cdots, \varepsilon_N)|^2 \leqq C_1 N^{-2u}$ *for* $p=1, 2, \cdots, m$ *and* $N=1, 2, \cdots$

(iii) *The sequence* $\{\varepsilon_t\}$ *fulfils the uniformly strongly mixing condition with the coefficients* $\phi_j \leqq C_2 j^{-u}$, *for* $j=0, 1, 2, \cdots$

(iv) *det* $V\neq 0$ *where* $V=(v_{ij})i, j=1, \cdots, m$ *with*

$$v_{ij}=\lim_{N\to\infty}\frac{1}{N}E\Big(\sum_{t,s=1}^{N} f_{it}\cdot f_{js}\Big).$$

Then the inequality

$$\operatorname*{Sup}_{B}\Big|P\Big(\frac{1}{\sqrt{N}}\sum_{t=1}^{N}\underset{\sim}{f}_t\in B\Big)-\Phi_v[B]\Big|\leqq C_3 N^{-1/2+\varepsilon}$$

is valid where B is a measurable convex set of the m-dimemoral Euclidian space R^m, $\Phi_v[B]=P(Z\in B)$ and $\varepsilon=3u^{-1/8}$. The constant C_3 depends on the dimension m, on the constants C_1, C_2, on u, and on the smallest eigen value $\lambda_{\min}$ of V. We note that

$$\hat{f}_N(x_p)=\frac{1}{Nh}\sum_{t=1}^{N}K\Big(\frac{x_p-X_t}{h}\Big), \qquad p=1, 2, \cdots, m.$$

As in section 2 we set,

$$f_{pt}=\frac{1}{\sqrt{h}}\Big[K\Big(\frac{x_p-X_{-t}}{h}\Big)-EK\Big(\frac{x_p-X_1}{h}\Big)\Big]=f_p(\varepsilon_{-t}, \varepsilon_{-t-1}, \cdots)$$

and

$$T_{pN}=\sqrt{Nh}[\hat{f}_N(x_p)-\bar{f}_N(x_p)]$$

where

$$\bar{f}_N(x_p)=\frac{1}{h}EK\Big(\frac{x_p-X_1}{h}\Big)=\frac{1}{\sqrt{N}}\sum_{t=-N}^{-1}f_{pt}.$$

Also

$$\sqrt{Nh}(\underset{\sim}{\hat{f}}_N-\underset{\sim}{\bar{f}}_N)=\sqrt{Nh}(\{\hat{f}_N(x_1)-\bar{f}_N(x_1)\}, \cdots, \{\hat{f}_N(x_p)-\bar{f}_N(x_p)\})^T$$
$$=\frac{1}{\sqrt{N}}\sum_{t=-N}^{-1}\underset{\sim}{f}_t$$

where $\underset{\sim}{f}_t=(f_{1t}, \cdots, f_{mt})^T$.

It may be noted that for the strict stationarity of $\{X_t\}$,

$$P\Big[(1/N)^{1/2}\sum_{t=-N}^{-1}f_{pt}<z\Big]=P\Big[(1/N)^{1/2}\sum_{t=1}^{N}f_{pt}<z\Big].$$

As in section 2 taking the Kernel to be a polinomial of degree two one can write

$$f_{pt}=\frac{1}{\sqrt{h}}[A_0+A_1X_{-t}+A_2X^2_{-t}-A_3]$$

where

$$A_0=a_0+a_1x_p/h+a_2x_p^2/h^2, \qquad A_1=-(a_1/h+2a_2x_p/h^2),$$

$$A_2=a_2/h^2, \qquad A_3=EK((x_p-X_1)/h).$$

Therefore,

$$f_{pt}=\frac{1}{\sqrt{h}}\left[A_1\sum_{j=0}^{\infty}\alpha_j\varepsilon_{-t-j}+A_2\sum_0^{\infty}\sum_0^{\infty}\alpha_i\alpha_j\varepsilon_{-t-i}\varepsilon_{-t-j}-A_4\right]$$

where $A_4=A_3-A_0$. Under this representation we have the following:

THEOREM 4.1. *For a stationary linear process $\{X_t\}$ satisfying*

(i) $E|\varepsilon_0|^{2u}<\infty$ *for a fixed number* $u\geqq 2$

(ii) $\sum_{i=N}^{\infty}|\alpha_i|\leqq K_1N^{-(u+3/5)}$, $h=O(N^{-1/5})$

(iii) $\det V\neq 0$ *where* $V=(v_{ij})_{i,j=1,\cdots,m}$ *with*

$$v_{ij}=\lim_{N\to\infty}Nh\operatorname{Cov}(\hat{f}_N(x_i),\hat{f}_N(x_j))$$

then

$$\operatorname*{Sup}_B|P\{(Nh)^{1/2}(\underset{\sim}{\hat{f}}_N-\underset{\sim}{\bar{f}}_N)\in B\}-\Phi_V[B]|\leqq K_2N^{-1/2+\varepsilon}$$

where B and ε are defined as above, and the constant K_2 depends on U, u, K_1, $E\varepsilon_0^4$, Var *X_t and the smallest eigen value $\lambda_{\min}$ of V.*

PROOF. Proceeding exactly similar way as in section 2 we see that

$$E[f_{p1}/\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-N}]=\frac{A_1}{\sqrt{h}}\sum_{j=0}^{N-1}\alpha_j\varepsilon_{-1-j}$$
$$+\frac{A_2}{\sqrt{h}}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\alpha_i\alpha_j\varepsilon_{-1-i}\varepsilon_{-1-j}+\frac{A_2}{\sqrt{h}}\sum_{i=N}^{\infty}\alpha_i^2-A_4$$

and

$$E^{1/2}(|f_{p1}-E(f_{p1}/\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-N})|^2)$$
$$\leqq\frac{A_1}{\sqrt{h}}\sum_{j=N}^{\infty}|\alpha_j|+2C\frac{A_2}{\sqrt{h}}\sum_{i=N}^{\infty}|\alpha_i|(E^{1/2}(\varepsilon_0^4)+1)$$
$$\leqq C\cdot N^{-u} \qquad \text{for}\quad u\geqq 2 \text{ by (i) and (ii)}.$$

Therefore

$$E(|f_{p1}-E(f_{p1}/\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-N})|^2)\leqq C_1N^{-2u}.$$

So condition (ii) of Lemma is satisfied. Condition (i) is also holds because of the assumption that $E|\varepsilon_0|^{2u}<\infty$. Condition (iii) is also fulfilled with any u and C_2 because of the independence of ε_t. Now

$$v_{ij}=\lim_{N\to\infty}Nh\operatorname{Cov}(\hat{f}_N(x_i),\hat{f}_N(x_j)), \qquad i\neq j$$
$$=0,$$

by (4.1). Also

$$v_{ii}=\lim_{N\to\infty}Nh\operatorname{Var}(\hat{f}_N(x_i))=f(x)\int K^2(u)du, \qquad \text{for}\quad i=1,2,\cdots,m.$$

Therefore, $\det V\neq 0$. So condition (iv) of Lemma is also satisfied and hence the theorem follows by applying the Lemma.

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