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FINITE DERIVATIVE CLOSURE OF PARTIAL FUNCTIONS AND A METHOD OF INFERENCE

By

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Abstract

We formulate a concept called a finite derivative closure of partial functions which is an abstraction of recursive structures in finite tree automata (transducers). Then we apply this concept to inference of functions from partial functions.

1. Preliminary Remarks and Notations

We can say that a partial function gives informations of a function partially. If a function $\tilde{\varphi}$ is decided by a recursive structure S and if a partial function φ which is a restriction of $\tilde{\varphi}$ contains the same structure S, then we can obtain $\tilde{\varphi}$ by naturally extending φ using the structure S. So we are interested in the problem to extract recursive structures from partial functions.

In this paper, we formulate a *finite derivative closure* as such a recursive structure. Under some restrictions, we discuss the methods for extracting such structures from given partial functions with finite domains.

We now give some notations needed in this paper. Let A and B be sets and φ be a partial function from A to B. $D(\varphi)$ denotes the domain of φ, that is, $D(\varphi)=\{x\in A;(\exists y\in B)[\varphi(x)=y]\}$. For a subset X of A, we set $\varphi(X)=\{y\in A;(\exists x\in X\cap D(\varphi))[\varphi(x)=y]\}$. For an element y of B and a subset Y of B, we set $\varphi^{-1}(y)=\{x\in A;\varphi(x)=y\}$ and $\varphi^{-1}(Y)=\bigcup_{y\in Y}\varphi^{-1}(y)$. φ is said to be *one-to-one* (or 1-1) if $\varphi(x)=\varphi(y)$ implies $x=y$ for each x, y in $D(\varphi)$. φ is said to be *onto* if $\varphi(A)=B$.

Let φ and ψ be partial functions. If $D(\varphi)\subseteqq D(\psi)$ and $\psi(x)$ is equal to $\varphi(x)$ for each x in $D(\varphi)$, then we say that φ is a *restriction* of ψ or ψ is an *extension* of φ and we denote the situation by $\varphi\subset\psi$.

For a nonempty set W and a nonnegative integer m, $\Pi_m(W)$ denotes the family of all partial functions from W to W^m. It should be noted that, in case of $m=0$, W^0 is a set containing only one element. Hence a partial function from W to W^0 is identified with a subset of W. For an $\vec{f}$ in $\Pi_m(W)$, m is denoted by $d(\vec{f})$ and called the *degree* of $\vec{f}$. We set $\Pi(W)=\bigcup_{m=0}^{\infty}\Pi_m(W)$. For an $\vec{f}$ in $\Pi(W)$, the notation $\vec{f}=(f_1,\cdots,f_m)$ means that

(1) $d(\vec{f})=m$,

(2) for each $i=1,\cdots,m$, f_i is in $\Pi_1(W)$ and $D(f_i)=D(\vec{f})$, and

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(3) for each x in $D(\vec{f})$, $\vec{f}(x)=(f_1(x), \cdots, f_m(x))$.

A partially ordered set $(W, \prec)$ is said to be a *well founded set* if there is no infinite decreasing sequence. Throughout this paper we assume the following (1)-(3):

(1) "partial function" is abbreviated to "pf".

(2) W and A are nonempty sets.

(3) i, j, k, l, m, n are nonnegative integers.

2. Derivatives of Partial Functions by Partial Functions

DEFINITION 1. Let ω be a pf from W to A and $\vec{f}$ be in $\Pi_m(W)$. We say that ω is *differentiable* by $\vec{f}$ iff $\vec{f}(x)=\vec{f}(y)$ implies $\omega(x)=\omega(y)$ for each x, y in $D(\vec{f})\cap D(\omega)$. When ω is differentiable by $\vec{f}$, the *derivative* of ω by $\vec{f}$, denoted by $\partial_{\vec{f}}\omega$, is defined by

(1) $\partial_{\vec{f}}\omega$ is a pf from W^m to A,

(2) $D(\partial_{\vec{f}}\omega)=\vec{f}(D(\omega))$, and

(3) for each $\vec{x}$ in W^m and each z in A, $\partial_{\vec{f}}\omega(\vec{x})=z$ iff there exists y in $D(\vec{f})\cap D(\omega)$ such that $\vec{f}(y)=\vec{x}$ and $\omega(y)=z$.

From the above definition, the following proposition obviously holds.

PROPOSITION 1. *Let $\vec{f}$ be in $\Pi_m(W)$ and ω be a pf from W to A.*

(1) *If ω is differentiable by $\vec{f}$, then $\partial_{\vec{f}}\omega(\vec{x})$ is the unique element of the set $\omega(\vec{f}^{-1}(\vec{x}))$ for each $\vec{x}$ in $D(\partial_{\vec{f}}\omega)$,*

(2) *If $\vec{f}$ is 1-1, then ω is differentiable by $\vec{f}$.*

(3) *In case that $\vec{f}$ is constant on $D(\vec{f})\cap D(\omega)$, ω is differentiable by $\vec{f}$ iff ω is also constant on $D(f)\cap D(\omega)$.*

(4) *If ω is a function and differentiable by an onto pf $\vec{f}$, then $\partial_{\vec{f}}\omega$ is also a function.*

It should be noted that if a pf ω from W to A is differentiable by $\vec{f}$ in $\Pi_0(W)$ then $\partial_{\vec{f}}\omega$ is an element of A or undefined because $\omega(D(\vec{f}))$ is a subset of A containing at most one element. For instance, if $\omega(D(\vec{f}))=\{a\}$, then $\partial_{\vec{f}}\omega=a$.

3. Case-Splitting Transformation

DEFINITION 2. A subfamily F of $\Pi(W)$ is said to be a *case-splitting transformation* (abbreviated to *cst*) over W iff the domains of pf's in F constitutes a finite division of W, that is,

(1) F is a finite family and the union of all $D(\vec{f})$'s for $\vec{f}$'s in F covers W, and

(2) all $D(\vec{f})$'s for $\vec{f}$'s in F are not empty and mutually disjoint.

DEFINITION 3. Let W be a well founded set with a partial order $\prec$ and F be a cst over W. Then F is said to be *descending* iff for each $\vec{f}=(f_1, \cdots, f_m)$ in F and for each x in $D(\vec{f})$, $f_i(x)\precneqq x$ for $i=1, \cdots, m$.

DEFINITION 4. A pf ω is said to be *differentiable by a cst F* iff ω is differentiable by every $\vec{f}$ in F.

If a pf ω from W to A is differentiable by a cst F over W, then the following logical formula is valid for each x in W:

$$\bigvee_{\vec{f}\in F}[x\in D(\vec{f})\wedge\omega(x)=\partial_{\vec{f}}\omega(\vec{f}(x))]$$

4. Finite Derivative Closure of Partial Functions

We denote the family of all functions from A^m to A by $\Omega_m(A)$ and set $\Omega(A)=\bigcup_{m=0}^{\infty}\Omega_m(A)$. It should be noted that $\Omega_0(A)$ is identified with A, and hence $\Omega(A)$ includes A. Let h be in $\Omega_m(A)$ and $\omega_1, \cdots, \omega_m$ be pf's from W to A. Then $h\circ(\omega_1, \cdots, \omega_m)$ denotes a pf from W^m to A such that the domain of the pf is $D(\omega_1)\times\cdots\times D(\omega_m)$ and the value of the pf is $h(\omega_1(x_1), \cdots, \omega_m(x_m))$ for $\vec{x}=(x_1, \cdots, x_m)$ in the domain.

For an $\vec{f}$ in $\Pi(W)$, a subset H of $\Omega(A)$ and pf's $\omega, \omega_0, \omega_1, \cdots, \omega_n$ from W to A, let $E(\vec{f}, \omega, H, \omega_0, \omega_1, \cdots, \omega_n)$ denote the set of all tuples $(h; j_1, \cdots, j_m)$, where $m=d(\vec{f})$, h is in $H\cap\Omega_m(A)$ and $j_1, \cdots, j_m$ are in $\{0, 1, \cdots, n\}$ such that $\partial_{\vec{f}}\omega\subset h\circ(\omega_{j_1}, \cdots, \omega_{j_m})$.

DEFINITION 5. A *finite derivative closure of pf's* (abbreviated to *fdcp*) is a system

$$\Gamma=(H; \omega_0, \omega_1, \cdots, \omega_n; F_0, F_1, \cdots, F_n),$$

where H is a subset of $\Omega(A)$, ω_i's are pf's from W to A and F_i's are cst's over W such that ω_i is differentiable by F_i for each $i=0, 1, \cdots, n$ and the set $E(\vec{f}, \omega_i, H, \omega_0, \omega_1, \cdots, \omega_n)$ is not empty for all $\vec{f}$ in F_i. It is also denoted by $E(\vec{f}, \omega_i, \Gamma)$.

If W is a well founded set and all F_i's are descending, then Γ is called a *halting fcdp*. Each ω_i is said to be *computable by* Γ.

A fdcp $\Gamma=(H; \omega_0, \omega_1, \cdots, \omega_n; F_1, F_1, \cdots, F_n)$ gives the following recursive program:

$$\overset{i=0}{\underset{n}{\{}} \bigvee_{\vec{f}\in F_i}[x\in D(\vec{f})\wedge\tilde{\omega}_i=h(\tilde{\omega}_{j_1}(f_1(x)), \cdots, \tilde{\omega}_{j_m}(f_m(x)))]$$

where $\vec{f}=(f_1, \cdots, f_m)$ and $(h; j_1, \cdots, j_m)$ is in $E(\vec{f}, \omega_i, \Gamma)$. If Γ is halting fdcp, then the above program defines functions $\tilde{\omega}_0, \tilde{\omega}_1, \cdots, \tilde{\omega}_n$ such that $\omega_i\subset\tilde{\omega}_i$ for $i=0, 1, \cdots, n$.

Thus, in some case, the problem of inference of functions from partial functions is solvable by finding a halting fdcp. The above program is called a *tree-automaton-like program generated by* Γ. If every $\vec{f}$ in each F_i satisfies that $d(\vec{f})\leqq 1$, then the program is called an *automation-like program*.

5. Simple Inference

In this section, we suppose that W is a well founded set with a partial order $\prec$.

DEFINITION 6. A subset H of $\Omega(A)$ is said to be *simple* iff, for each a in A, $h^{-1}(a)$ is finite for all h in H and there exist only a finite number of functions h in H such that $h^{-1}(a)$ is nonemyty.

For a subset D of W, we set $C(D)=\{x\in W; (\exists y\in D)[x\prec y]\}$. Let H be a simple subset of $\Omega(A)$, F is a descending cst over W and ω is a pf from W to A with a finite domain. Then we can obtain all ω_i's such that $\partial_{\vec{f}}\omega\subset h\circ(\omega_1, \cdots, \omega_m)$ for some $\vec{f}$ in F owing to the simplicity of H and the finiteness of $D(\omega)$, where $m=d(f)$. Such ω_i's are said to be *obtainable by* (ω, F, H).

Now suppose that a pf ω from W to A such that $C(D(\omega))$ is finite, a simple subset H of $\Omega(A)$ and a finite family G of descending cst's over W are given. Then, checking all possibility, we can easily construct all fdcp $\Gamma=(H; \omega_0, \omega_1, \cdots, \omega_n; F_0, F_1, \cdots, F_n)$ such that $\omega=\omega_0$, all F_i's are in G and for $i=1, \cdots, n$, ω_i is obtainable by (ω_j, F_j, H) for some $j<i$. An fdcp which is constructed as above is said to be *simply constructable*

by (ω, G, H).

A simply constructable fdcp is a representation of a method of inference of a function $\tilde{\omega}$ such that $\omega \subset \tilde{\omega}$. This inference is called *simple inference*.

We can use the simple inference for many sorts of objects, for example, regular languages, linear context-free languages, recognizable tree sets, generalized sequential machine (abbreviated to gsm) mappings and so on.

EXAMPLE 1 (Lonear context-free languages).

Let $W=\Sigma^*$ for some alphabet Σ and $A=\{true, false\}$. We introduce a partial order $\prec$ on W such that $x \prec y$ means that x is a subword of y. We define pf's f_σ and g_σ in $\Pi_1(W)$ for each σ in Σ by setting $f_\sigma(\sigma x)=x$ and $g_\sigma(x\sigma)=x$ for all x in W. Then $D(f_\sigma)=\sigma\cdot W$ and $D(g_\sigma)=W\cdot\sigma$. We also consider a pf $\bar{\varepsilon}$ in $\Pi_0(W)$ such that $D(\bar{\varepsilon})=\{\varepsilon\}$. We set $F^{(l)}=\{\bar{\varepsilon}\}\cup\} f_\sigma ; \sigma\in\Sigma\}$ and $F^{(r)}=\{\bar{\varepsilon}\}\cup\{g_\sigma ; \sigma\in\Sigma\}$, and set $G=\{F^{(l)}, F^{(r)}\}$.

Suppose that two disjoint finite subsets S_1, S_2 of W are given. The pair (S_1, S_2) is identified with a pf ω_0 from W to A with $D(\omega_0)=S_1\cup S_2$ defined by $\omega_0(x)=true$ if $x\in S_1$, and $\omega_0(x)=false$ if $x\in S_2$. We set $H=\{id\}\cup A$, where id stands for the identity mapping on A. Then, clearly we can take an fdcp $\Gamma=(H; \omega_0, \omega_1, \cdots, \omega_n; F_0, F_1, \cdots, F_n)$ simply constructable by (ω_0, H, G) with minimum length n.

The automaton-like program generated by Γ computes functions $\tilde{\omega}_0, \tilde{\omega}_1, \cdots, \tilde{\omega}_n$ satisfying $\omega_i\subset\tilde{\omega}_i$ for $i=0, 1, \cdots, n$, and if we set $L_1=\{x\in W ; \tilde{\omega}_0(x)=true\}$ and $L_2=\{x\in W ; \tilde{\omega}_0(x)=false\}$, then L_1 and L_2 are linear context-free languages such tnat $S_1\subset L_1$ and $S_2\subset L_2$. The linear context-free grammars G_1 and G_2 generating L_1 and L_2 respectively are obtained from Γ as follows: We take nonterminals $N_0, N_1, \cdots, N_n$. N_0 is the start symbol. If $F_i=F^{(l)}$ and $\partial_{f_\sigma}\omega_i=\omega_j$ then both G_1 and G_2 contain a production $N_i\to\sigma N_j$. If $F_i=F^{(r)}$ and if $\partial_{g_\sigma}\omega_i=\omega_j$ then both G_1 and G_2 contain a production $N_i\to N_j\sigma$. If $\partial_{\bar{\varepsilon}}\omega_i=true$, then G_1 contains $N_i\to\varepsilon$. If $\partial_{\bar{\varepsilon}}\omega_i=false$, then G_2 contains $N_i\to\varepsilon$. G_1 and G_2 contain just only productions as mentioned above.

EXAMPLE 2 (gsm Mappings).

We take the same W, f_σ's and $F^{(l)}$ as in Example 1. We set $A=\Delta^*$ for some alphabet Δ and set $H=A\cup\{h_x ; x\in A\}$, where $h_x(y)=xy$ for all x, y in A. Then clearly H is simple. We put $F=F^{(l)}$ and $G=\{F\}$. Suppose that a pf ω_0 from W to A with a finite domain is given satisfying the condition that ω_0 can be a restriction of some gsm mapping from W to A. Then clearly we can take an fdcp $\Gamma=(H; \omega_0, \omega_1, \cdots, \omega_n; F, F, \cdots, F)$ simply constructable by (ω_0, H, G) with minimum length n.

The automaton-like program generated by Γ computes functions $\tilde{\omega}_0, \tilde{\omega}_1, \cdots, \tilde{\omega}_n$ such that $\omega_i\subset\tilde{\omega}_i$ for $i=0, 1, \cdots, n$. The gsm computing $\tilde{\omega}_0$ is obtained from Γ as follows: We take states $q_0, q_1, \cdots, q_n$, where q_0 is the initial state. Let δ and λ be the state-transition function and the output function of the gsm, respectively. If $\partial_{f_\sigma}\omega_i\subset h_x\circ\omega_j$, then $\delta(q_i, \sigma)=q_j$ and $\lambda(q_i, \sigma)=x$.

For other interesting examples, the inference of pure LISP functions can be treated by considering cst's constituted by variously restricted *car* and *cdr*.

If we give a suitable H which is not simple and a suitable G such that we can construct an fdcp using the H and G, then we can make an nonsimple inference.

The inference using halting fdcp's is very powerful and it is significant to supply many examples although we mentioned in this paper only two of them.

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