ON THE ORDERS OF MAX-MIN FUNCTIONALS

Hisano, Hiroshi
Nakamuragakuen Sanyou High School

Maruyama, Fumio
Department of Mathematics, Kyushu University

ON THE ORDERS OF MAX-MIN FUNCTIONALS

By

Hiroshi Hisano* and Fumio Maruyama**

Abstract

We construct a continuous function on $[0,1]^n$ that orders the set of Max-min functionals on $C([0,1]^n)$ according to an arbitrary given extended order of the natural partial order.

1. Introduction

Let $S_1, S_2, \cdots, S_n$ be compact spaces and let $\prod_{i=1}^{n} S_i$ be the direct product with the product topology. For a continuous real valued function $f(x_1, x_2, \cdots, x_n)$ on $\prod_{i=1}^{n} S_i$

$$\text{Max}_{x_i} f(x_1, x_2, \cdots, x_n) \quad \text{and} \quad \min_{x_i} f(x_1, x_2, \cdots, x_n) \quad (i=1, 2, \cdots, n)$$

are continuous functions of $(n-1)$-variables. Inductively, an n-length sequential maximizing or minimizing of a continuous function gives a mapping:

$$C\left(\prod_{i=1}^{n} S_i\right) \longrightarrow R,$$

where $C\left(\prod_{i=1}^{n} S_i\right)$ is the set of all real valued continuous functions on $\prod_{i=1}^{n} S_i$. We call this mapping a Max-min functional on $C\left(\prod_{i=1}^{n} S_i\right)$.

For a subset $T \subset C\left(\prod_{i=1}^{n} S_i\right)$, there might be no function of T that mutually distinguishes all Max-min functionals. It depends on the cardinality and the topology of S_i $(i=1, 2, \cdots, n)$ and the subset T. The most well-kown non-trivial example is the following minimax theorem of von Neumann [2].

THEOREM (VON NEUMANN). Let

$$S_1 = \{(\xi_1, \xi_2, \cdots, \xi_m) \in [0, 1]^m \mid \xi_1 + \xi_2 + \cdots + \xi_m = 1\},$$

$$S_2 = \{(\eta_1, \eta_2, \cdots, \eta_n) \in [0, 1]^n \mid \eta_1 + \eta_2 + \cdots + \eta_n = 1\},$$

and let T be the set of all bilinear forms. Then

$$\forall f \in T \ \text{Max}_{x_1 \in S_1} \min_{x_2 \in S_2} f(x_1, x_2) = \min_{x_2 \in S_2} \text{Max}_{x_1 \in S_1} f(x_1, x_2).$$

In this paper we consider the case $S_1 = S_2 = \cdots = S_n = [0, 1]$ under its usual topology. In section 3, where the role of each variable is fixed, we construct a polynomial of

---

* Nakamuragakuen Sanyou High School, Fukuoka, Japan.
** Department of mathematics, Kyushu University 33, Fukuoka 812, Japan.

quite simple form whose degree is not greater than the number of Max-min functionals those are to be distinguished. A function determines an order on the set of Max-min functionals according to thier values at this function. In section 4 we construct a continuous function that determines the same arbitrary given order on the set of Max-min functionals as long as this order is consistent with the natural partial order.

As seen in [1] and [2] there is a close relation between Max-min functionals and the theory of games. The systematic structure of the constructed functions in this paper will give some information about the character of the games with which these Max-min functionals are concerned.

2. Preliminaries

We give some basic definitions and notations. For a fixed natural number n, $C([0, 1]^n)$ denotes the set of all continuous real valued functions on $[0, 1]^n$ under its usual product topology and Σ denotes the set of all Max-min functionals on $C([0, 1]^n)$. Let r be the cardinality of Σ.

We will use i, j, k, as natural numbers less than or equal to n,

$\qquad l, m, p, q$, as natural numbers less than or equal to r,

$\qquad\qquad d$, as an integer,

$\qquad\qquad x, y, t$, as real numbers in the closed interval $[0, 1]$,

$\qquad\qquad\qquad \alpha$, as a non-empty subset of $\{1, 2, \cdots, n\}$,

$\qquad\qquad\qquad \sigma, \mu$, as elements of Σ.

A Max-min functional σ can be represented uniquely as

$$\sigma = \prod_{k \leq m(\sigma)} \prod_{i \in A_k(\sigma)} \text{Max}_{x_i} \prod_{j \in B_k(\sigma)} \min_{x_j},$$

where

$$A_k(\sigma), B_k(\sigma) \subset \{1, 2, \cdots, n\}, \quad A_k(\sigma) \cup B_k(\sigma) \neq \phi \quad (k=1, 2, \cdots, m(\sigma)),$$

and

$$\bigcup_{k \leq m(\sigma)} (A_k(\sigma) \cup B_k(\sigma)) = \{1, 2, \cdots, n\}.$$

DEFINITION 2.1. $n(\sigma, i) = k$ iff $i \in A_k(\sigma) \cup B_k(\sigma)$ $(i=1, 2, \cdots, n)$.

DEFINITION 2.2.

(1) $$\bar{X}(\sigma) = \bigcup_{k \leq m(\sigma)} A_k(\sigma).$$

(2) $$\underline{X}(\sigma) = \bigcup_{k \leq m(\sigma)} B_k(\sigma).$$

(3) $$\bar{X}(\sigma, i) = \bigcup_{k \leq n(\sigma, i)} A_k(\sigma) \qquad \text{for} \quad i \in \underline{X}(\sigma).$$

(4) $$\underline{X}(\sigma, i) = \bigcup_{k < n(\sigma, i)} B_k(\sigma) \qquad \text{for} \quad i \in \bar{X}(\sigma).$$

DEFINITION 2.3.

(1) $$S(\sigma) = \{\mu \mid \exists i \in \bar{X}(\sigma) \exists j \in \underline{X}(\mu) \cap \underline{X}(\sigma, i) \ i \in \bar{X}(\mu, j)\}.$$

(2) $$T(\sigma) = \{\mu \mid \bar{X}(\sigma) \cap \underline{X}(\mu) \neq \phi\}.$$

For the natural order $<_\Sigma$ on Σ (i. e. $\mu <_\Sigma \sigma$ iff $\mu \neq \sigma$ and $\forall f \in C([0, 1]^n)$ $\mu(f) \leq \sigma(f)$) it is easy to observe that $\mu <_\Sigma \sigma$ implies $\mu \in S(\sigma) \cup T(\sigma)$.

3. Restricted Case

In this section we fix $A \subset \{1, 2, \cdots, n\}$ and consider $\sum_A = \{\sigma \mid \bar{X}(\sigma) = A\}$. If the cardinal number of $A = a$, then there is no loss of generality in assuming $A = \{1, 2, \cdots, a\}$. Let $b = n - a$, $y_k = x_{a+k}$, $n'(\sigma, k) = n(\sigma, a+k)$, $\beta(\sigma, k) = \bar{X}(\sigma, a+k)$ $(k=1, 2, \cdots, b)$ and $B_i' = \{j - a \mid j \in B_i\}$, where $B_i \subset B = \{a+1, a+2, \cdots, n\}$.

LEMMA 3.1.

$$\prod_l \forall t_l \prod_i \exists x_i (\min_l \min_i |t_l - a^{-1} x_i| > \delta, \min_{\alpha \neq \alpha'} |a^{-1} \sum_{i \in \alpha} x_i - a^{-1} \sum_{i \in \alpha'} x_i| > 2\delta)$$

for some positive δ that depends only on n.

PROOF. For each $i \leq a$ choose $n_{i,l} \in \{1, 2, 3\}$ inductively so that the inequalities

$$|t_l - a^{-1} \sum_{p \leq l} n_{i,p} 5^{-ri-p}| \geq a^{-1} 5^{-ri-l} \qquad (l=1, 2, \cdots, r)$$

are satisfied. Put

$$x_i = \sum_l n_{i,l} 5^{-ri-l} \qquad (i=1, 2, \cdots, a).$$

Then

$$|t_l - a^{-1} x_i| > a^{-1} 5^{-ri-l} \cdot (1 - 5^{-1} \cdot 3 \cdot (1 - 5^{-1})^{-1})$$

$$> a^{-1} 5^{-ri-l-1} \geq a^{-1} 5^{-ra-r-1} \qquad (i=1, 2, \cdots, a) \ (l=1, 2, \cdots, r).$$

If $\alpha \neq \alpha'$ then $j \in (\alpha \smallsetminus \alpha') \cup (\alpha' \smallsetminus \alpha)$ for some $j \leq a$. It follows that

$$|a^{-1} \sum_{i \in \alpha} x_i - a^{-1} \sum_{i \in \alpha'} x_i| > a^{-1} 5^{-rj-1} > 2a^{-1} 5^{-ra-r-1}.$$

The lemma is proved for $\delta \leq a^{-1} 5^{-ra-r-1}$.

LEMMA 3.2. *Let* $\sigma = \prod_{k \leq m(\sigma)} \prod_{i \in A_k(\sigma)} \text{Max} \prod_{x_i} \min_{j \in B_k'} \min_{y_j}$ *and let*

$$f_A = \prod_{k \leq b} \prod_\alpha (c_{k,\alpha}(y_k - a^{-1} \sum_{j \in \alpha} x_j)^2 + 1),$$

where $c_{k,\alpha} > 2$ *for* $k \leq b$, $\phi \neq \alpha \subset A$ *and* $c_{k,\alpha} < c_{k,\alpha'}$ *for* $\alpha \subsetneqq \alpha'$ $(f_A \equiv 1$ *in the case* $a = n$ *or* $a = 0$). *Then there exists a positive number* ε *which depends only on* n *such that for all* σ

$$\sigma(f_A) > \varepsilon \prod_{k \leq b} c_{k,\beta(\sigma,k)}^{-1} \prod_\alpha c_{k,\alpha'}$$

where $c_{k,\phi} = 1$ *in the case* $\beta(\sigma, k) = \phi$.

PROOF. The inequalities

$$\min_{\alpha \neq \alpha'} |a^{-1} \sum_{j \in \alpha} x_j - a^{-1} \sum_{j \in \alpha'} x_j| > 2\delta, \quad |y_k - a^{-1} \sum_{j \in \alpha} x_j| < \delta$$

imply

$$\forall \alpha' \neq \alpha \ |y_k - a^{-1} \sum_{j \in \alpha'} x_j| > \delta.$$

Consequently, since $c_{k,\alpha} \leq c_{k,\alpha'}$ for $\alpha \subset \alpha'$, if the inequalities

$$\min_{k \leq b} \min_{\alpha \subset \beta(\sigma,k)} |y_k - a^{-1} \sum_{j \in \alpha} x_j| > \delta, \quad \min_{\alpha \neq \alpha'} |a^{-1} \sum_{j \in \alpha} x_j - a^{-1} \sum_{j \in \alpha'} x_j| > 2\delta$$

are satisfied, then

$$f_A(x_1, x_2, \cdots, x_n) > \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha (\delta^2 c_{k, \alpha} + 1) > \delta^{2b(2^a - 1)} \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha c_{k, \alpha}.$$

But from Lemma 3.1

$$\prod_{k \leq m(\sigma)} \prod_{i \in A_k(\sigma)} \exists x_i \prod_{j \in B'_k(\sigma)} \forall y_j$$

$$\min_{k \leq b} \min_{\alpha = \beta(\sigma, k)} |y_k - a^{-1} \sum_{j = \alpha} x_j| > \delta, \quad \min_{\alpha \neq \alpha'} |a^{-1} \sum_{j = \alpha} x_j - a^{-1} \sum_{j = \alpha'} x_j| > 2\delta)$$

for some $\delta > 0$. Therefore the lemma is proved for $\varepsilon \leq \delta^{2b(2^a - 1)}$.

LEMMA 3.3.

$$\sigma(f_A) < 2^{b(2^a - 1)} \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha c_{k, \alpha}.$$

PROOF. For each $k \leq b$, let $y_k = a^{-1} \sum_{j = \beta(\sigma, k)} x_j$, then

$$f_A(x_1, x_2, \cdots, x_n) < \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha (c_{k, \alpha} + 1)$$

$$< \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha 2 c_{k, \alpha} = 2^{b(2^a - 1)} \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha c_{k, \alpha}.$$

It follows that

$$\sigma(f_A) < 2^{b(2^a - 1)} \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha c_{k, \alpha}.$$

THEOREM 3.4. *There exists a positive number ε which depends only on n such that if*

$$\text{Max}_{\mu \neq \sigma} \min \{ \prod_{k \leq b} c_{k, \beta(\sigma, k)} c_{k, \beta(\mu, k)}^{-1}, \prod_{k \leq b} c_{k, \beta(\mu, k)} c_{k, \beta(\sigma, k)}^{-1} \} < \varepsilon,$$

then $\mu(f_A) < \sigma(f_A)$ follows from $\prod_{k \leq b} c_{k, \beta(\sigma, k)} < \prod_{k \leq b} c_{k, \beta(\mu, k)}$.

PROOF. By Lemma 3.2, for some $\delta > 0$,

$$\delta \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha c_{k, \alpha} < \sigma(f_A).$$

By Lemma 3.3

$$\mu(f_A) < 2^{b(2^a - 1)} \prod_{k \leq b} c_{k, \beta(\mu, k)}^{-1} \prod_\alpha c_{k, \alpha}.$$

Then

$$\prod_{k \leq b} c_{k, \beta(\sigma, k)} c_{k, \beta(\mu, k)}^{-1} < 2^{-b(2^a - 1)} \delta \equiv \varepsilon$$

implies

$$\mu(f_A) < 2^{b(2^a - 1)} \prod_{k \leq b} c_{k, \beta(\mu, k)}^{-1} \prod_\alpha c_{k, \alpha} < \delta \prod_{k \leq b} c_{k, \beta(\sigma, k)}^{-1} \prod_\alpha c_{k, \alpha} < \sigma(f_A).$$

COROLLARY 3.5. *There exists a polynomial of degree $\min \{2b(2^a - 1)\}, 2a(2^b - 1)\}$ such that $\sigma(f) \neq \mu(f)$ for any different $\sigma, \mu \in \sum_A$.*

PROOF. Let f_A and f_B be the polynomials in Theorem 3.4. Then the degrees of f_A and f_B are $2b(2^a - 1)$ and $2a(2^b - 1)$ respectively. For $\sigma \in \sum_A$,

$$\sigma(-f_B) = \prod_{k \leq m(\sigma)} \prod_{i \in A(\sigma, k)} \text{Max}_{x_i} \prod_{j \in B(\sigma, k)} \min_{x_j} (-f_B)$$

$$= - \prod_{k \leq m(\sigma)} \prod_{i \in A(\sigma, k)} \min_{x_i} \prod_{j \in B(\sigma, k)} \text{Max}_{x_j} (f_B).$$

Consequently f_A or $-f_B$ is the desired function.

REMARK. Since there is no positive numbers c_1, c_2, c_3 such that $c_1 < c_2$, $c_2 c_3 < c_1 c_3$, no function f of the type in Theorem 3.4 satisfies the inequalities

$$\min_{y_1} \min_{y_3} \text{Max}_x \min_{y_2} (f) < \min_{y_2} \min_{y_3} \text{Max}_x \min_{y_1} (f),$$

$$\min_{y_2} \text{Max}_x \min_{y_1} \min_{y_3} (f) < \min_{y_1} \text{Max}_x \min_{y_2} \min_{y_3} (f).$$

According to the general results in section 4, there exists a continuous function satisfying these inequalities.

4. General Case

In this section we fix a linear order $<_r$ on $\sum$ which is an extension of the natural order $<_\Sigma$ and number the elements of $\sum$ according to $<_r$. I.e. $p < q$ iff $\sigma_p <_r \sigma_q$.

DEFINITION 4.1.

(1) $$h_i(x, y) = \text{Max}\{\min\{2\delta_i^{-1} \min_{d \leq 2^{n_i}} | |3x - y| - d2^{-n_i}| - 1, 1\}, 0\},$$

where $n_i = i(n+3)$, $\delta_i = 2^{1-n_{i+1}}$ $(i = 1, 2, \cdots, n)$.

(2) $$f_{<_r}(x_1, x_2, \cdots, x_n) = \text{Max}_{q < r} \min_{p \leq q} \text{Max}_{i \in \underline{X}(\sigma_p)} \text{Max}_{j \in \bar{X}(\sigma_p, i)} l h_j(x_i, x_j)$$

$(h_0(x) \equiv \text{Max}_{j \in \phi} h_j(x_i, x_j) \equiv \min\{2x_i, 1\}$ in the case $\bar{X}(\sigma_p, i) = \phi)$.

LEMMA 4.2.

$$\forall i \prod_{j \neq i} \forall x_j \exists x_i \prod_k \forall y_k \min_{j \neq i} h_i(x_j, x_i) = \min_k h_k(x_i, y_k) = h_0(x_i) = 1.$$

PROOF. Let μ denote Lebesgue measure on $[0, 1]$. For arbitrary i, x,

$$\mu(\{y \mid h_i(x, y) = 1\}) = \mu(\{y \mid \min_{d \leq 2^{n_i}} | |3x - y| - d2^{-n_i}| \geq \delta_i\})$$

$$= \mu(\{y \mid |3x - y| \notin \bigcup_{d \leq 2^{n_i}} [d2^{-n_i} - \delta_i, d2^{-n_i} + \delta_i]\})$$

$$\geq 1 - 2^{n_i+1}\delta_i = 1 - 2^{i(n+3)+1-(i+1)(n+3)+1} = 1 - 2^{-n-1} \geq 1 - 4^{-1}n^{-1}.$$

Consequently

$$\forall i \prod_{j \neq i} \forall x_j \mu(\{y \mid \min_{j \neq i} h_i(x_j, y) = 1\}) \geq 1 - (n-1)(4n)^{-1} > 4^{-1} \cdot 3,$$

and so

$$\forall i \prod_{j \neq i} \forall x_j \exists y > 4^{-1} \cdot 3(\min_{j \neq i} h_i(x_j, y) = 1). \qquad (1)$$

But $y > 4^{-1} \cdot 3$ implies

$$\forall k \forall x \, h_k(y, x) = 1, \qquad h_0(y) = 1. \qquad (2)$$

Therefore the lemma follows from (1) and (2).

LEMMA 4.3. $\forall p \, \sigma_p(f_{<_r}) \geq p - 1$.

PROOF. We claim

$$\prod_{k \leq m(\sigma_p)} \prod_{i \in A(\sigma_p, k)} \exists x_i \prod_{j \in B(\sigma_p, k)} \forall x_j \forall q < p \, g_q = 1, \qquad (1)$$

where $g_q = \text{Max}_{i \in \underline{X}(\sigma_q)} \text{Max}_{j \in \bar{X}(\sigma_q, i)} h_j(x_i, x_j)$ $(q = 1, 2, \cdots, r-1)$.

168

H. Hisano and F. Maruyama

From Lemma 4.2

$$\prod_{k\leq m(\sigma_p)}\prod_{i\in A(\sigma_p,k)}\exists x_i\prod_{j\in B(\sigma_p,k)}\forall x_j$$
$$(\forall i\in\overline{X}(\sigma_p)\forall j\in\underline{X}(\sigma_p,i)\forall k\ h_i(x_j,x_i)=h_k(x_i,x_k)=h_0(x_i)=1).\qquad(2)$$

Since $q<p$ implies $\sigma_q\in S(\sigma_p)\cup T(\sigma_p)$, there are two cases.

CASE 1. $\sigma_q\in S(\sigma_p)$.

Then
$$\exists j\in\underline{X}(\sigma_q)\exists i\in\overline{X}(\sigma_q,j)\ (i\in\overline{X}(\sigma_p),\ j\in\underline{X}(\sigma_p,i)).$$
So
$$\forall i\in\overline{X}(\sigma_p)\forall j\in\underline{X}(\sigma_p,i)\ h_i(x_j,x_i)=1$$
implies
$$\exists j\in\underline{X}(\sigma_q)\exists i\in\overline{X}(\sigma_q,j)\ h_i(x_j,x_i)=1.$$

CASE 2. $\sigma_q\in T(\sigma_p)$.

Then
$$\exists i\in\overline{X}(\sigma_p)\cap\underline{X}(\sigma_q).$$
So
$$\forall i\in\overline{X}(\sigma_p)\forall k\ h_k(x_i,x_k)=h_0(x_i)=1$$
implies
$$\exists i\in\underline{X}(\sigma_q)\operatorname*{Max}_{j\in\overline{X}(\sigma_q,i)}h_j(x_i,x_j)=1.$$

In both cases
$$\forall i\in\overline{X}(\sigma_p)\forall j\in\underline{X}(\sigma_p,i)\forall k\ h_i(x_j,x_i)=h_k(x_i,x_k)=h_0(x_i)=1$$
implies $g_q=1$.

Therefore (1) follows from (2). Then
$$\prod_{k\leq m(\sigma_p)}\prod_{i\in A_k(\sigma_p,k)}\exists x_i\prod_{j\in B_k(\sigma_p,k)}\forall x_j(\operatorname*{min}_{q\leq p-1}(p-1)g_q=p-1).$$
Thus
$$\prod_{k\leq m(\sigma_p)}\prod_{i\in A_k(\sigma_p,k)}\exists x_i\prod_{j\in B_k(\sigma_p,k)}\forall x_j(\operatorname*{Max}_{l<r}\operatorname*{min}_{q\leq l}lg_q\geq p-1).$$

LEMMA 4.4.
$$\prod_i\forall x_i\exists x\operatorname*{Max}_i h_i(x,x_i)=0.$$

PROOF. Since $\delta_1=2^{1-n_2}=2^{1-2(n+3)}<2^{-1}$ and $2^{-n_1}=2^{-(n+3)}<2^{-1}$, there exists a $d_1\leq2^{n_1}$ such that
$$I_1=[x_1+d_12^{-n_1}-2^{-1}\delta_1,\ x_1+d_12^{-n_1}+2^{-1}\delta_1]\subset[0,1].$$

If $i>1$ and I_{i-1} is a closed interval of length δ_{i-1} contained in $[0,1]$,
$$\delta_i<2^{-1}\delta_{i-1}\ \text{and}\ 2^{-n_i}=2^{-1}\delta_{i-1}\ \text{imply}$$
$$\exists d_i\leq2^{n_i}[x_i+d_i2^{-n_i}-2^{-1}\delta_i,\ x_i+d_i2^{-n_i}+2^{-1}\delta_i]\subset I_{i-1}.$$

Thus, inductively, there exists a sequence of closed intervals
$$I_n\subset I_{n-1}\subset\cdots\subset I_1\subset[0,1],$$
where for each i
$$\exists d_i\leq2^{n_i}I_i=[x_i+d_i2^{-n_i}-2^{-1}\delta_i,\ x_i+d_i2^{-n_i}+2^{-1}\delta_i].$$

If $3x \in I_n = \bigcap_i I_i \subset [0, 1]$, then $\prod_i \forall x_i \operatorname{Max}_i h_i(x, x_i) = 0$.

LEMMA 4.5. $\forall p \ \sigma_p(f_{<_r}) = p - 1$.

By Lemma 4.3, it suffices to prove $\forall p \ \sigma_p(f_{<_r}) \leqq p - 1$.

Lemma 4.4 and $h_0(0) = 0$ imply

$$\forall \sigma \forall i \in \underline{X}(\sigma) \prod_{j \in \bar{X}(\sigma, i)} \forall x_j \exists x_i \operatorname*{Max}_{j \in \bar{X}(\sigma, i)} h_j(x_i, x_j) = 0.$$

Thus, for $p < r$,

$$\prod_{k \leqq m(\sigma_p)} \prod_{j \in A_k(\sigma_p)} \forall x_j \prod_{i \in B_k(\sigma_p)} \exists x_i \operatorname*{Max}_{i \in \underline{X}(\sigma_p)} \operatorname*{Max}_{j \in \bar{X}(\sigma_p, i)} h_j(x_i, x_j) = 0.$$

I. e. $\sigma_p(g_p) = 0$. Consequently, for $p < r$,

$$\sigma_p(f_{<_r}) = \sigma_p(\operatorname*{Max}_{l < r} \operatorname*{min}_{q \leqq l} lg_q) \leqq \sigma_p(\operatorname{Max}\{p - 1, (r - 1)g_p\})$$

$$= \operatorname{Max}\{p - 1, (r - 1)\sigma_p(g_p)\} = p - 1.$$

The inequality $\sigma_r(f_{<_r}) \leqq r - 1$ is obvious.

THEOREM 4.6. *Let $<_s$ be an extension of the natural order on Σ. Then for some $f \in C([0, 1]^n)$ and arbitrary $\sigma, \mu \in \Sigma$, $\sigma <_s \mu$ implies $\sigma(f) < \mu(f)$.*

PROOF. Extend $<_s$ to a linear order $<_r$ on Σ. Then the theorem follows from Lemma 4.5.

Acknowledgement

References

[1] IWAMOTO, S.: *Dynamic Programming*, Kyushu University Press, (1987) (Japanese).

[2] VON NEUMANN, J.: *Zur Theorie der Gesellschaftssiele*, Mathematische Annalen, **100** (1928), 295-320.