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Sasagawa, Fumiyoshi
International Institute for Advanced Study of Social Information Science, Fujitsu Limited.

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THE DYNAMICS ON THE CELL SPACES WITH p STATES UNDER THE LOCAL TRANSFORMATIONS SATISFYING THE PRINCIPLE OF LOCAL MAJORITY

By

Fumiyoshi Sasagawa*

Abstract

In order to understand biological phenomena, a new mathematical field was proposed by T. Kitagawa in 1970 and it was named biomathematics. T. Kitagawa and M. Yamaguchi discussed cell spaces which are one of approaches to establish biomathematics.

More than two states are introduced in the cell space. By algebraic methods, in a cell space with four or two states it is proved that a set of all configurations is a vector space and a set of all possible stable configurations is a vector subspace. Thus, in the cell space with four or two states any stable configuration is represented by a superposition of elementary stable configurations. Furthermore, it is proved that in the cell space with two states, there exists one to one correspondence between the set of stable configurations and the set of garden of Eden configurations. The invariant boundary is introduced in the cell space with two states, which generate some variant cell subspaces whose configuration oscillates among definite configurations. The possible positions of this cell subspace are investigated.

1. Introduction

Biological phenomena are thought to be dynamic phenomena in huge and complex systems. Thus, we consider that synthetic comprehensions about biological phenomena may be difficult problems to solve. In order to overcome these difficulties, T. Kitagawa [1] proposed a new mathematical field which was named biomathematics in 1970. T. Kitagawa discussed the methodology of biomathematics which is expected to give us a mathematical framework for synthetic comprehensions about biological phenomena. In the field of biomathematics, the principal strategy of research programs should consist of the five elements: discrete, combinatorial, dynamical, evolutionary and design mathematics [1].

These principal strategies of biomathematics can be satisfied by the cell space [2] and are important bases when we make mathematics connect with biology. The cell space is a finite, two-dimensional iterative array of finite state cell automata, where each unit cell may be a square, an equilateral triangle or a regular hexagon [2, 5]. Several basic concepts and mathematical formulations based on these cell spaces with

* International Institute for Advanced Study of Social Information Science, Fujitsu Ltd., 17-25, Shinkamata 1-chome, Ohta-ku, Tokyo 144, Japan.

two states were given by T. Kitagawa and M. Yamaguchi [2-8]. We introduce more than two states in the cell space and study the structure of attractors. Furthermore, we introduce an invariant cell and investigate an oscillation phenomena.

2. Basic Structure and Notions of Cell Space

An $m\times n$ cell space C is an $m\times n$ rectangular arrangement of mn square cells. The cell space is represented by the matrix notation $(c_{i,j})$ $(1\leqq i\leqq m;\ 1\leqq j\leqq n)$, where the notation $c_{i,j}$ means (i, j) cell. Each cell in the cell space has one and only one state which belongs to a certain fixed set of states. This set is called a state space. A state space S_p consists of p elements $\{a, b, c, \cdots\}$ $(p\geqq 2)$. A set of all configurations X in the $m\times n$ cell space C is denoted by $\mathfrak{X}$. Similarly, for any cell subspace $T\subset C$, the set of all configurations is denoted by $\mathfrak{X}(T)$. If a configuration of entire cell space is X, the configuration restricted to the cell subspace T is denoted by $T[X]$ and a state of cell $c_{i,j}$ is similarly denoted by $c_{i,j}[X]$. Thus, $T[X]\in\mathfrak{X}(T)$ and $c_{i,j}[X]\in S_p$. Any 2×2 cell subspace is said to be a basic cell subspace denoted by B. The set of all basic cell spaces is denoted by $\mathscr{B}$. In this paper, we consider two kinds of cell which are an ordinary and invariant cell regarding to local majority transformation (LMT) described below. Any state of the later can not be changed by a local majority transformation which is described below, but the former's state is subject to LMT. Any configuration X changes by the following processes:

(1) Choose one basic cell space B belonging to $\mathscr{B}$ by a firing system which is an independent stochastic process with identical probability distribution [2, 3, 5].

(2) Let be $B[X]=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. A configuration $B[X]$ changes by the local majority transformation [2, 3] as follows;

$$LMT:\ B[X]=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \tag{2.1}$$

where for ordinary cell,

$$y_i=\begin{cases} s & \text{if } x_i\neq x_j\ (i\neq j) \text{ and there exists a state } s \\ & \quad \text{such that } |\{j\,|\,x_j=s \text{ and } 1\leqq j\leqq 4\}|\geqq 2, \\ x_i & \text{otherwise,} \end{cases}$$

and for invariant cell, its state can not be changed i.e., $y_i=x_i$.

(3) Processes (1) and (2) are repeated.

If the number of all states in a state space S_p is two, the above local majority transformation rule is identical with T. Kitagawa's LMT [3]. Various local transformation rules satisfying the principle of local majority were extensively proposed by T. Kitagawa [5] in the case that the state space S_2 is $\{0, 1\}$.

In this paper, we deal with an extension of their works of cell spaces. That is, the state space S_p $(p\geqq 2)$ and invariant cells are introduced in the cell space, but the local transformation rule is limited to LMT. We extensively investigate a change of the dynamic properties of cell spaces under LMT when the number of all states in the state space or the boundary condition is changed.

3. Stable Configurations and Garden of Eden Configurations in the $m\times n$ Cell Space with State Space S_p

In this section, we assume that all cells are ordinary in the cell space. A configuration X_S is stable if and only if $LMT(B[X_S])=B[X_S]$ for any basic cell space B [3]. A configuration X_E is a garden of Eden configuration if and only if $LMT(B[X_E])\neq B[X_E]$ for any basic cell space B [6]. The set of all possible stable configurations of the $m\times n$ cell space C is denoted by $\mathscr{X}_S$ and the set of all possible garden of Eden configurations by $\mathscr{X}_E$. This section is devoted to the investigations of stable configurations and garden of Eden configurations in the $m\times n$ cell space. Firstly, we present a pattern of any stable configuration in the following theorem.

THEOREM 3.1. *In the $m\times n$ cell space C with state space S_p, a stable configuration has a following checked pattern $CP(m_1, m_2, \cdots, m_k|n_1, n_2, \cdots, n_l|S_p)$ where m_i and n_j $(1\leqq i\leqq k;\ 1\leqq j\leqq l)$ are positive integers and partitions of m and n satisfying that $m=m_1+m_2+\cdots+m_k,\ n=n_1+n_2+\cdots+n_l$:*

(1) *The $m\times n$ cell space is divided into kl cell subspaces by the correspondence with the partitions of two integers m and n. Each cell subspace is denoted by $C(m_i, n_j)$ $(i=1, 2, 3, \cdots, k;\ j=1, 2, 3, \cdots, l)$ and $C(m_i, n_j)$ is the $m_i\times n_j$ cell subspace.* (see Fig. 5 in the reference [2].)

(2) *The states of all cells in the cell subspace $C(m_i, n_j)$ are entirely the same state. A cell subspace $C(m_i, n_j)$ in which states of all cells are $s_{i,j}$ is called type $s_{i,j}$.*

(3) (a) *Two cell subspaces $C(m_i, n_j)$ which own a common side are different types.*

(b) *Adjacent four cell subspaces $C(m_i, n_j)$, $C(m_i, n_{j+1})$, $C(m_{i+1}, n_j)$ and $C(m_{i+1}, n_{j+1})$ are all different type, or there exist two pairs of the same type in these four cell subspaces.*

PROOF. By using the definition of LMT, it is clear that for any $X_S\in\mathscr{X}_S$ and $B\in\mathscr{B}$,

$$B[X_S]=\begin{pmatrix}x_1 & x_1\\ x_1 & x_1\end{pmatrix}, \begin{pmatrix}x_1 & x_1\\ x_2 & x_2\end{pmatrix}, \begin{pmatrix}x_1 & x_2\\ x_1 & x_2\end{pmatrix}, \begin{pmatrix}x_1 & x_2\\ x_2 & x_1\end{pmatrix} \text{ or } \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$$

where $x_i\in S_p$ and $x_i\neq x_j$ $(i\neq j)$. From this fact, we obtain Theorem 3.1. Q.E.D.

In the case $p=2$, T. Kitagawa and M. Yamaguchi [3] showed that any stable configuration has a checked pattern. From the above theorem, it is clear that in a cell space with a state space S_p for $p=3, 4, \cdots$, a stable configuration has a checked pattern, but a state allocated to each cell subspace $C(m_i, n_j)$ is generally different. Next, we consider several properties of stable and garden of Eden configurations in the cell space with S_p for $p=2, 3, 4, \cdots$, respectively.

3.1. The $m\times n$ cell space with state space S_2

The asymptotic behaviour of an $m\times n$ cell space with the state space S_2 was obtained by T. Kitagawa. Each of all possible configurations in an $m\times n$ cell space changes into some stable configurations by sequential application of LMT [3, 6]. The stable configurations in an $m\times n$ cell space were extensively studied by T. Kitagawa and M. Yamaguchi [2, 3]. In this cell space, the state space is $\{a, b\}$ in our notation, while in T. Kitagawa's notation the state space is $\{0, 1\}$. T. Kitagawa and M.

Yamaguchi [3, 6] introduced an addition in mod. 2 in the state space. We introduce the following operation on the state space S_2: $a+b=b+a=b$, $a+a=a$, $b+b=a$, $ab=ba=a$, $bb=b$, $aa=a$. Thus, the state space $S_2=\{a, b\}$ is a finite field. Then, a necessary and sufficient condition for stable configurations is that $\sum_{c\in B} c[X_S]=a$. It has been known that the pattern of the stable configuration is a checked pattern $CP(m_1, \cdots, m_k | n_1, \cdots, n_l | S_2)$ and the number of all possible stable configurations is 2^{m+n-1} [3].

3.1.1. One to one correspondence between stable configurations and garden of Eden configurations

Here, we show that there is one to one correspondence between $\mathcal{X}_S$ and $\mathcal{X}_E$ in an $m\times n$ cell space and all garden of Eden configurations are obtained from stable configurations by an algebraic method. Let us introduce two operations on the space $\mathcal{X}$: for any X, $X'\in\mathcal{X}$ and $s\in S_p$, $X+X'$ and sX are defined by $c_{i,j}[X+X']=c_{i,j}[X]+c_{i,j}[X']$ and $c_{i,j}[sX]=sc_{i,j}[X]$ for any $c_{i,j}$, respectively.

THEOREM 3.2. *A set $\mathcal{X}$ is a vector space over the finite field S_2 and a set $\mathcal{X}_S$ is a vector subspace of $\mathcal{X}$.*

PROOF. From the definition of operations between configurations and states it is clear that a set of $\mathcal{X}$ is a vector space. For two stable configurations X_S and X'_S, $\sum_{c\in B} c[X_S+X'_S]=\sum_{c\in B} c[X_S]+\sum_{c\in B} c[X'_S]=a+a=a$. Furthermore, for $s\in S_p$ and $X_S\in\mathcal{X}_S$, $\sum_{c\in B} c[sX_S]=s\sum_{c\in B} c[X_S]=sa=a$. Thus, $\mathcal{X}_S$ is a vector subspace. Q. E. D.

Here, we present two propositions about algebraic properties for stable configurations and garden of Eden configurations. Firstly, we show that an addition X_S+X_E of a stable configuration X_S and a garden of Eden configuration X_E is a garden of Eden configuration. A necessary and sufficient condition for garden of Eden configurations is that $\sum_{c\in B} c[X_E]=b$ for any $B\in\mathcal{B}$. We find that $\sum_{c\in B} c[X_S+X_E]=\sum_{c\in B} c[X_S]+\sum_{c\in B} c[X_E]=a+b=b$. Thus, an addition X_S+X_E is a garden of Eden configuration. Secondly, by the similar argument, we find that an addition $X_E+X'_E$ of two garden of Eden configurations X_E and X'_E is stable. By the above facts, we can show that there is one to one correspondence between stable configurations and garden of Eden configurations. It should be noted that at least one garden of Eden configuration exists in an $m\times n$ cell space.

THEOREM 3.3. *Let $X_{0,E}$ be a given garden of Eden configuration. Then, $\mathcal{X}_E$ is given by*

$$\mathcal{X}_E=\{X_E | X_E=X_S+X_{0,E},\ X_S\in\mathcal{X}_S\}\,. \tag{3.1}$$

PROOF. Each element of a set in the righthand side of (3.1) is a garden of Eden configuration. For $X_E\in\mathcal{X}_E$, we obtain the identical equations that $c_{i,j}[X_E]=c_{i,j}[X_{0,E}]+c_{i,j}[X_{0,E}]+c_{i,j}[X_E]$. By the previous proposition that an addition of two garden of Eden configurations is stable, a configuration $X_S=X_E+X_{0,E}$ is stable. Thus, the garden of Eden configuration X_E can be represented by $X_E=X_S+X_{0,E}$. From these discussions, we find that Eq. (3.1) is valid. Q. E. D.

Theorem 3.3 secures that one to one correspondence between each garden of Eden configuration in the set $\mathcal{X}_E$ and each stable configuration in the set $\mathcal{X}_S$ exists. Thus, the number of all possible garden of Eden configurations is equal to the number of all possible stable configurations. From this fact, it is obtained that $|\mathcal{X}_E|$ is 2^{m+n-1}.

3.1.2. Superpositions of elementary stable configurations

M. Yamaguchi [4] introduced a determinative cell subspace. The cell subspace D is said to be a determinative cell subspace if and only if for any $X^D \in \mathcal{X}(D)$ there exists only one stable configuration $X_S \in \mathcal{X}_S$ such that $D[X_S]=X^D$. There exists one to one correspondence between a configuration of the determinative cell subspace and a stable configuration in $\mathcal{X}_S$. Furthermore, in the cell space she [4] constructed a set of elementary stable configurations from a set of specific configurations of a determinative cell subspace and proved a theorem such that any stable configuration can be represented by a superposition of elementary stable configurations in mod. 2.

It has been known that $|D|=m+n-1$ in an $m\times n$ cell space with the state space S_2 [2]. We number each cell in a determinative cell subspace D as $D=\{d_1, d_2, \cdots, d_{m+n-1}\}$. For $i=1, 2, \cdots, m+n-1$, we define an elementary stable configuration $X_S^{(i)}$. $X_S^{(i)} \in \mathcal{X}_S$ is called to be an elementary stable configuration iff $d_i[X_S^{(i)}]=b$ and $d_j[X_S^{(i)}]=a$ for $j=1, 2, \cdots, m+n-1$ $(j\neq i)$. It should be noted that one of simple determinative cell subspaces in an $m\times n$ cell space is $\{c_{1,1}, c_{1,2}, \cdots, c_{1,n}, c_{2,1}, c_{3,1}, c_{4,1}, \cdots, c_{m,1}\}$.

THEOREM 3.4. *For a determinative cell subspace $D=\{d_1, d_2, \cdots, d_{m+n-1}\}$, any stable configuration $X_S \in \mathcal{X}_S$ is represented by a superposition of elementary stable configurations $X_S^{(i)}$ $(i=1, 2, \cdots, m+n-1)$ as*

$$X_S=\sum_{i=1}^{m+n-1} s_i X_S^{(i)} \tag{3.2}$$

where $s_i=d_i[X_S]$ $(i=1, 2, \cdots, m+n-1)$.

PROOF. From Theorem 3.2, it follows that $\sum_{i=1}^{m+n-1} s_i X_S^{(i)} \in \mathcal{X}_S$ $(s_i \in S_2)$ because $\mathcal{X}_S$ is a vector subspace over the finite field S_2. On the other hand, we find that

$$\begin{aligned} d_j\left[\sum_{i=1}^{m+n-1} s_i X_S^{(i)}\right] &= \sum_{i=1}^{m+n-1} s_i d_j[X_S^{(i)}] \\ &= s_1 a + s_2 a + \cdots + s_j b + \cdots + s_{m+n-1} a \\ &= s_j b = s_j = d_j[X_S], \end{aligned} \tag{3.3}$$

then $D\left[\sum_{i=1}^{m+n-1} s_i X_S^{(i)}\right]=D[X_S]$.

From the definition of the determinative cell subspace D, we find that $X_S=\sum_{i=1}^{m+n-1} s_i X_S^{(i)}$. Q. E. D.

That is to say, a set $\mathcal{X}_S$ is a vector subspace over the finite field S_2 of $\mathcal{X}$ in which the set of all elementary stable configurations is the set of the fundamental vectors. A basis is here, we have given another proof of her theorem. By using our method for the above proof, Theorem 3.4 is proved not only in an $m\times n$ cell space with the state space S_2 but in more general cell spaces with two or four states in which any cell belongs to at least one basic cell space.

3.2. The $m\times n$ cell space with the state space S_3

We easily find that in the $m\times n$ cell space with the state space S_3, each possible stable configuration is a checked pattern $CP(m_1, m_2, \cdots, m_k | n_1, n_2, \cdots, n_l | S_3)$ and three states a, b and c can not coexist in stable configurations except checked patterns $CP(m | n_1, n_2, \cdots, n_l | S_3)$ and $CP(m_1, m_2, \cdots, m_k | n | S_3)$. In these considerations, it turns out that the number of all possible stable configurations is given by $3(2^{m+n-1}+3^{m-1}+3^{n-1}-2^m-2^n+1)$.

3.3. The $m\times n$ cell space with the state space S_4

In order to study stable configurations and garden of Eden configurations, we introduce an addition and a product in the state space S_4 as is shown in Tables 3.1 and 3.2. By using these additions, we obtain tractable necessary and sufficient conditions for $X_S\in\mathcal{X}_S$ and $X_E\in\mathcal{X}_E$. For a stable configuration, this condition is given by $\sum_{c\in B} c[X_S]=a$, which is similar as in the cell space with S_2, but the necessary and sufficient condition for a garden of Eden configuration is given by $\sum_{c\in B} c[X_E]=b$, c, or d.

Table 3.1. The additions between four states $\{a, b, c, d\}$.

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Table 3.2. The products between four states $\{a, b, c, d\}$.

	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

3.3.1. Any stable configuration can be represented by a superposition of elementary stable configurations

We prove a theorem that any stable configuration in an $m\times n$ cell space with the state space S_4 can be represented by a superposition of elementary stable configurations in addition defined in Table 3.1. A determinative cell subspace defined by M. Yamaguchi is also well-defined for an $m\times n$ cell space with the state space S_4. It should be noted that a cell subspace $\{c_{1,1}, c_{1,2}, c_{1,3}, \cdots, c_{1,n}, c_{2,1}, c_{3,1}, \cdots, c_{m,1}\}$ is one of determinative cell subspaces. Thus, the number of all possible stable configurations is 4^{m+n-1}. An addition $X+X'$ for $X, X'\in\mathcal{X}$ and a product sX for $s\in S_p$ and $X\in\mathcal{X}$ are defined by the similar way as the case of two states.

THEOREM 3.5. *The set $\mathcal{X}$ is a vector space over the finite field S_4 and the set $\mathcal{X}_S$ is a vector subspace of $\mathcal{X}$.*

PROOF. From the definition of operation between configurations and states, we can prove the above theorem by using similar method as proof of Theorem 3.2. Q. E. D.

By the similar way as in the case of S_2, the elementary stable configurations $X_S^{(i)}$ is defined. Thus, for the determinative cell subspace $D=\{d_1, d_2, \cdots, d_{m+n-1}\}$, $d_i[X_S^{(i)}]=b$ and $d_j[X_S^{(i)}]=a$ $(j=1, 2, \cdots, m+n-1;\ j\neq i)$.

THEOREM 3.6. *In an* $m\times n$ *cell space with the state space* S_4, *any stable configuration* $X_S\in\mathcal{X}_S$ *is represented by a superposition* (3.4) *of elementary stable configurations* $X_S^{(i)}$ $(i=1, 2, \cdots, m+n-1)$ *as*

$$X_S=\sum_{i=1}^{m+n-1} s_i X_S^{(i)} \tag{3.4}$$

where $s_i=d_i[X_S]$ *and the addition and the product are shown in Table* 3.1 *and* 3.2.

PROOF. By the similar method as the proof of Theorem 3.4, we can prove the above theorem. Q. E. D.

It should be noted that $\mathcal{X}$ is a vector space over the finite field S_4 and $\mathcal{X}_S$ is a vector subspace of $\mathcal{X}$ in which the set of all elementary stable configuration is a basis.

3.3.2. Garden of Eden configurations in an $m\times n$ cell space with the state space S_4

By using properties of the finite field $\{a, b, c, d\}$, we can obtain the sufficient and necessary condition such that a configuration X_E is a garden of Eden configuration. That is, a configuration X_E is a garden of Eden configuration, if and only if four states in any basic cell space satisfy one of three algebraic equations (3.5), (3.6) and (3.7):

$$c_{i,j}[X_E]+c_{i,j+1}[X_E]+c_{i+1,j}[X_E]+c_{i+1,j+1}[X_E]=b \tag{3.5}$$

$$c_{i,j}[X_E]+c_{i,j+1}[X_E]+c_{i+1,j}[X_E]+c_{i+1,j+1}[X_E]=c \tag{3.6}$$

$$c_{i,j}[X_E]+c_{i,j+1}[X_E]+c_{i+1,j}[X_E]+c_{i+1,j+1}[X_E]=d \tag{3.7}$$

where $1\leqq i\leqq m-1$ and $1\leqq j\leqq n-1$. The above algebraic conditions can be used to count the number of all possible garden of Eden configurations. We allocate arbitrary state to each cell belonging to a cell subspace $\{c_{1,1}, c_{1,2}, \cdots, c_{1,n}, c_{2,1}, c_{3,1}, \cdots, c_{m,1}\}$. The number of all possible configurations in this cell subspace is 4^{m+n-1}. A state of the cell $c_{2,2}$ must satisfy one of Eqs. (3.5), (3.6) and (3.7) when $i=j=1$. Thus, a state of the cell $c_{2,2}$ must be one of three possible states. By the same procedure, we determine states of the cells from left to right in the second row. In this way, other cells' states are completely determined. From this procedure for constructing a garden of Eden configuration we know that the number of all garden of Eden configurations is $4^{m+n-1}3^{(m-1)(n-1)}$.

In the case of other state spaces S_p $(p\geqq 5)$, it is known that the notion of generative determinative cell subspace [2, 5] is not well-defined, but the pattern of the stable configuration is a checked one in an $m\times n$ cell space.

4. The $m\times n$ Cell Space with Invariant Boundary and the State Space S_2

In this section, we will investigate the effect of the fixed boundary condition for an $m\times n$ cell space on dynamic properties of the $m\times n$ cell space. In an $m\times n$ cell space $(m\geqq 3,\ n\geqq 3)$, a cell subspace $\{c_{1,i}, c_{m,i}, c_{j,1}, c_{j,n} \mid i=1, 2, \cdots, n,\ j=2, 3, \cdots, m-1\}$ consists of invariant cells, and other cells except all cells of this cell subspace are ordinary cells. Then, this cell subspace is called an invariant boundary.

From direct observations, we can know that a cell subspace whose configuration oscillates among definite configurations is generated after long time [3, 7, 8]. Such a cell subspace is called a variant cell subspace, which is defined later. Emergence of the variant cell subspaces is one of characteristic phenomena which occur by the effect of a fixed boundary condition. For the purpose of investigating the possible emerging positions of the variant cell subspaces, we prepare the following terminology.

(T1) Two cells $c_{i,j}$ and $c_{k,l}$ are said to be connected, if and only if these two cells own jointly the same side.

(T2) A cell subspace T is said to be connected, if and only if for arbitrary two cells $c_{i,j}$, $c_{k,l}$ in T, the cell subspace $\{c_{i,j}, c_{u(1),v(1)}, c_{u(2),v(2)}, \cdots, c_{u(p),v(p)}, c_{k,l}\}$ exists in T such that $c_{u(q),v(q)}$ and $c_{u(q+1),v(q+1)}$ are connected ($q=0, 1, 2, \cdots, p$; $c_{u(0),v(0)}=c_{i,j}$ and $c_{u(p+1),v(p+1)}=c_{k,l}$).

(T3) For a cell $c_{i,j}$, a union of all possible basic cell subspaces which include a cell $c_{i,j}$ is denoted by $U(c_{i,j})$.

(T4) A circumference of the cell subspace T is defined by

$$\Gamma(T)=\bigcup_{c_{i,j}\in T}\Delta(c_{i,j}) \tag{4.1}$$

where $\Delta(c_{i,j})=\{c_{k,l}\mid c_{k,l}\in U(c_{i,j}) \text{ and } c_{k,l}\notin T\}$.

(T5) A configuration of a cell subspace T is said to be invariant, if and only if configuration of T can not be changed by any finite sequence of applications of LMT.

(T6) A cell subspace of circumference of a $k\times l$ cell subspace T is said to be a row (column) beam of T, if and only if it consists of all cells in $\Gamma(T)$ which lie a common row (column).

By using the above terminology (T1)-(T6), we present the definition of the variant cell subspace.

DEFINITION 4.1. In an $m\times n$ cell space with invariant boundary and the state space S_2, a cell subspace V is said to be variant, if and only if it satisfies the following three conditions:

(1) The cell subspace V is connected.

(2) For any configuration of the cell subspace V, a configuration of a circumference $\Gamma(V)$ is invariant.

(3) For any configuration of the cell subspace V, a state of each cell in V is changed by applications of LMT to a finite adequate sequence of basic cell spaces.

It becomes evident from the above definition that a variant cell subspace V is a $k\times l$ cell subspace ($1\leqq k\leqq m-2$ and $1\leqq l\leqq n-2$). By using this fact and the condition (3) of Definition 4.1, it can be concluded that all possible configurations of a circumference $\Gamma(V)$ of a variant cell subspace V are

$$\begin{pmatrix} s & x & x & \cdots & x & t \\ y & & & & & \bar{y} \\ y & & & V & & \bar{y} \\ \vdots & & & & & \vdots \\ y & & & & & \bar{y} \\ u & \bar{x} & \bar{x} & \cdots & \bar{x} & v \end{pmatrix} \tag{4.2}$$

where x, y, s, t, u and v are a or b and $\bar{a}(\bar{b})=b(a)$, and $s=\bar{v}=x$ if $x=y$ and $t=\bar{u}=x$ if $x\neq y$. The more detail of proof is omitted. Now, we present the following theorem which describes the possible positions in which variant cell subspaces emerge.

THEOREM 4.1. *At least, two beams of a variant cell subspace in an $m\times n$ cell space with invariant boundary are included by an invariant boundary.*

PROOF. By reductio and absurdum, we can prove Theorem 4.1. We assume that at least one beam of variant cell subspace V is included by an invariant boundary. Then, there is a basic cell space which consists of three cells c_1, c_2, c_3 belonging to $\Gamma(V)$ and one cell c_4 belonging to V, where two cells c_1 and c_2 have the state x and one cell c_3 has the state $\bar{x}$ ($x=a$ or b). Due to condition (3) of Definition 4.1, we can assume that a state of the cell c_4 belonging to V in this basic cell space is x, the state $\bar{x}$ of the cell belonging to $\Gamma(V)$ is changed to x by an application of LMT to this basic cell space. This contradicts Definition 4.1. It implied that at least two beams are included by invariant boundary. Q. E. D.

Under suitable conditions, variant cell subspaces are generated by sequential applications of LMT. This phenomenon represents a differentiation of functional ability of cells in an $m\times n$ cell space.

A cell subspace is said to be an internal cell subspaces, if it consists of all cells which do not belong to an invariant boundary. The restriction of a stable configuration X_S in the internal cell subspace is a checked pattern because the restriction of a stable configuration in this cell subspace is also stable. The patterns of stable configurations are similar for both boundary conditions in the $m\times n$ cell space. However, the variant cell subspaces can be generated in the $m\times n$ cell space with the fixed boundary condition, but not in the $m\times n$ cell space with free boundary condition.

5. Discussion and Summary

In this paper, we have investigated properties of the dynamical systems based on cell spaces under the various conditions. Especially, it has been evident that the checked pattern is universal one for stable configurations through state space S_p ($p=1, 2, 3, \cdots$). For the state spaces S_2 and S_4, we have found that the set of all stable configurations $\mathscr{X}_S$ is a vector subspace of $\mathscr{X}$ and any stable configuration is represented by a superposition of elementary stable configurations. We have found that the dynamical behaviour of cell spaces with invariant boundary is remarkably different from cell spaces without invariant boundary. That is, in contrast with the case of free boundary condition, the variant cell subspaces may be generated after the long time in the cell space with invariant boundary.

Recently, the investigation for the cellular automata is revived by many authors [9]. It is expected that the formulation for the cellular automata gives us the mathematical model of natural systems instead of differential equations. Especially, the biological phenomena are hoped to be understood by using the framework of biomathematics including cell space approach, which was extensively discussed by T. Kitagawa. We have constructed some concepts and mathematical formulations based on the $m \times n$ cell space, which is expected to give us an implication to the leading principle for constructing biomathematics.

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