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AN OPTIMAL STOPPING PROBLEM IN THE PRESENCE OF COSTS OF OBSERVATIONS

By

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Abstract

The present paper deals with an optimal stopping problem which permits the cost of obserbation in the case of continuous time Markov processes. Under certain conditions of the terminal functions and the running cost functions, we show the existence of a finite optimal stopping time. Moreover we also discuss a free boundary problem concerning the optimal stopping problem.

1. Introduction

Discrete time optimal stopping problems have been studied by many authors. Chow and Robbins [2] have studied the case where optimal Markov times are actually finite. Furukawa [4] has studied the problems with general reward systems and especially has given sufficient conditions for the sequence of value iterations to converge to an optimal return without assuming the uniqueness of the solution of the optimality equation. In continuous time case Fakeev [8, 9] and Thompson [10] have treated optimal stopping problems, using martingale theory, in which the finiteness of optimal stopping times has been assumed. In [5], Shiryayev has treated optimal stopping problems of continuous time Markov process in the case where the cost of observation is incurred. It is stated as follows: E is locally compact space with a countable basis. $X=(X_t, \mathfrak{F}, \mathfrak{F}_t, \theta_t, P^x)$ is a standard Markov process with state space $(E, \mathfrak{E})$. f is a continuous function on E and C is a bounded nonnegative universally measureable function on E which satisfies $\int_0^\infty C(X_t)dt=\infty$ a. s. Then the problem is to find a finite stopping time τ maximizing

$$(1) \qquad E^x\left[f(X_\tau)-\int_0^\tau C(X_s)ds\right] \qquad x\in E.$$

Shiryayev has given an optimal stopping time for this problem in Chapter III of [5] under certain cumbersome conditions (see [5, P. 107, Theorem 3]).

In the present paper we shall give an optimal stopping time of Problem (1) without assuming Shiryayev's conditions and show that the optimal value of Problem (1) is a unique solution of Stefan problem if it exists.

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2. Optimal Stopping of a Markov Process in the Presence of the Cost of Observation

Let T be the set of all nonnegative real numbers, the time space. Let $\{P_t\}_{t\geqq 0}$ be the family of the transition functions of the process X defined in Section 1. Let $\bar{\mathcal{M}}$ (resp. $\mathcal{M}$) be the family of all Markov times (resp. the family of all stopping times, that is, all Markov times which are finite with probability one for every starting point). We assume that the function C satisfies $\int_0^\infty C(X_s)ds=\infty$ almost surely. Define an optimal value function V by

$$V(x)=\sup_{\tau\in\mathcal{M}} E^x\left[f(X_\tau)-\int_0^\tau C(X_s)ds\right] \tag{2}$$

for $x\in E$. Obviously the value of (2) does not change even if we replace $\mathcal{M}$ with $\bar{\mathcal{M}}$ in (2). A stopping time τ^* is said to be optimal if

$$V(x)=E^x\left[f(X_{\tau^*})-\int_0^{\tau^*} C(X_s)ds\right]$$

for all $x\in E$. Let $\|\ \|$ be the supremum norm on E. Let $b\mathfrak{E}^*$ be the family of all bounded universally measurable functions on E. Define, for each $t\in T$, an operator R_t on $b\mathfrak{E}^*$ by

$$R_t k(x)=E^x\left[k(X_t)-\int_0^t C(X_s)ds\right]$$

for $x\in E$. Then the next lemma is immediate.

LEMMA 2.1. *$\{R_t\}_{t\geqq 0}$ is a semigroup on $b\mathfrak{E}^*$.*

Define, for each $t\in T$, operators Q_t and Q_t^N by

$$Q_t k=k\vee R_t k$$

and

$$Q_t^{N+1}k=Q_tQ_t^N k$$

for natural numbers N and $k\in b\mathfrak{E}^*$.

LEMMA 2.2. *Let a natural number n be arbitrary but fixed. Then $V_n=\lim_{N\to\infty} Q_{2^{-n}}^N f$ is the smallest excessive majorant of f for the Markov chain with the time space $\{m\cdot 2^{-n}: m$ is a nonnegative integer.$\}$, that is, V_n satisfies* (a), (b) *and* (c):

(a) $V_n\geqq f$.

(b) $V_n\geqq R_{2^{-n}}V_n$.

(c) If $k\in b\mathfrak{E}^*$ satisfies $k\geqq f$ and $k\geqq R_{2^{-n}}k$, then $k\geqq V_n$.

PROOF. We can easily check the above results in the same line as Chapter II of Shiryayev [5]. □

LEMMA 2.3. *$\{V_n\}_n$ is an increasing sequence.*

PROOF. $R_{2^{-n}}V_{n+1}=R_{2^{-(n+1)}}R_{2^{-(n+1)}}V_{n+1}\leqq R_{2^{-(n+1)}}V_{n+1}\leqq V_{n+1}$. Since $V_{n+1}\geqq f$, Lemma 2.2 implies $V_{n+1}\geqq V_n$. □

LEMMA 2.4. *$\bar{V}=\lim_{n\to\infty} V_n$ is the smallest excessive majorant of f with respect to the semigroup $\{R_t\}_{t\geqq 0}$, that is, $\bar{V}$ satisfies* (a)~(d):

(a) $\bar{V}\geqq f$.

(b) $\bar{V}\geqq R_t\bar{V}$ *for all* $t\in T$.

(c) *$\bar{V}$ is finely lower-semicontinuous and nearly Borel measurable.*

(d) *If $k \in b\mathfrak{E}^*$ satisfies $k \geqq f$ and $k \geqq R_t k$ for all $t \in T$, then $k \geqq \bar{V}$.*

REMARK. The definitions of the finely topology and nearly Borel measurability are referred to Blumenthal and Getoor [1].

PROOF. We have $f \leqq V_n \leqq \|f\|$ for all natural numbers n, therefore (a) follows from Lemma 2.3. Next from Lemmas 2.2 and 2.3, we have that

$$(3) \qquad \bar{V} \geqq R_{k' \cdot 2^{-k}} \bar{V} \text{ for all natural numbers } k \text{ and } k'.$$

From the proof of Lemma III. 1 in Shiryayev [5], if ϕ is a bounded nearly Borel measurable finely lower-semicontinuous function on E such that the process $\{\phi(X_t)\}_{t \geqq 0}$ is separable, then $P_t\phi$ is bounded, nearly Borel measurable and finely lower-semicontinuous on E for each $t \in T$. On the other hand, for each $t \in T$, $\xi = \int_0^t C(X_r)dr$ is bounded and $\xi \circ \theta_s = \int_s^{t+s} C(X_r)dr$ converges to ξ almost surely as $s \downarrow 0$, therefore $E^x\left[\int_0^t C(X_r)dr\right]$ is finely continuous on E according to Corollary 4.16 in Dynkin [3]. Moreover

$$E^x\left[\int_0^t e^{-\alpha r} C(X_r)dr\right] = U^\alpha C(x) - e^{-\alpha t} P_t U^\alpha C(x)$$

is nearly Borel measurable on E for each positive real number α, where $U^\alpha k = \int_0^\infty e^{-\alpha r} P_r k \, dr$ for $k \in b\mathfrak{E}^*$. By the monotone convergence theorem, when letting α to 0, $E^x\left[\int_0^t C(X_r)dr\right]$ is bounded, finely continuous and nearly Borel measurable on E for each $t \in T$. Consequently $R_t\phi$ is bounded, finely lower-semicontinuous and nearly Borel measurable on E for each $t \in T$, and so is $Q_t\phi$. From the definition of $\bar{V}$, $\bar{V}$ is finely lower-semicontinuous and nearly Borel measureable. Finally we can prove (b) and (d) according to the same line as Lemma III. 1 in Shiryayev [5]. □

Now we examine properties of the optimal value function V.

LEMMA 2.5. *V satifies* (a) *and* (b):

(a) *$V \geqq f$.*

(b) *$V \geqq R_\tau V$ for each $\tau \in \mathcal{M}$.*

PROOF. (a) is tririal from (2). Fix any $\tau \in \mathcal{M}$ and $x \in E$ such that $R_\tau V(x) > -\infty$. Then $P_\tau(x, \cdot)$ is a measure on $(E, \mathfrak{E}^*)$. Now $K = \{R_\sigma f : \sigma \in \mathcal{M}\}$ is filtering upward, namely for any $R_{\sigma_i} f \in K$ $(i=1, 2)$ we define

$$\sigma = \begin{cases} \sigma_1 \text{ on } X_0 \in \{R_{\sigma_1} f > R_{\sigma_2} f\} \\ \sigma_2 \text{ otherwise,} \end{cases}$$

then we have $\sigma \in \mathcal{M}$ and $R_\sigma f = R_{\sigma_1} f \vee R_{\sigma_2} f$. We can choose a sequence $\{\sigma_n\} \subset \mathcal{M}$ such that $R_{\sigma_n} f$ converges increasingly to V almost surely $P_\tau(x, \cdot)$ as $n \to \infty$. Hence by using $E^x\left[\int_0^\tau C(X_s)ds\right] < \infty$ and a slight modification of Lemma 2.1, we obtain

$$\begin{aligned} R_\tau V(x) &= \lim_{n \to \infty} R_\tau R_{\sigma_n} f(x) \\ &= \lim_{n \to \infty} R_{\tau + \sigma_n \circ \theta_\tau} f(x) \leqq V(x). \quad \square \end{aligned}$$

PROPOSITION 2.6 *It holds that* $V=\overline{V}$.

PROOF. Lemma 2.4 (d) and Lemma 2.5 imply $V\geqq\overline{V}$. If we have $\overline{V}\geqq R_\tau\overline{V}$ for any $\tau\in\mathcal{M}$, then $\overline{V}\geqq R_\tau\overline{V}\geqq R_\tau f$ for any $\tau\in\mathcal{M}$. This relation implies $\overline{V}\geqq V$. Therefore it is sufficient to prove $\overline{V}\geqq R_\tau\overline{V}$ for any $\tau\in\mathcal{M}$ to complete the proof of this proposition.

Now we fix any $x\in E$ and $\tau\in\mathcal{M}$. Let t and α be positive real numbers. From Lemma 2.4 (b) we have

$$\begin{aligned}&E^x\left[\int_0^\tau \frac{e^{-\alpha r}P_t\overline{V}(X_r)-e^{-\alpha r}\overline{V}(X_r)}{t}dr\right]\\ &\leqq E^x\int_0^\tau\frac{e^{-\alpha r}}{t}E^{X_r}\left[\int_0^t C(X_s)ds\right]dr.\end{aligned}\tag{4}$$

We shall calculate the limitting forms of each side of (4) as $t\downarrow 0$ and $\alpha\downarrow 0$.

$$\begin{aligned}&E^x\left[\int_0^\tau e^{-\alpha r}P_t\overline{V}(X_r)dr\right]\\ &=E^x\left[\int_0^\infty e^{-\alpha r}P_t\overline{V}(X_r)dr\right]-E^x\left[\int_\tau^\infty e^{-\alpha r}P_t\overline{V}(X_r)dr\right]\end{aligned}\tag{5}$$

and then we define

$$\phi_s(y)=E^y\left[\int_0^\infty e^{-\alpha r}P_s\overline{V}(X_r)dr\right]$$

for $y\in E$ and $s\in T$. From the Markov property we can show

$$\phi_s(y)=e^{\alpha s}\int_s^\infty e^{-\alpha r}E^y[\overline{V}(X_r)]dr\tag{6}$$

for each $y\in E$ and $s\in T$. On the other hand, by using the strong Markov property and the definition of ϕ_s, we have

$$\begin{aligned}(5)&=\phi_t(x)-E^x\left[\int_0^\infty e^{-\alpha\tau}e^{-\alpha r}P_t\overline{V}(X_{\tau+r})dr\right]\\ &=\phi_t(x)-E^x[e^{-\alpha\tau}\phi_t(X_\tau)].\end{aligned}$$

Therefore the left hand side of (4) is equal to

$$t^{-1}\cdot\{\phi_t(x)-\phi_0(x)\}-E^x[e^{-\alpha\tau}t^{-1}\cdot\{\phi_t(X_\tau)-\phi_0(X_\tau)\}].\tag{7}$$

From (6) for each $y\in E$ we obtain

$$\begin{aligned}&t^{-1}\{\phi_t(y)-\phi_0(y)\}\\ &=t^{-1}(e^{\alpha t}-1)\int_t^\infty e^{-\alpha r}P_r\overline{V}(y)dr-t^{-1}\int_0^t e^{-\alpha r}P_r\overline{V}(y)dr.\end{aligned}\tag{8}$$

Then it holds that the left hand side of (8) converges to $\alpha\int_0^\infty e^{-\alpha r}P_r\overline{V}(y)dr-\overline{V}(y)$ as $t\downarrow 0$, by using Lemma 2.4 (b) and (c). Hence from (8) we have $t^{-1}\|\phi_t-\phi_0\|\leqq 2\cdot\|\overline{V}\|$ for any $t>0$. Therefore by the bounded convergence theorem, (7) converges to

(9) $$\alpha U^{\alpha}\bar{V}(x)-\bar{V}(x)-E^{x}[e^{-\alpha\tau}\cdot\alpha U^{\alpha}\bar{V}(X_{\tau})-e^{-\alpha\tau}\bar{V}(X_{\tau})]$$

as $t\downarrow 0$, where the operator U^{α} is defined in the proof of Lemma 2.4. Now (9) is equal to

(10) $$\alpha E^{x}\Big[\int_{0}^{\tau}e^{-\alpha r}\bar{V}(X_{r})dr\Big]-\bar{V}(x)+P_{\tau}^{\alpha}\bar{V}(x).$$

Since $\tau\in\mathcal{M}$ and

(11) $$\Big|\alpha E^{x}\Big[\int_{0}^{\tau}e^{-\alpha r}\bar{V}(X_{r})dr\Big]\Big|\leqq E^{x}[1-e^{-\alpha\tau}]\cdot\|\bar{V}\|,$$

the left hand side of (11) converges to 0 as $\alpha\downarrow 0$. Consequently, according to the bounded convergence theorem, (10) converges to $-\bar{V}(x)+P_{\tau}\bar{V}(x)$ as $\alpha\downarrow 0$, which implies the left hand side of (4) converges to $-\bar{V}(x)+P_{\tau}\bar{V}(x)$ as $t\downarrow 0$ and $\alpha\downarrow 0$.

Next we shall consider the right hand side of (4) as $t\downarrow 0$ and $\alpha\downarrow 0$. The right hand side of (4) is equal to

(12) $$E^{x}\Big[\int_{0}^{\tau}e^{-\alpha r}E^{X_{r}}\Big[\frac{1}{t}\int_{0}^{t}C(X_{s})ds\Big]dr\Big].$$

After some calculations on (12) similar to those on (5), (12) becomes to

(13) $$t^{-1}\Big\{\int_{0}^{\infty}e^{-\alpha r}\int_{r}^{r+t}E^{x}[C(X_{s})]ds\,dr-\int_{0}^{\infty}e^{-\alpha r}\int_{r}^{r+t}E^{x}[e^{-\alpha\tau}E^{X_{\tau}}[C(X_{s})]]ds\,dr\Big\}.$$

Hence (13) converges to

(14) $$E^{x}\Big[\int_{0}^{\tau}e^{-\alpha r}C(X_{r})dr\Big]$$

as $t\downarrow 0$. Since $C\geqq 0$, (14) converges to

(15) $$E^{x}\Big[\int_{0}^{\tau}C(X_{r})dr\Big]$$

as $\alpha\downarrow 0$, which implies the right hand side of (4) converges to (15) as $t\downarrow 0$ and $\alpha\downarrow 0$. Consequently it holds that $-\bar{V}(x)+P_{\tau}\bar{V}(x)\leqq E^{x}\Big[\int_{0}^{\tau}C(X_{r})dr\Big]$. Since $x\in E$ and $\tau\in\mathcal{M}$ are arbitrary, we obtain $R_{\tau}\bar{V}\leqq\bar{V}$ for any $\tau\in\mathcal{M}$. This completes the proof of Proposition 2.6. □

Now we may identify $\bar{V}$ with V, therefore V has the properties of $\bar{V}$. We can prove the following Corollary 2.7 in the same line as Proposition 2.6.

COROLLARY 2.7. *Let σ, $\tau\in\mathcal{M}$ such that $\sigma\leqq\tau$ almost surely. Then it holds that $R_{\sigma}V\geqq R_{\tau}V$.*

The proof of the following lemma is same as that of Lemma II. 6 in Shiryayev [5].

LEMMA 2.8. $\limsup_{t\to\infty}V(X_{t})=\limsup_{t\to\infty}f(X_{t})$ *almost surely.*

Let ε be a nonnegative real number. Define a nearly Borel set B_{ε} and its entry time σ_{ε} by

$$B_\varepsilon=\{x\in E:\ V(x)\leqq f(x)+\varepsilon\},$$

$$\sigma_\varepsilon=\inf\{t:\ X_t\in B_\varepsilon\}$$

for each $\varepsilon\geqq 0$.

LEMMA 2.9. *It holds that*

$$f\leqq R_{\sigma_\varepsilon\wedge N}V$$

for any $\varepsilon>0$ and any positive real number N.

PROOF. Let $\varepsilon>0$ and $N>0$ be fixed. For any $\tau\in\mathcal{M}$ it holds that

$$\begin{aligned}\tau+(\sigma_\varepsilon\wedge N)\circ\theta_\tau&=(\tau+\sigma_\varepsilon\circ\theta_\tau)\wedge(\tau+N)\\&\geqq\sigma_\varepsilon\wedge N.\end{aligned}$$

From Corollary 2.7 and a slight modification of Lemma 2.1 we have

$$R_\tau R_{\sigma_\varepsilon\wedge N}V=R_{\tau+(\sigma_\varepsilon\wedge N)\circ\theta_\tau}V\leqq R_{\sigma_\varepsilon\wedge N}V \tag{16}$$

for any $\tau\in\mathcal{M}$. Define a constant d by

$$d=\sup_{x\in E}[f(x)-R_{\sigma_\varepsilon\wedge N}V(x)].$$

we should pay attention to $d\leqq\|f\|+\|V\|+\|C\|\cdot N<\infty$. Now we can prove this lemma by (16) and the same method as Lemma III. 8 in Shiryayev [5]. □

LEMMA 2.10. $V=R_{\sigma_\varepsilon}V$ *for any* $\varepsilon>0$.

PROOF. Let $x\in E$ and $\varepsilon>0$ be fixed. From Lemma 2.9 it holds that

$$\begin{aligned}f(x)-\|V\|&\leqq R_{\sigma_\varepsilon\wedge N}V(x)-\|V\|\\&\leqq E^x\Big[(V-\|V\|)(X_{\sigma_\varepsilon})-\int_0^{\sigma_\varepsilon}C(X_s)ds:\ \sigma_\varepsilon\leqq N\Big]\end{aligned}$$

for each $N>0$. Now we have $\sigma_\varepsilon<\infty$ almost surely according to Lemma 2.8. By means of the monotone convergence theorem, when N increases to infinite, (17) converges to $R_{\sigma_\varepsilon}V(x)-\|V\|$. Therefore $f\leqq R_{\sigma_\varepsilon}V$. For any $\tau\in\mathcal{M}$, by using Corollary 2.7, we obtain

$$R_\tau f\leqq R_\tau R_{\sigma_\varepsilon}V\leqq R_{\tau+\sigma_\varepsilon\circ\theta_\tau}V\leqq R_{\sigma_\varepsilon}V$$

for any $\tau\in\mathcal{M}$. From this result and Lemma 2.5 we have $V=R_{\sigma_\varepsilon}V$ for $\varepsilon>0$. □

COROLLARY 2.11. *For any* $\varepsilon>0$ *the time* σ_ε *is an* ε*-optimal stopping time.*

The proof is immediate from Lemma 2.10.

THEOREM 2.12. *The time* σ_0 *is an optimal stopping time.*

PROOF. Denine τ by $\tau=\lim_{\varepsilon\to 0}\sigma_\varepsilon$. Then $\tau\in\mathcal{M}$ and $\tau\leqq\sigma_0$. Now we show $V=R_\tau f$. Let $x\in E$ be fixed. From Lemma 2.10, Corollary 2.11 and the definition of σ_ε, it holds that

$$\begin{aligned}V(x)&=\lim_{\varepsilon\to 0}R_{\sigma_\varepsilon}V(x)\\&\leqq\lim_{\varepsilon\to 0}[R_{\sigma_\varepsilon}f(x)+\varepsilon]\\&=\lim_{\varepsilon\to 0}E^x\Big[f(X_{\sigma_\varepsilon})-\int_0^{\sigma_\varepsilon}C(X_s)ds\Big].\end{aligned} \tag{18}$$

Since X is quasi-left-continuous, by using convergence theorems, it holds that

$$(18)\leqq E^x\left[f(X_\tau)-\int_0^\tau C(X_s)ds:\ \tau<\infty\right]$$
$$+E^x\left[\limsup_{t\to\infty} f(X_t)-\int_0^\infty C(X_s)ds:\ \tau=\infty\right].$$

However $P^x(\tau<\infty)=1$ from the boundedness of V and the assumption of C. Consequently $V(x)=R_\tau f(x)$. Since $x\in E$ is arbitrary, it holds that

$$V=R_\tau V=R_\tau f\,. \tag{19}$$

(19) implies that $E^\cdot\left[\int_0^\tau C(X_s)ds\right]$ is bounded on E. Therefore (19) is reduced to $E^x[V(X_\tau)]=E^x[f(X_\tau)]$ for each $x\in E$, which implies $\tau\geqq\sigma_0$ almost surely. So $\tau=\sigma_0$ almost surely and (19) implies

$$V=R_{\sigma_0}f\,. \tag{20}$$

Finally from boundedness of V, f and c, we obtain $\sigma_0<\infty$ almost surely. This completes the proof of Theorem 2.12. □

REMARK 1. In this paper we may assume that f is upper-semicontinuous and finely continuous instead of continuity of f.

REMARK 2. If we find an optimal value V, then an optimal stopping time σ_0 is given by Theorem 2.12. Therefore we are interested in properties of V. From Lemma 2.10, if C is nearly Borel measurable and finely continuous, then V satisfies the following conditions

$$\begin{cases} \mathcal{A}V-C=0 & \text{on}\quad B_0 \\ V=f & \text{off}\quad B_0, \end{cases} \tag{21}$$

where $\mathcal{A}$ is the characteristic operator in the finely topology. You may refer to Dynkin [3] about the definition of $\mathcal{A}$.

In the rest of this paper we shall consider the uniqueness of the solutions of (21). Let A be the weak infinitesimal generator and D_A be its domain.

PROPOSITION 2.13. *Let f, c and V be taken as before. Moreover we assume that C satisfies $C(x)=\lim_{t\downarrow 0}P_tC(x)$ for each $x\in E$. If $\overline{V}\in D_A$ satisfies conditions* (i)~(iii):

(i) $A\overline{V}-C\leqq 0$

(ii) $V\geqq f$

(iii) $A\overline{V}-C=0$ on $\{\overline{V}>f\}$,

then $\overline{V}=V$.

PROOF. From Dynkin's formula and (iii) we have

$$R_{\tau\wedge N}\overline{V}=\overline{V} \tag{22}$$

for any positive constant N. Where τ is the first entry time of $\{x\in E:\ \overline{V}(x)=f(x)\}$. Let $x\in E$ be fixed. Then (22) implies that

$$\bar{V}(x)-\|\bar{V}\|=E^x\left[(\bar{V}-\|\bar{V}\|)(X_\tau)-\int_0^\tau C(X_s)ds:\ \tau\leqq N\right]$$
$$+E^x\left[(\bar{V}-\|\bar{V}\|)(X_N)-\int_0^N C(X_s)ds:\ \tau>N\right].$$

By virtue of convergence theorems, we have

$$\bar{V}(x)-\|\bar{V}\|\leqq E^x\left[(\bar{V}-\|\bar{V}\|)(X_\tau)-\int_0^\tau C(X_s)ds:\ \tau<\infty\right]$$
$$+E^x\int_0^\infty C(X_s)ds:\ \tau=\infty\Big].$$

This implies $P^x(\tau<\infty)=1$ and $\bar{V}(x)\leqq R_\tau\bar{V}(x)$. Since x is arbitrary, $\tau\in\mathcal{M}$ and $\bar{V}\leqq R_\tau\bar{V}$.

On the other hand, from $\tau\in\mathcal{M}$ and the right-continuity of X and continuities of $\bar{V}$ and f, we obtain $R_\tau\bar{V}=R_\tau f$. Consequently $\bar{V}\leqq R_\tau\bar{V}=R_\tau f\leqq V$. Conversely $\bar{V}\geqq V$ is easily obtained from (i) and (ii). □

Remark. The proof of next case follows in the same line as that of Proposition 2.13. Let $C\equiv 0$. Let f be a continuous function on E such that $\lim_{t\to\infty}P_tf(x)=0$ for each $x\in E$. If $V'\in D_A$ satisfies conditions (i')~(iv'):

(i') $AV'\leqq 0$
(ii') $V'\geqq f$
(iii') $AV'=0$ on $\{V'>f\}$
(iv') $\lim_{t\to\infty}P_tV'(x)=0$ for each $x\in E$,

then $V'=V^*$, where $V^*(x)=\sup_{\tau\in\mathcal{M}}E^x[f(X_\tau)]$ for $x\in E$.

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