ZERO-SUM GAMES WITH STOPPING TIMES AND VARIATIONAL INEQUALITIES

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ZERO-SUM GAMES WITH STOPPING TIMES AND VARIATIONAL INEQUALITIES

By

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Abstract

This paper deals with a zero-sum game, whose state changes correspondingly with a kind of Markov process. In the game the two players' strategies are when to stop the game to maximize their own rewards.

In this paper we investigate optimal stopping times, optimal rewards and the conditions which should be satisfied. To describe more precisely, it is as follows.

Let M be an m-symmetric Hunt process and let $(\mathfrak{F}, \mathfrak{E})$ be the corresponding Dirichlet space. For $\alpha>0$ and $W\in\mathfrak{F}$ we consider a variational inequality:

(1.1) $$U^{\alpha}\in\mathfrak{K},\quad \mathfrak{E}_{\alpha}(U^{\alpha}-W, V-U^{\alpha})\geqq 0 \qquad \text{for all } V\in\mathfrak{K},$$

where $\mathfrak{K}$ is a certain subset of $\mathfrak{F}$. In this paper we investigate properties of solutions U^{α} of (1.1) and we show that solutions U^{α} has a quasi-continuous version which becomes to be a value of a certain zero-sum game associated with regions $X-F_i$ $(i=1,2)$ not to permit stopping the game. The aim of this paper is to discuss these in the case of m-symmetric Hunt processes, which have m-symmetric diffusion processes and m-symmetric jump processes as examples. When the processes are transient, we can treat the case of $\alpha=0$ and we discuss it similarly.

0. Introduction

Zero-sum Markov games with stopping times are studied by several authors. N. V. Krylov [6] and L. Stettner [8] have studied them by making use of penalty methods. J. M. Bismut [2] has proved that a zero-sum game possesses a value from the standpoint of convex analysis. Especially A. Bensoussan and A. Friedman [1] and A. Friedman [4] have investigated a relation between a zero-sum Markov game with stopping times and a certain variational inequality in the case of diffusion processes. On the other hand H. Nagai [7] has used Dirichlet space to discuss a relation between a stopping problem and a variational inequality. Moreover J. Zabczyk [9] has discussed zero-sum Markov games in this situation. We deal with them in the case of an m-symmetric Hunt process and the corresponding regular Dirichlet space $(\mathfrak{F}, \mathfrak{E})$ by a different way from [9] and also discuss them in the transient case.

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In the latter part of this section we state the notations of an m-symmetric process and the related results. In Section 1 we investigate a zero-sum game with a discounted rate α in the case of an m-symmetric Hunt process with a general regular Dirichlet space. In Section 2 we discuss a non-discounted zero-sum game in the case of an m-symmetric Hunt process with a transient regular Dirichlet space. In Section 3 we consider the case of an m-symmetric jump process as an example of Section 1 and we also consider the case of a diffusion in a bounded domain as an example of Section 2.

Let X be a locally compact separable Hausdorff space and let m be a positive Radon measure on X. Let $M=\{\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, P_x, \theta_t, \zeta\}$ be an m-symmetric Hunt process on X and let P_t and R_α be its transition function and resolvent respectively. In this paper we use notations and definitions of M. Fukushima [5] concerning Dirichlet space corresponding to m-symmetric Hunt process M. $L^2(X; m)$ denotes the real L^2-space with the inner product $(\cdot, \cdot)$. Let $T_t(G_\alpha)$ be the L^2-transition function (L^2-resolvent) induced from P_t (R_α resp.). $(\mathfrak{F}, \mathfrak{E})$ denotes a Dirichlet space associated with the process M. In this paper we assume that $(\mathfrak{F}, \mathfrak{E})$ is regular. $\|\cdot\|_\infty$ denotes the essential supremum norm on X and $d(\cdot, \cdot)$ expresses a metric on X. We express by $\{\mathfrak{F}_t\}_{0\leqq t\leqq\infty}$ the minimum completed admissible nondecreasing family of sub-σ-algebras of $\mathfrak{M}$.

Hence for $\alpha\geqq 0$ we define a continuous additive functional $A=\{A(t)\}_{t\geqq 0}$ associated with a multiplicative functional $\{e^{-\alpha t}\}_{t\geqq 0}$ as follows:

$A(t, \omega)$ is a real valued function on $[0, \infty)\times\Omega$ which is $\{\mathfrak{F}_t\}$-adapted and for which there exist a set $\Lambda\in\mathfrak{F}$ and an exceptional set N such that $P_x(\Lambda)=1$ for each $x\in X-N$ and for which the following properties (F1)~(F4) are satisfied for each $\omega\in\Lambda$:

(F1) $A(t, \omega)$ is finite and continuous in $t\geqq 0$,

(F2) $A(0, \omega)=0$,

(F3) $A(t, \omega)=A(\zeta(\omega), \omega)$ for each $t\geqq\zeta(\omega)$,

(F4) $A(t+s, \omega)=A(t, \omega)+e^{-\alpha t}A(s, \theta_t\omega)$ for each $t, s\geqq 0$.

Especially we note the property (F4), which is induced from [3], is different from additive functionals in [5]. For $\alpha\geqq 0$ we express by $\mathfrak{A}_c^\alpha$ the set of all continuous additive functionals associated with a multiplicative functional $\{e^{-\alpha t}\}_{t\geqq 0}$.

$C_0(X)$ denotes the space of all continuous functions on X with compact supports. We define $\langle\nu, U\rangle=\int_X U(x)\nu(dx)$ for $U\in\mathfrak{F}$ and Radon measures ν on X and we write by $\tilde{U}$ a quasi-continuous version of U for $U\in\mathfrak{F}$ in accordance with [5]. In this paper we also use the following notations:

$$U\vee V=max\{U, V\} \quad\text{and}\quad U\wedge V=min\{U, V\} \qquad \text{for } U, V\in\mathfrak{F},$$

$$U^+=U\vee 0 \quad\text{and}\quad U^-=-(U\wedge 0) \qquad \text{for } U\in\mathfrak{F}.$$

1. Zero-sum Games and Variational Inequalities

Let α be a fixed positive number. Let F_1 and F_2 be closed subsets of X. Let $f_1, f_2\in\mathfrak{F}\cap C_0(X)$ and put $W=R_0 g$ for $g\in\mathfrak{F}\cap L^\infty(X; m)$. We assume (A1) and (A2):

(A1) $X-F_i$ $(i=1, 2)$ are relatively compact.

(A2) $f_1 \leqq f_2$ m-a.e. on $F_1 \cap F_2$.

We call $F_1(F_2)$ a stoppable reageon for player I (II resp.). Of courese it is possible to take $F_1=F_2=X$. Hence we put

$$\mathfrak{K}=\{V \in \mathfrak{F}: f_1 \leqq V \ m\text{-a.e. on } F_1 \text{ and } f_2 \geqq V \ m\text{-a.e. on } F_2\}.$$

LEMMA 1. *There exists a unique solution U^α of $\mathfrak{K}$ which satisfies the variational inequality* (1.1):

$$\mathfrak{E}_\alpha(U^\alpha - W, V-U^\alpha) \geqq 0 \quad \text{for all } V \in \mathfrak{K}. \tag{1.1}$$

PROOF. Since $\mathfrak{K}$ is a closed convex subset of Hilbert space $(\mathfrak{F}, \mathfrak{E}_\alpha)$, there exists a unique element U^α of $\mathfrak{K}$ such that

$$\mathfrak{E}_\alpha(U^\alpha - W, U^\alpha - W) \leqq \mathfrak{E}_\alpha(V-W, V-W) \quad \text{for all } V \in \mathfrak{K}. \tag{1.1'}$$

Since $\mathfrak{K}$ is convex in $(\mathfrak{F}, \mathfrak{E}_\alpha)$, (1.1′) is equivalent to (1.1). Therefore we obtain Lemma 1. □

Now we note $\mathfrak{K}$ possesses the following property (R):

(R) $U \vee V \in \mathfrak{K}$ and $U \wedge V \in \mathfrak{K}$ for each $U, V \in \mathfrak{K}$.

LEMMA 2. *U^α satisfies both inequalities* (1.2) *and* (1.3):

$$\mathfrak{E}_\alpha(U^\alpha - W, (V-U^\alpha)^+) \geqq 0 \quad \textit{for all } V \in \mathfrak{K}, \tag{1.2}$$

$$\mathfrak{E}_\alpha(U^\alpha - W, (V-U^\alpha)^-) \geqq 0 \quad \textit{for all } V \in \mathfrak{K}. \tag{1.3}$$

PROOF. We obtain (1.2) ((1.3)), by taking $U^\alpha \vee V$ ($U^\alpha \wedge V$ resp.) instead of V in (1.1) and noting the property (R). □

The following lemma is a modification of Lemma 5.1.2 in [5].

LEMMA 3. *Let $\{U_n\}_n$ be an $\mathfrak{E}_1$-Cauchy sequence of quasi-continuous functions in $\mathfrak{F}$ such that $\|U_n\|_\infty$ is bounded uniformly in n. Then there exists a subsequence $\{U_{n_k}\}_k$ satisfying the condition that for q.e. $x \in X$,*

$$P_x(\{e^{-\alpha t} U_{n_k}(X_t)\}_k \textit{ converges uniformly in } t \textit{ on } [0, \infty]) = 1.$$

PROOF. For each $\varepsilon > 0$ and $U \in \mathfrak{F}$ we take $T = 1/\alpha \cdot \log(\|U\|_\infty/\varepsilon)$. Then for all $t > T$ and a.a. $\omega \in \Omega$ we have

$$\varepsilon = e^{-\alpha t} \|\tilde{U}\|_\infty \geqq |e^{-\alpha t} \tilde{U}(X_t \omega)|.$$

Hence by noting this fact, in the same line as the proof of Lemma 5.1.2 in [5] we obtain this lemma. □

We also need the following lemma and its proof, which are modifications of Theorem 5.2.2 in [5], in order to prove Theorem 1.

LEMMA 4. *For each $U \in \mathfrak{F} \cap L^\infty(X; m)$ there exists an additive functienal $A \in \mathfrak{A}_c^\alpha$ such that for q.e. $x \in X$ and Markov time τ* (1.4) *hold:*

$$\begin{cases} \tilde{U}(x) = E_x[A(\infty)] \quad \textit{and} \\ \tilde{U}(x) = E_x[A(\tau)] + E_x[e^{-\alpha\tau} \tilde{U}(X_\tau)]. \end{cases} \tag{1.4}$$

PROOF. We show this lemma, by modifying the proof of Theorem 5.1.1 in [5]. Fix an arbitrary $U\in\mathfrak{F}\cap L^{\infty}(X;m)$. Then by applying Theorem 4.3.2 in [5], there exists a quasi-continuous Borel measurable version $\tilde{U}$ of U and a properly exceptional set N such that $\tilde{U}$ vanishes on N for which (1.5) and (1.6) hold:

$$nR_{n+\alpha}\tilde{U}(x)\to\tilde{U}(x) \text{ as } n\to\infty \quad \text{for each } x\in X-N, \tag{1.5}$$

$$nR_{n+\alpha}\tilde{U}\to\tilde{U} \text{ as } n\to\infty \text{ strongly with respect to } \mathfrak{E}_1. \tag{1.6}$$

For each n we define

$$g_n(x)=\begin{cases} n(\tilde{U}(x)-nR_{n+\alpha}\tilde{U}(x)) & \text{if } x\in X-N,\\ 0 & \text{if } x\in N.\end{cases} \tag{1.7}$$

Then from (1.5), (1.6) and the resolvent equation we obtain (1.5′) and (1.6′):

$$R_\alpha g_n(x)\longrightarrow\tilde{U}(x) \text{ as } n\to\infty \quad \text{for each } x\in X-N, \tag{1.5′}$$

$$R_\alpha g_n\longrightarrow\tilde{U} \text{ as } n\to\infty \text{ strongly with respect to } \mathfrak{E}_1. \tag{1.6′}$$

Moreover for each n and $\omega\in\Omega$ we define an additive functional $A_n=\{A_n(t)\}_{t\ge 0}\in\mathfrak{A}_c^{\alpha}$ by (1.8):

$$A_n(t,\omega)=\int_0^t e^{-\alpha s}g_n(X_s\omega)ds \quad \text{for } (t,\omega)\in[0,\infty)\times\Omega. \tag{1.8}$$

Then we can prove this lemma in similar way to Theorem 5.1.1 of [5]. □

Let $\mathcal{M}$ be the family of all Markov times. Define an operator Q_τ and subspaces $\mathfrak{G}^+$ and $\mathfrak{G}^-$ by

$$Q_\tau k(x)=E_x[e^{-\alpha\tau}k(X_\tau)] \quad \text{for } \tau\in\mathcal{M},\ x\in X \text{ and } k\in\mathfrak{F},$$

$$\mathfrak{G}^+=\{(V-U^\alpha)^+ : V\in\mathfrak{R}\}, \qquad \mathfrak{G}^-=\{(V-U^\alpha)^- : V\in\mathfrak{R}\}.$$

For each Borel subset B of X, σ_B (τ_B) expresses the first entry time (the first hitting time resp.) of B:

$$\sigma_B=\inf\{t\geqq 0 : X_t\in B\}, \qquad \tau_B=\inf\{t>0 : X_t\in B\}.$$

THEOREM 1. *Uuder (A1) and (A2), there exists a properly exceptional set N for which the solution U^α satisfies (1.9) and (1.10):*

$$\tilde{U}^\alpha-\tilde{W}\geqq Q_{\tau\wedge\sigma_{B_2}}(\tilde{U}^\alpha-\tilde{W}) \ m\text{-a.e.} \quad \text{for each } \tau\in\mathcal{M}, \tag{1.9}$$

$$\tilde{U}^\alpha-\tilde{W}\leqq Q_{\sigma_{B_2}\wedge\sigma}(\tilde{U}^\alpha-\tilde{W}) \ m\text{-a.e.} \quad \text{for each } \sigma\in\mathcal{M}, \tag{1.10}$$

where $B_i=\{\tilde{U}^\alpha=\tilde{f}_i\}\cap(F_i-N)$ are stopping reageons for $i=1, 2$.

PROOF. First we show $U^\alpha\in\mathfrak{F}\cap L^\infty(X;m)$. Put $a=\|f_1\|_\infty+\|f_2\|_\infty+\|W\|_\infty+1$ and define $U_a=((W-a)\vee U^\alpha)\wedge(W+a)\in\mathfrak{R}$. Noting $U_a-W=((-a)\vee(U^\alpha-W))\wedge a$ are using Markov property of $(\mathfrak{F},\mathfrak{E}_\alpha)$, we obtain

$$\mathfrak{E}_\alpha(U_a-W, U_a-W)\leqq\mathfrak{E}_\alpha(U^\alpha-W, U^\alpha-W).$$

By virtue of Lemma 1, we obtain $U^\alpha\in\mathfrak{F}$ and $U_a=U^\alpha-W$ m-a.e. This fact shows U^α

$\in\mathfrak{F}\cap L^{\infty}(X;m)$.

Next for $i=1, 2$ and $\varepsilon\geqq 0$ we define

$$F_{i,\varepsilon}=\{x\in X: d(x, y)\leqq\varepsilon \text{ for some } y\in F_i\}.$$

Then we show that there exists a sequence $\{f_{2,n}\}_{n\geqq 1}$, modifications of f_2, which satisfies (1.11):

$$\text{(1.11)}\qquad \begin{cases} f_{2,n}\geqq f_2 \text{ on } X & \text{for each } n\,, \\ f_{2,n}=f_2 \text{ on } F_2 & \text{for each } n \quad\text{and} \\ \tilde{U}^{\alpha}\leqq\tilde{f}_{2,n}-1/n \text{ q.e. on } \overline{X-F_{2,1/n}} & \text{for each } n\,. \end{cases}$$

Here $\bar{B}$ denotes the closure of a subset B of X. Since $\overline{X-F_{2,\varepsilon}}$ is compact for each $\varepsilon>0$, there exists $W_{\varepsilon}\in C_0(X)$ which satisfies (1.12):

$$\text{(1.12)}\qquad \begin{cases} W_{\varepsilon}\geqq 0 & \text{on } X\,, \\ W_{\varepsilon}=a & \text{on } \overline{X-F_{2,\varepsilon}} \\ \text{and } W_{\varepsilon}=0 & \text{on } F_2\,, \end{cases}$$

where $a=\|f_1\|_{\infty}+\|f_2\|_{\infty}+\|W\|_{\infty}+1$. By virtue of Lemma 1.4.2 of [5], for each $\varepsilon>0$ there exists a sequence $\{W_{\varepsilon,n}\}_n$ of $\mathfrak{F}\cap C_0(X)$ which satisfies (1.13):

$$\text{(1.13)}\qquad \begin{cases} \text{Supp}\,[W_{\varepsilon,n}]\subset\{W_{\varepsilon}\neq 0\}\subset X-F_2 & \text{for each } n \\ \text{and } \|W_{\varepsilon,n}-W_{\varepsilon}\|_{\infty}\leqq 1/n & \text{for each } n\,. \end{cases}$$

Hence the subsequence $\{W_{1/n,n}\}_n$ posesses the property (1.14):

$$\text{(1.14)}\qquad \begin{cases} W_{1/n,n}=0 \text{ on } F_2 \text{ for each } n \text{ and} \\ W_{1/n,n}\geqq a+1/n \text{ on } X-F_{2,1/n} \text{ for each } n\,. \end{cases}$$

Therefore by putting $f_{2,n}+f_2+(W_{1,1}\vee W_{1/2,2}\vee\cdots\vee W_{1/(n+1),n+1})^{+}\in\mathfrak{F}$, we have (1.15):

$$\text{(1.15)}\qquad \begin{cases} f_{2,n}\geqq f_2 \text{ on } X \quad \text{for each } n\,, \\ f_{2,n}=f_2 \text{ on } F_2 \quad \text{for each } n \text{ and} \\ \tilde{f}_{2,n}\geqq\tilde{f}_2+a+1/(n+1)\geqq\tilde{U}^{\alpha}+1/n \ m\text{-a.e.} \\ \text{on } X-F_{2,1/(n+1)} \quad \text{for each } n\,. \end{cases}$$

By using Lemma 3.1.4 in [5], we obtain (1.11).

Now we define a sequence $\{V_n\}_n$ of $\mathfrak{F}$ by

$$V_n=(f_{2,n}-U^{\alpha})^{+}\wedge(1/n) \qquad \text{for } n\geqq 1\,.$$

Then we show the following (1.16):

(1.16) $\qquad V_n h\in\mathfrak{G}^{+}$ for each natural number n and $h\in\mathfrak{F}$ satisfying $0\leqq h\leqq 1$.

Let a natural number n be arbitrary but fixed. Define $V'=(V_n h+U^{\alpha})\wedge f_{2,n}$. Then we have

$$f_2=f_{2,n}\geqq V' \text{ on } F_2 \quad\text{and}\quad V'\geqq U^\alpha\wedge f_2\geqq f_1 \ m\text{-a.e. on } F_1.$$

Therefore $(V'-U^\alpha)^+=(V_nh\wedge(f_{2,n}-U^\alpha))^+=(V_nh)^+\wedge(f_{2,n}-U^\alpha)^+=V_nh$. Moreover it holds that $V_n\in\mathfrak{F}\wedge L^\infty(X;m)$, since $f_{2,n}$, $U^\alpha\in\mathfrak{F}$. Therefore we obtain $V_nh\in\mathfrak{F}$ and $V'\in\mathfrak{R}$, by applying Theorem 1.4.2 in [5]. Consequently (1.16) holds.

On the other hand we take additive functionals A, $A_n\in\mathfrak{A}_c^\alpha$ $(n\geqq 1)$ and functions $g_n\in\mathfrak{F}$ $(n\geqq 1)$, whose definitions are (1.4), (1.8) and (1.7) respectively, corresponding to $U^\alpha-W$ instead of U. Since $U^\alpha\in\mathfrak{R}$, there exists an exceptional set N_1 satisfying (1.17):

$$\tilde{f}_2\geqq\tilde{U}^\alpha \text{ on } F_2-N_1 \quad\text{and}\quad \tilde{f}_1\leqq\tilde{U}^\alpha \text{ on } F_1-N_1. \tag{1.17}$$

And we put $B_{2,n}=\{\tilde{U}^\alpha\geqq\tilde{f}_{2,n}-1/n\}\cap(F_{2,1/n}-N_1)$ for natural numbers n. Hence concerning the expression of $U^\alpha-W$ by additive functionals we shall prove (1.18):

(1.18) There exists an m-negligible Borel subset N_2 containing N_1, such that for each $x\in X-N_2$ natural number l and Borel subset B containing $B_{2,l}$ it holds that

$$\begin{aligned}\tilde{U}^\alpha(x)-\tilde{W}(x)-Q_{\sigma_B}(\tilde{U}^\alpha-\tilde{W})(x)&=\tilde{U}^\alpha(x)-\tilde{W}(x)-Q_{\tau_B}(\tilde{U}^\alpha-\tilde{W})(x)\\&=E_x[A(\sigma_B)]=E_x[A(\tau_B)]\geqq 0.\end{aligned}$$

Fix an arbitrary Borel measurable function $h_0\in L^2(X;m)$ satisfying $0\leqq h_0\leqq 1$, an arbitrary natural number l and an arbitrary Borel subset B containing $B_{2,l}$. Hence we put

$$h_1=\alpha R_\alpha h_0-\alpha Q_{\tau_B}R_\alpha h_0. \tag{1.19}$$

Since the right hand side of (1.19) equals to $\alpha\cdot E.\left[\int_0^{\tau_B}e^{-\alpha t}h_0(x_t)dt\right]$ and both $R_\alpha h_0$ and $Q_{\tau_B}R_\alpha h_0$ are α-excessive, we have $h_1\in\mathfrak{F}$ and $0\leqq h_1\leqq 1$. Moreover we put by $\tilde{V}_l$ a quasi-continuous Borel version of V_l. Then from (1.16) and (1.19), we have

$$\left\{\begin{aligned}\mathfrak{E}_\alpha(nG_{n+\alpha}(U^\alpha-W),\ V_lh_1)&=\mathfrak{E}_\alpha(\tilde{U}^\alpha-\tilde{W},\ nG_{n+\alpha}V_lh_1)\\&=\mathfrak{E}_\alpha(nR_{n+\alpha}(\tilde{U}^\alpha-\tilde{W}),\ \tilde{V}_lh_1)\\&=\mathfrak{E}_\alpha(R_\alpha g_n,\ \tilde{V}_lh_1)\\&=\alpha\cdot(g_n,\ \tilde{V}_lh_1)\\&=\alpha\cdot(g_n\tilde{V}_l,\ R_\alpha h_0)-\alpha\cdot(g_n\tilde{V}_l,\ Q_{\tau_B}R_\alpha h_0)\\&=\alpha\cdot(R_\alpha(g_n\tilde{V}_l),\ h_0)-\alpha\cdot(Q_{\sigma_B}R_\alpha(g_n\tilde{V}_l),\ h_0)\quad\text{for each } n.\end{aligned}\right. \tag{1.20}$$

Here we used that $\{Q_{\tau_B}R_\alpha(g_n\tilde{V}_l)\neq Q_{\sigma_B}R_\alpha(g_n\tilde{V}_l)\}$ is m-negligible. Moreover from (1.11) and the definition of $B_{2,l}$, there exists an exceptional set N_2 containing N_1 such that for each $s>0$ and $\omega\in\Omega$ satisfying $\sigma<s<\sigma_B(\omega)$ we have

$$X_s(\omega)\in\{U^\alpha\leqq f_{2,l}-1/l\}\cup N_2\subset\{\tilde{V}_l=1/l\}\cup N_2.$$

The equations (1.20), together with this fact, imply

$$\mathfrak{E}_\alpha(nG_{n+\alpha}(U^\alpha - W),\ V_l h_1) = \alpha \cdot E_{h_0 \cdot m}\left[\int_0^{\sigma_B} e^{-\alpha s}(g_n \tilde{V}_l)(X_s)\,ds\right]$$

$$= \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}\left[\int_0^{\sigma_B} e^{-\alpha s} g_n(X_s)\,ds\right]$$

$$= \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}[A_n(\sigma_B)]$$

$$= \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}[A_n(\tau_B)] \qquad \text{for each n.}$$

By considering a subsequence $\{n_k\}$ of $\{n\}$ in the same manner as the proof of Lemma 4 and tending n_k infinite, we obtain

$$\mathfrak{E}_\alpha(U^\alpha - W,\ V_l h_1) = \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}[A(\sigma_B)] = \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}[A(\tau_B)]\,.$$

Consequently we obtain (1.18) from this fact, (1.20), Lemma 2 and Lemma 4.

Next concerning α-excessivity of $U^\alpha - W$ we shall show (1.21):

$$\text{(1.21)} \qquad \begin{cases} \tilde{U}^\alpha(x) - \tilde{W}(x) \geqq Q_{\tau \wedge \sigma_{B_{2,l}}}(\tilde{U}^\alpha - \tilde{W})(x) & \text{for q.e. } x \in X, \\ \text{each } \tau \in \mathcal{M} \text{ and each natural number } l\,. \end{cases}$$

Fix an arbitrary natural number l. Since $Q_{\tau_B}(\tilde{U}^\alpha - \tilde{W})$ is quasi-continuous for each Borel subset B containing $B_{2,l}$ (see Theorem 4.4.1 in [5]), by applying Lemma 3.1.4 and Theorem 4.2.1 of [5] to (1.18), there exists a properly exceptional set N_3 containing N_2 such that

$$\text{(1.22)} \qquad \begin{cases} \tilde{U}^\alpha(x) - \tilde{W}(x) - Q_{\tau_B}(\tilde{U}^\alpha - \tilde{W})(x) = E_x[A(\tau_B)] \geqq 0 & \text{for each } x \in X - N_3 \\ \text{and for each Borel subset } B \text{ containing } B_{2,l}\,. \end{cases}$$

On the other hand from Theorem 4.3.2 in [5], there exists a properly exceptional set N_4 for which $\{\tilde{U}^\alpha < \tilde{f}_{2,l} - 1/l\} \cap (X - N_4)$ is finely open and Borel measurable. Hence we put a properly exceptional set $N_5 = N_3 \cup N_4$ and define a sub-process $\overline{M} = \{\overline{X}_t\}$ by

$$\overline{X}_t(\omega) = \begin{cases} x & \text{if } X_0(\omega) = x \in N_5\,, \\ X_t(\omega) & \text{if } X_0(\omega) \notin N_5 \text{ and } t < \sigma_{B_{2,l}}(\omega)\,, \\ \Delta & \text{if } X_0(\omega) \notin N_5 \text{ and } t \geqq \sigma_{B_{2,l}}(\omega)\,, \end{cases}$$

where Δ is the death point for M. Then $\overline{M}$ becomes to be the standard Markov process in the sense of Blumenthal and Getoor [3] by virtue of Theorems 4.1.3 and 4.1.4 in [5], since $X - N_5 - B_{2,l} = \{\tilde{U}^\alpha < \tilde{f}_{2,l} - 1/l\} \cap (X - F_{2,1/l}) \cap (X - N_5)$ is a finely open Borel set. Now we define

$$U(x) = \begin{cases} E_x[A(\tau_{B_{2,l}})] & \text{if } x \in X - N_5 - B_{2,l}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then we shall show that U is an α-excessive function with respect to the process $\overline{M}$. Because U is nonnegative and Borel measurable. Moreover for each $x \in X - N_5$ and each Borel subset B of X,

$$\begin{aligned}\bar{Q}_{\bar{\tau}_B}U(x) &= E_x[e^{-\alpha\tau_B}U(X_{\tau_B}) : \tau_B < \sigma_{B_{2,l}}] \\ &= E_x[e^{-\alpha\tau_B}E_{X_{\tau_B}}[A(\tau_{B_{2,l}})] : \tau_B < \sigma_{B_{2,l}}] \\ &= E_x[e^{-\alpha\tau_B}A(\tau_{B_{2,l}})\circ\theta_{\tau_B} : \tau_B < \sigma_{B_{2,l}}] \\ &= E_x[A(\tau_B + \tau_{B_{2,l}}\circ\theta_{\tau_B}) - A(\tau_B) : \tau_B < \sigma_{B_{2,l}}] \\ &= E_x[A(\tau_{B_{2,l}}) - A(\tau_B) : \tau_B < \tau_{B_{2,l}}] \\ &= E_x[A(\tau_{B_{2,l}}) - A(\tau_{(B\cup B_{2,l})})] \\ &\leqq E_x[A(\tau_{B_{2,l}})] = U(x)\,,\end{aligned}$$

where $\bar{Q}$ and $\bar{\tau}_B$ imply the transition function function and the hitting time corresponding to the process $\bar{M}$. And while for each $x \in N_5$ and each Borel subset B of X,

$$\bar{Q}_{\bar{\tau}_B}U(x) = \bar{E}_x[e^{-\alpha\bar{\tau}_B}U(X_{\bar{\tau}_B})] = E_x[e^{-\alpha\tau_B}U(x)] \leqq U(x)\,,$$

where $\bar{E}_x$ denotes the expectation starting at x corresponding to $\bar{M}$. Therefore by virtue of Theorem 5.1 in [3, Chapter II] we obtain

$$U \geqq \beta\bar{R}_{\beta+\alpha}U \qquad \text{for each } \beta > 0\,, \tag{1.23}$$

where $\{\bar{R}_\alpha\}_{\alpha>0}$ denotes the resolvent corresponding to $\bar{M}$. On the other hand

$$\bar{Q}_tU(x) = \begin{cases} E_x[A(\tau_{B_{2,l}}) - A(t\wedge\tau_{B_{2,l}})] & \text{if } x \in X - N_5\,, \\ \bar{E}_x[e^{-\alpha t}U(x)] & \text{if } x \in N_5\,, \end{cases}$$

which converges to $U(x)$ as $t \downarrow 0$ in both cases. This fact and (1.23) imply that U is α-excessive with respect to the process $\bar{M}$ in the sense of Blumenthal and Getoor [3]. By virtue of Proposition 2.8 in [3, Chapeter II], for each $x \in X - N_5 - B_{2,l}$ and $\tau \in \mathfrak{M}$

$$\begin{aligned}E_x[A(\sigma_{B_{2,l}})] = U(x) \geqq \bar{Q}_\tau U(x) &= E_x[e^{-\alpha\tau}U(X_\tau) : \tau < \sigma_{B_{2,l}}] \\ &= E_x[e^{-\alpha\tau}E_{X_\tau}[A(\sigma_{B_{2,l}})] : \tau < \sigma_{B_{2,l}}] \\ &= Q_{\tau\wedge\sigma_{B_{2,l}}}(E.[A(\sigma_{B_{2,l}})])(x)\,.\end{aligned}$$

Therefore

$$\tilde{U}^\alpha - \tilde{W} - Q_{\sigma_{B_{2,l}}}(\tilde{U}^\alpha - \tilde{W}) \geqq Q_{\tau\wedge\sigma_{B_{2,l}}}[(\tilde{U}^\alpha - \tilde{W}) - Q_{\sigma_{B_{2,l}}}(\tilde{U}^\alpha - \tilde{W})] \text{ q.e.}.$$

This shows (1.21).

Finally we shall let l infinite in (1.21). Since $\{f_{2,l}\}_l$ is nondecreasing, $\{B_{2,l}\}_l$ is nonincreasing and $B_{2,l} \supset B_2$ for all l. Therefore

$$\lim_{l\to\infty}\sigma_{B_{2,l}} = \sup_l \sigma_{B_{2,l}} \leqq \sigma_{B_2}\,. \tag{1.24}$$

Hence we express the left term in (1.24) by σ. Since $F_{2,n}$ is closed and both $\tilde{U}^\alpha$ and $\tilde{f}_{2,l}$ $(l \geqq 1)$ are finely continuous q.e., for any natural numbers n, l $(n \geqq l)$ and any $x \in X - N_5$,

$$X_{\sigma_{B_{2,n}}}\in B_{2,n}\subset B_{2,l} \quad \text{a.s. } P_x \text{ on } \{\sigma<\infty\}.$$

By letting n infinite,

$$X_\sigma\in B_{2,l} \quad \text{a.s. } P_x \text{ on } \{\sigma<\infty\}.$$

Moreover by tending l infinite, for each $x\in X-N_5$ we obtain

$$X_\sigma\in B_2 \quad \text{a.s. } P_x \text{ on } \{\sigma<\infty\},$$

This, together with (1.24), proves

$$(1.25) \qquad P_x(\sigma=\sigma_{B_2})=1 \quad \text{for q.e. } x\in X.$$

Since $\{\tilde{U}^\alpha-\tilde{W}>\|\tilde{U}^\alpha-\tilde{W}\|_\infty\}$ is exceptional from Lemmas 4.3.2 and 4.2.4 in [5], by virtue of the bounded convergence theorem and (1.25), (1.21) follows

$$(1.26) \qquad \begin{cases} \tilde{U}^\alpha(x)-\tilde{W}(x)\geqq Q_{\tau\wedge\sigma_B}(\tilde{U}^\alpha-\tilde{W})(x) \\ \qquad \text{for q.e. } x\in X \text{ and each } \tau\in\mathcal{M}. \end{cases}$$

This means (1.9). Moreover (1.10) follows in the same line as the proof of (1.9), by changing the signs of (1.9). Consequently Theorem 1 holds. □

Now we consider the zero-sum game associated with (1.13) and (1.14). Fix a Borel measurable function $g\in L$ and put $W=R_\alpha g$. For $i=1, 2$ we define

$$\mathcal{M}_i=\{\tau\in\mathcal{M}: P_x(X_\tau\in F_i)=1 \ m\text{-a.a. } x\in X\}.$$

Let $h\in\mathfrak{F}\cap L^\infty(X; m)$. Hence we assume (A3) in addition to (A1) and (A2):

$$(\text{A3}) \qquad f_1\leqq h\leqq f_2 \ m\text{-a.e.} \quad \text{on } F_1\cap F_2.$$

For τ, $\sigma\in\mathcal{M}$ we define a function

$$J(\tau, \sigma)(x)=E_x\left[e^{-\alpha(\tau\wedge\sigma)}L(X_{\tau\wedge\sigma})+\int_0^{\tau\wedge\sigma}e^{-\alpha t}g(X_t)dt\right],$$

where

$$L(\tau, \sigma)=\begin{cases} \tilde{f}_1(X_\tau) & \text{on } \{\tau<\sigma\}, \\ \tilde{h}(X_\tau) & \text{on } \{\tau=\sigma\}, \\ \tilde{f}_2(X_\sigma) & \text{on } \{\tau>\sigma\}. \end{cases}$$

Hence we consider the following zero-sum game with stopping times:

$$(1.27) \qquad \begin{cases} \text{Find a pair of optimal Markov times } (\tau^*, \sigma^*) \\ \in\mathcal{M}_1\times\mathcal{M}_2 \text{ attaining an optinal value function } U^* \text{ such that} \\ \tilde{U}^*=\max\limits_{\tau\in\mathcal{M}_1}\min\limits_{\sigma\in\mathcal{M}_2}J(\tau, \sigma) \\ \quad =\min\limits_{\sigma\in\mathcal{M}_2}\max\limits_{\tau\in\mathcal{M}_1}J(\tau, \sigma) \ m\text{-a.e.} \end{cases}$$

Then the following theorem is derived from Theorem 1.

THEOELEM 2. *Under* (A1)~(A3), U^α *is an optimal value function and* $(\sigma_{B_1}, \sigma_{B_2})$ *is a pair of optimal Markov times of the problem* (1.27), *where* $B_i=\{\tilde{U}^\alpha=\tilde{f}_i\}\cap(F_i-N)$ *for*

$i=1, 2$. *Moerover* $(\sigma_{B_1}, \sigma_{B_2})$ *is a saddle point of* (1.27).

PROOF. From (1.27) and the definition of W, for m-a. a. $x\in X$ and each $\tau\in\mathcal{M}$,

$$\tilde{U}^{\alpha}(x)\geqq E_x\left[e^{-\alpha(\tau\wedge\sigma_{B_2})}\tilde{U}^{\alpha}(X_{\tau\wedge\sigma_{B_2}})+\int_0^{\tau\wedge\sigma_{B_2}}e^{-\alpha t}g(X_t)dt\right]. \tag{1.28}$$

Since $U^{\alpha}\in\mathfrak{R}$, the right term of (1.28) is greater than $J(\sigma, \tau_{B_2})(x)$. By considering about (1.10) similarly, for each $(\tau, \sigma)\in\mathcal{M}_1\times\mathcal{M}_2$ we obtain

$$J(\tau, \sigma_{B_2})\leqq\tilde{U}^{\alpha}=J(\sigma_{B_1}, \sigma_{B_2})\leqq J(\tau_{B_1}, \sigma) \ m\text{-a. e.} \tag{1.29}$$

This completes the proof of Theorem 2. □

2. Transient Case

In this section we treat the transient case with both $\alpha=0$ and $W=0$. Let $(\mathfrak{F}, \mathfrak{E})$ be a regular transient Dirichlet space and let $(\mathfrak{F}_e, \mathfrak{E})$ be an extended transient Dirichlet space in the sense of [5]. Let F_1 and F_2 be closed subsets of X and let f_1 and f_2 be elements of $\mathfrak{F}\cap L^{\infty}(X; m)$. We assume (A4) in addition to (A1) and (A2):

(A4) $\{f_2>f_1\}$ is relatively compact.

Define $\mathfrak{R}_e=\{V\in\mathfrak{F}_e: f_2\geqq V$ m-a. e. on F_2 and $V\leqq f_1$ m-a. e. on $F_1\}$. Then the following lemma holds in similar way to Lemmas 1 and 2.

LEMMA 5. *There exists a unique element* U^0 *of* $\mathfrak{R}_e$ *such that*

$$\mathfrak{E}(U^0, U^0)\leqq\mathfrak{E}(V, V) \quad \text{for all } V\in\mathfrak{R}_e. \tag{2.1}$$

Moreover U^0 *satisfies* (2.2) *and* (2.3):

$$\mathfrak{E}(U^0, V-U^0)\geqq 0 \quad \text{for all } V\in\mathfrak{R}_e, \tag{2.2}$$

$$\begin{cases}\mathfrak{E}(U^0, (V-U^0)^+)\geqq 0 & \text{for all } V\in\mathfrak{R}_e,\\ \mathfrak{E}(U^0, (V-U^0)^-)\geqq 0 & \text{for all } V\in\mathfrak{R}_e,\end{cases} \tag{2.3}$$

The solutions $\{U^{\alpha}\}_{\alpha>0}$ in Lemma 1 have the following properties.

PROPOSITION 1. $\{U^{\alpha}\}_{\alpha>0}$ *converges to* U^0 *in the norm* $\mathfrak{E}$ *as* $\alpha\downarrow 0$. *Moreover for each* $\beta>0$, $\{U^{\alpha+\beta}\}_{\alpha>0}$ *converges to* U^{β} *as* $\alpha\downarrow 0$ *in the norm* $\mathfrak{E}_1$.

PROOF. Define $a=\|f_1\|_{\infty}+\|f_2\|_{\infty}+1$ and $b=\inf_{V\in\mathfrak{R}}\mathfrak{E}(V, V)$. Then

$$\begin{aligned} b&=\inf_{V\in\mathfrak{R}}\inf_{\alpha>0}\mathfrak{E}_{\alpha}(V, V)\\ &=\inf_{\alpha>0}\inf_{V\in\mathfrak{R}}\mathfrak{E}_{\alpha}(V, V)\\ &=\lim_{\alpha\downarrow 0}\inf_{V\in\mathfrak{R}}\mathfrak{E}_{\alpha}(V, V)\\ &=\lim_{\alpha\downarrow 0}\mathfrak{E}_{\alpha}(U^{\alpha}, U^{\alpha})\\ &=\lim_{\alpha\downarrow 0}\{\mathfrak{E}(U^{\alpha}, U^{\alpha})+\alpha(U^{\alpha}, U^{\alpha})\}.\end{aligned} \tag{2.4}$$

Since $U^{\alpha}\in\mathfrak{R}$ and $\overline{X-F_i}$ $(i=1, 2)$ are compact, we have

$$(2.5)\quad \begin{cases} (U^\alpha, U^\alpha) \leqq \int_{\{U^\alpha>0\}\cap F_2} f_2^2 dm + \int_{\{U^\alpha<0\}\cap F_1} f_1^2 dm \\ \qquad + \int_{\{U^\alpha>0\}\cap(X-F_2)} (U^\alpha)^2 dm + \int_{\{U^\alpha<0\}\cap(X-F_1)} (U^\alpha)^2 dm \\ \qquad \leqq \|f_2\|_2^2 + \|f_1\|_2^2 + a^2\{m(X-F_2)+m(X-F_1)\} < \infty . \end{cases}$$

Therefore (2.4), together with this inequality, shows

$$b = \lim_{\alpha\downarrow 0} \mathfrak{E}(U^\alpha, U^\alpha) .$$

From Lemma 5 and this fact there exists a subsequence $\{U^{\alpha(n)}\}_n$ of $\{U^\alpha\}_{\alpha>0}$ such that

$$U^{\alpha(n)} \longrightarrow U^0 \quad \text{as } n\to\infty \text{ in the norm } \mathfrak{E} ,$$

However we can extract a subsequence from any subsequence of $\{U^\alpha\}_{\alpha>0}$ in the same manner, therefore

$$U^\alpha \longrightarrow U^0 \quad \text{as } \alpha\downarrow 0 \text{ in the norm } \mathfrak{E} .$$

Next as for $\{U^{\alpha+\beta}\}_{\alpha>0}$ we get the desired property similarly, by replacing U^α, U^0 and $\mathfrak{E}$ with $U^{\alpha+\beta}$, U^β and $\mathfrak{E}_1$ respectively. □

In the transient case the following lemma is trivial but essential for the main theorem.

LEMMA 6. *Fix an arbitrary Borel subset B of X such that $m(B)<\infty$. Then it holds that*

$$E.[\sigma_{(X-B)}] < \infty \quad m\text{-a.e.} \quad \text{on } X .$$

PROOF. Fix an arbitrary Borel subset B of X such that $m(B)<\infty$. By virtue of Lemma 1.5.1 in [5], we have

$$E_x[\sigma_{(X-B)}] \leqq E_x\left[\int_0^\infty I_B(X_t)dt\right] < \infty \quad m\text{-a.a. } x\in X ,$$

Here I_B denotes the indicator function of B. This completes Lemma 6. □

THEOREM 3. *Under* (A1), (A2) *and* (A4), *there exists a properly exceptional set N for which the solution U^0 satisfies* (2.6) *and* (2.7):

$$(2.6)\quad \tilde{U}^0 \geqq P_{\tau\wedge\sigma_{B_2}} \tilde{U}^0 \quad m\text{-a.e.} \quad \text{for each } \tau\in\mathcal{M} ,$$

$$(2.7)\quad \tilde{U}^0 \leqq P_{\sigma_{B_1}\wedge\sigma} \tilde{U}^0 \quad m\text{-a.e.} \quad \text{for each } \sigma\in\mathcal{M} ,$$

where $B_i = \{\tilde{U}^\alpha = \tilde{f}_i\}\cap(F_i - N)$ *for* $i=1, 2$.

PROOF. We shall sketch this theorem following the proof of Theorem 1. We note $U^0\in\mathfrak{F}_e\cap L^\infty(X; m)$ in the same line as the first part of the proof of Theorem 1. Since (2.5) holds in the case of $\alpha=0$, we obtain $U^0\in\mathfrak{F}\cap L^\infty(X; m)$. Hence we take $F_{i,\varepsilon}$ ($i=1, 2$; $\varepsilon\geqq 0$) and $\{f_{2,n}\}_{n\geqq 1}$ in the same way as the proof of Theorem 1 and put

$$V_n = (f_{2,n} - U^0)^+ \wedge (1/n) \qquad \text{for } n\geqq 1$$

and

$$\mathfrak{G}_e^+ = \{(V-U^0)^+ :\ V\in\mathfrak{F}_e\} .$$

Then in the same line as the proof of (1.16), we obtain

$$(2.8)\qquad \begin{cases} V_n h \in \mathfrak{G}_\varepsilon^- & \text{for each natural number } n \\ \text{and each } h \in \mathfrak{F} \text{ satisfying } 0 \leqq h \leqq 1 \end{cases}$$

Take an exceptional set N_1 satisfy (2.9):

$$(2.9)\qquad \tilde{f}_2 \geqq \tilde{U}^0 \text{ on } F_2 - N_1 \quad \text{and} \quad \tilde{f}_1 \leqq \tilde{U}^0 \text{ on } F_1 - N_1 .$$

And for natural numbers n define

$$B_{2,n} = \{\tilde{U}^0 \geqq \tilde{f}_{2,n} - 1/n\} \cap \{F_{2,1/n} - N_1\} .$$

Moreover for each $\alpha > 0$ we take the additive functional $A^\alpha \in \mathfrak{A}_c^\alpha$, which is defined by (1.4), corresponding to U^α instead of U. Then in the same line as the proof of Theorem 1 we obtain

$$\mathfrak{E}_\alpha(U^\alpha, V_l h_1) = \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}[A^\alpha(\sigma_B)] = \alpha \cdot l^{-1} \cdot E_{h_0 \cdot m}[A^\alpha(\tau_B)] ,$$

where h_1 is defined by (1.19). Hence we have $\mathfrak{E}(U^0, V_l h_1) \geqq 0$ and $|(U^0, V_l h_1)| \leqq \|U^0\|_\infty \cdot l^{-1}$. Therefore, from Lemma 4,

$$(2.10)\qquad \begin{cases} 0 \leqq \lim_{\alpha \downarrow 0} E_{h_0 \cdot m}[A^\alpha(\sigma_B)] = \lim_{\alpha \downarrow 0} E_{h_0 \cdot m}[A^\alpha(\tau_B)] \\ \quad = \lim_{\alpha \downarrow 0} \{(\tilde{U}^0, h_0) - E_{h_0 \cdot m}[e^{-\alpha \sigma_B} \tilde{U}^0(X_{\sigma_B})] \\ \quad = \lim_{\alpha \downarrow 0} \{(\tilde{U}^0, h_0) - E_{h_0 \cdot m}[e^{-\alpha \tau_B} \tilde{U}^0(X_{\tau_B})] . \end{cases}$$

While from (A2), (A3) and Lemma 6 we have

$$E_{h_0 \cdot m}[\sigma_B] < \infty \quad \text{and} \quad E_{h_0 \cdot m}[\tau_B] < \infty$$

for all Borel subsets B containing $B_{2,l}$. Consequently by virtue of the bounded convergence theorem, (2.10) follows

$$\begin{aligned} 0 &\leqq (\tilde{U}^0, h_0) - E_{h_0 \cdot m}[\tilde{U}^0(X_{\sigma_B})] \\ &= (\tilde{U}^0, h_0) - E_{h_0 \cdot m}[\tilde{U}^0(X_{\tau_B})] . \end{aligned}$$

Therefore we get Theorem 3 in the same line as the proof of Theorem 1, by using Theorem 5.2.2 in [5] instead of Lemma 4 and by noting $E_x[\sigma_{B_2}] < \infty$ m-a. a. $x \in X$. □

Now we shall state the zero-sum game associated with (2.1). For $i = 1, 2$ we define

$$\mathcal{M}_{f,i} = \{\tau \in \mathcal{M} : P_x(X \in F_i \text{ and } \tau < \infty) = 1 \ m\text{-a. a. } x \in X\} .$$

Let $h \in \mathfrak{F} \cap L^\infty(X; m)$. Hence we assume (A3) in addition to (A1), (A2) and (A4). For $\tau, \sigma \in \mathcal{M}$ satisfying that $P_y(\tau < \infty) = P_y(\sigma < \infty) = 1$ m-a. a. $y \in X$, we define

$$J^0(\tau, \sigma)(x) = E_x[L(X_{\tau \wedge \sigma})] \qquad \text{for } x \in X ,$$

where L is defined in Section 1. Hence we consider the following problem:

$$(2.11)\quad \begin{cases} \text{Find a pair of optimal stopping times } (\tau^*, \sigma^*) \\ \in \mathcal{M}_{f,1}\times\mathcal{M}_{f,2} \text{ attaining an optinal value function } U^* \text{ such that} \\ \tilde{U}^* = \max\limits_{\tau\in\mathcal{M}_{f,1}} \min\limits_{\sigma\in\mathcal{M}_{f,2}} J^0(\tau, \sigma) \\ \qquad \max\limits_{\sigma\in\mathcal{M}_{f,2}} \min\limits_{\tau\in\mathcal{M}_{f,1}} J^0(\tau, \sigma) \ m\text{-a. e.} \end{cases}$$

THEOREM 4. *Under* (A1)~(A4), U^0 *is an optimal value function and* $(\sigma_{B_1}, \sigma_{B_2})$ *is a pair of optimal stopping times of the problem* (2.11), *where* $B_i=\{\tilde{U}^\alpha=\tilde{f}_i\}\cap(F_i-N)$ *for* $i=1, 2$. *Moreover* $(\sigma_{B_1}, \sigma_{B_2})$ *is a saddle point of* (2.11).

PROOF. This proof is in the same line as the proof of Theorem 2. □

3. Examples

First we consider about Section 1. When the process M is a diffusion, the relation between the variational inequality (1.1) and the game (1.27) is discussed in [1], [4] and [6]. Here we treat the case when M is a jump process. Let λ be a positive number and let $q(x, y)$ be a Borel measurable function on $X\times X$, which satisfies (i)~(iii):

(i) $q(x, y)=q(y, x)$ for each $x, y\in X$,

(ii) $q(x, x)=0$ for each $x\in X$,

(iii) $\int_X q(x, y)m(dy)=1$ for each $x\in X$.

According to [3, pp. 63-68] we put

$$P_tU(x)=\int_X \lambda e^{-\lambda t}q(x, y)U(y)m(dy)$$

for $t\geqq 0$, $x\in X$ and $U\in L^2$. Then

$$\mathfrak{E}(U, V)=\int_Y (U(x)-U(y))(V(x)-V(y))\frac{\lambda^2}{2}q(x, y)m(dx)m(dy)$$

for $U, V\in L^2$, where $Y=\{(x, y)\in X\times X: x\neq y\}$. From the properties (i)~(iii) of q we have

$$\mathfrak{E}(U, U)\leqq 2\lambda\cdot(U, U)<\infty \qquad \text{for } U\in L^2.$$

Therefore from this fact and (1.2.24) in [5] we obtain $\mathfrak{F}=L^2$. On the other hand $(\mathfrak{F}, \mathfrak{E}_\alpha)$ is Dirichlet space from Example 1.2.4 in [5]. Hence we can easily check $(\mathfrak{F}, \mathfrak{E}_\alpha)$ is regular, by using $\mathfrak{F}=L^2$. For a Borel measurable $g\in L^2$ we took $W=R_\alpha g$ in Section 1. Now we can easily reduce (1.1) to (3.1):

$$(3.1)\quad \begin{cases} \dfrac{\lambda^2}{2}\displaystyle\int_Y (U^\alpha(x)-U^\alpha(y))(V(x)-U^\alpha(x)-V(y)+U^\alpha(y))q(x, y)m(dx)m(dy) \\ \qquad +\displaystyle\int_X (\alpha U^\alpha(x)-g(x))(V(x)-U^\alpha(x))m(dx)\geqq 0 \qquad \text{for all } V\in\mathfrak{R}. \end{cases}$$

Next we consider about Section 2. Let $X=D$ be a bounded domain of R^n with the smooth boundary. Let m be Lebegue measure on R^n. Now we consider M is a diffusion as follows:

$$\mathfrak{F}=H_0^1(D)=\left\{U\in L^2(D):\frac{\partial U}{\partial x_i}\in L^2(D)\ (1\leqq i\leqq n)\text{ and } U=0 \text{ on } R^n-D\right\}$$

$$\mathfrak{E}(U,\ V)=\sum_{i,j=1}^{n}\int_D\frac{\partial U}{\partial x_i}\frac{\partial V}{\partial x_j}dx \qquad \text{for } U,\ V\in\mathfrak{F}.$$

Then $(H_0^1(D), \mathfrak{E}_\alpha)$ is transient from Example 1.5.2 in [5]. Now (2.2) is reduced to (3.2):

$$\text{(3.2)}\qquad \sum_{i,j=1}^{n}\int_D\frac{\partial U^\alpha}{\partial x_i}\left(\frac{\partial V}{\partial x_j}-\frac{\partial U^\alpha}{\partial x_j}\right)dx-\int_D g(x)(V(x)-U^\alpha(x))dx\geqq 0 \qquad \text{for all } V\in\mathfrak{R}.$$

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