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By

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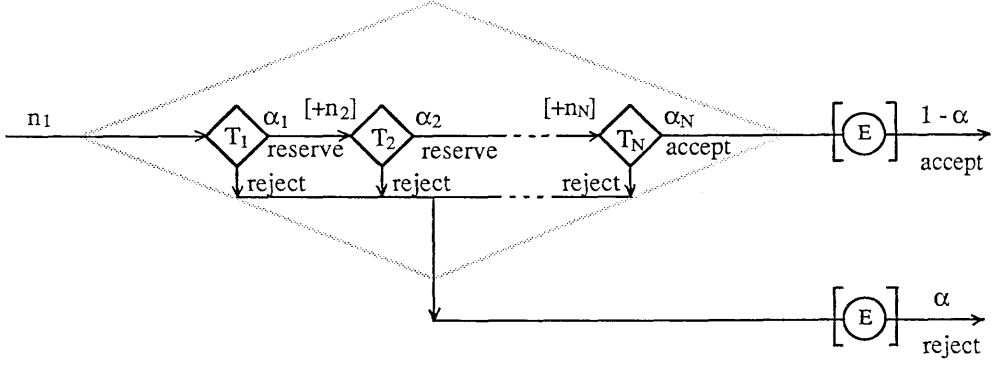
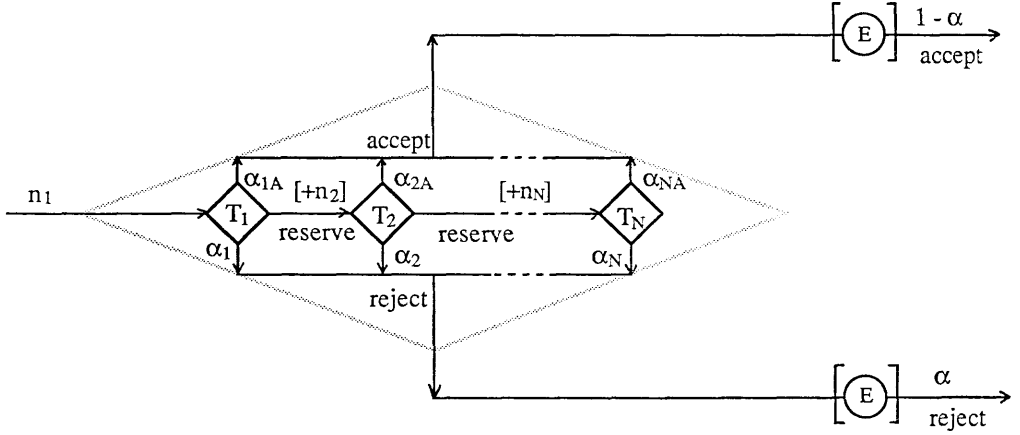
Abstract

Successive processes of statistical methods are usually applied to data analysis in practice. Excepting some studies of prototypes, however, mathematical properties of such many inference processes are still remained unclear, although their investigations are strongly needed in routine. In the present paper, inference processes are generally classified to three categories, and several plans proposed by previous authors are extended with wide flexibility, and are discussed on their properties. For their numerical evaluation, knowledge functions developed in NISAN system are applied.

1. Introduction

Theories of serial application of statistical methods have been studied in the fields of multistage sampling inspection plans, sequential analyses of statistical inference, successive processes of statistical methods, and so on. Since such serial procedures for inference are very attractive in view of sample size, sampling cost, significance level, power and relative efficiency of the overall process of tests, and the performed precision for estimation of parameters, there exist still various schemes to be investigated, excepting rather simple procedures already discussed by many authors, e.g. Kitagawa [8], Wald [14], Asano et al. [2, 3], McPherson and Armitage [10], Miller [11], Aroian [1], Pocock [13], Gould and Pecore [7] and so on. Regarding such inference procedures, there exist three categories, depending on the aim of actual process of statistical recognition [2]. The first kind of categories is process via identical tests by adding samples, the second is process via different tests with a similar purpose, and the third is inference process via different tests with different purposes. The respective categories contain two types, say Types 1 and 2 shown in Figures 1 and 2, with one-by-one, partial, group or adaptive sampling rule. In the present paper, several serial processes of inference are extended theoretically and their numerical evaluations are precisely investigated about three categories. For these evaluations, knowledge functions in NISAN system are effectively used. That is to say, PRTP command gives the power of an overall test process, expected number of samples (ASN) and expected cost (AC), which consists in both costs of sampling set up and samples, for the respective sampling

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Figure 1 N -stage test process plan of Type 1.Figure 2 N -stage test process plan of Type 2.

rules. PRTE command gives estimation of parameters, mean square error, and value of risk function, basing on an test procedure. PRDIST command is to give a nominal significance level for a conditional distribution of the final test statistics due to a test process. Such PR-family of commands in NISAN system is possible to examine properties of about 90 papers in the recent 300 papers related to inference processes.

2. Fundamental Properties of Statistical Inference Processes

Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{in_i})$ be a random variable vector with n_i samples, and let $T_i(\mathbf{X}_i)$ be a statistic at i -th stage, $i=1, 2, \dots, N$.

DEFINITION 1. N stage process with an overall significance level α is defined as follows,

$$Pr\left[\bigcup_{i \in I} (T_i(\mathbf{X}_i) \in C_i | H_{0i})\right] \leq \alpha, \quad (2.1)$$

where C_i is a critical region of i -th test for a null hypothesis H_{0i} , $i=1, 2, \dots, N$, and I is a non-empty index set $I \subset [1, 2, \dots, N]$.

Let α_i be a nominal significance level for critical region C_i and let α_{iA} a nominal acceptance probability for acceptance region A_i , $i=1, 2, \dots, N$. Let $Pr(C_i)$ and $Pr(A_i)$ denote $Pr(T_i(\mathbf{X}_i) \in C_i)$ and $Pr(T_i(\mathbf{X}_i) \in A_i)$, respectively, in short. The following Lemma 1 is led by Bonferroni's inequality.

LEMMA 1. *Let T be an N stage test process with $\alpha_i = \alpha/N$, $i=1, 2, \dots, N$. Then, the overall significance level of T is less or equal to α*

PROOF. By using additive theorem of probability, the lemma is derived.

LEMMA 2. *Let T_1 and T_2 be N stage process of Type 1 and Type 2, respectively, with the same critical region C_i , $i=1, 2, \dots, N$. Then, the overall significance level of T_1 is greater than or equal to T_2 .*

PROOF.

$$\begin{aligned} \alpha(T_1) &= Pr(\bigcup_{i \in I} C_i | T_1) = 1 - Pr(\bigcap_{i \in I} A_i | T_1) = 1 - Pr(A_N | T_1) \\ &\geq 1 - Pr(\bigcup_{i \in I} A_i | T_2) = Pr(\bigcup_{i \in I} C_i | T_2) = \alpha(T_2), \end{aligned} \quad (2.2)$$

where I is index set defined in Definition 1.

Usually, evaluation for the process with α_i by Lemma 1 is conservative, so that exact evaluation for overall significance level is to be calculated.

For N stage test process of Type 1, the overall hypothesis H_0 is rejected in case that at least one stage is rejected in N stages with nominal significance level α_i , $i=1, 2, \dots, N$.

On the other hand, there are 2^N patterns in the N stage test process basing on significance and nonsignificance patterns. In the process of Type 1, the overall α is given by the sum of probabilities for $2^N - 1$ patterns under a null hypothesis, excepting the probabilities of N nonsignificances. Thus, the following theorem is derived.

THEOREM 1. *Let T be an N stage test process in which the overall hypothesis H_0 is rejected in case that at least k -th stage, $k=0, 1, \dots, N$, is rejected in N stages with nominal significance level α_i , $i=1, 2, \dots, N$, and tests at each stage are independent each other, then the overall significance level δ for T is given by*

$$\delta = \sum_{i=k}^N \prod_{\substack{q_i \subseteq \Omega \\ j \in \omega_i}} [\alpha_j \prod_{t \in \Omega - \omega_i} (1 - \alpha_t)], \quad (2.3)$$

where Ω is a set $[1, 2, \dots, N]$ and ω_i is the subset of Ω .

PROOF. Under the independence of each stages, the probability that the H_0 is rejected at k -th stage in N stages plan is given by

$$\prod_{j \in \omega_k} [\alpha_j \prod_{t \in \Omega - \omega_k} (1 - \alpha_t)].$$

Thus, the probability for rejecting at least k -th stage in N stages is given by the sum of probabilities up to the N stages.

COROLLARY 1. *Under the condition of Theorem 1, if each nominal significance level is equals to α^* , the overall significance level is given by the sum of binomial $B(N, \alpha^*)$ probabilities up to N -th stage from k -th stage.*

PROOF. Every i -th stage, $i=1, 2, \dots, N$, independently has the same probability α^* for reject, so that the desired overall significance level is given by the upper probability

of binomial distribution with parameters, N and α^* .

In the actual situations, however, the tests at each stage have some associations, so that the decision for the overall α with nominal α_i is a difficult problem.

For the power, ASN and AC, it is usually very laborious and complicated to obtain numerically theoretical values for the conditional distribution of test statistics in test processes, based on mathematical formulations. For certain practical situations, asymptotic theories are applied. In the case of small sample size or closed plan, however, direct evaluation is to be preferred.

3. Inference Processes via Identical Tests

The inference processes in this section are the processes via identical tests, for instance, repeated significance tests (RST) and sequential probability ratio tests (SPRT). These inference processes are most popular, in which one statistical test is repeatedly applied by adding samples at every stage, when any decision is reserved. It is laborious and complicated to obtain numerical values of probability for the mathematical properties of multi-stage procedure. In the case of small sample size, however, direct methods for evaluation is to be preferred. In this section, two direct methods for closed SPRT and RST are evaluated.

3.1 Direct method for normal mean test

Aroian [1] presented a method of numerical evaluation of the power and ASN for SPRT one sample mean test (T) of a closed process. That is, let T be a one sample normal mean test of a closed type at N samples from a normal population $N(\mu, \sigma^2)$, σ^2 : known, based on one by one sampling SPRT. The hypothesis is $H_0: \mu = \mu_0$ against the alternative $H_1: \mu = \mu_1$ and the critical value at i -th stage, $i=1, 2, \dots, N-1$, is used the value for SPRT. Then, the procedure of T is shown in the following way.

PROCEDURE 1. At i -th stage, $1 \leq i \leq N-1$,

$$\begin{aligned} &\text{accept } H_0, && \text{if } \sum_{j=1}^i x_j \geq X_{i1}, \\ &\text{accept } H_1, && \text{if } \sum_{j=1}^i x_j \leq X_{i0}, \\ &\text{continue sampling,} && \text{otherwise,} \end{aligned}$$

the final stage, i.e. $i=N$

$$\begin{aligned} &\text{accept } H_0, && \text{if } \sum_{j=1}^i x_j \geq (X_{i0} + X_{i1})/2, \\ &\text{accept } H_1, && \text{otherwise,} \end{aligned}$$

where

$$\begin{aligned} X_{ij} &= \sigma^2 / (\mu_1 - \mu_0) \log A_j + i(\mu_1 - \mu_0)/2, \\ A_0 &= \beta / (1 - \alpha), \quad A_1 = (1 - \beta) / \alpha, \quad (i=1, 2, \dots, N), (j=0, 1). \end{aligned}$$

The mathematical formulations of the power, ASN and AC for this process are shown as below.

THEOREM 2. Let T be an N stage test process which has Procedure 1. Then, power,

ASN and AC are given by the following formulae respectively, where K_1 and K_2 are sample and setup costs.

$$L(\mu) = \sum_{i=1}^N C^i \left[\int_{-\infty}^{L_i - Y_{i-1}} \dots \int_{L_1}^{U_1} f_i(\mu) \prod_{j=1}^i dx_j \right], \quad (3.1)$$

$$E_1(\mu) = \sum_{i=1}^N i C^i \left[\int_{U_i - Y_{i-1}}^{\infty} + \int_{-\infty}^{L_i - Y_{i-1}} \right] \int_{L_{i-1} - Y_{i-2}}^{U_{i-1} - Y_{i-2}} \dots \int_{L_1}^{U_1} f_i(\mu) \prod_{j=1}^i dx_j, \quad (3.2)$$

$$E_2(\mu) = K_1 \sum_{i=1}^{N-1} C^i \left[\int_{L_i - Y_{i-1}}^{U_i - Y_{i-1}} \dots \int_{L_1}^{U_1} f_i(\mu) \prod_{j=1}^i dx_j \right] + K_2 E_1(\mu), \quad (3.3)$$

where

$$f_i(\mu) = \exp \left[- \sum_{j=1}^i (x_j - \mu)^2 / (2\sigma^2) \right], \quad \int_{L_0}^{U_0} f_0(\mu) d\mu = 1,$$

$$C = (\sqrt{2\pi}\sigma)^{-1}, \quad Y_i = \sum_{j=1}^i x_j, \quad U_i = X_{i1}, \quad L_i = X_{i0}.$$

PROOF. The probability of accept H_1 at i -th stage is given by the conditional probability

$$C^i \left[\int_{-\infty}^{L_i - Y_{i-1}} \dots \int_{L_1}^{U_1} f_i(\mu) \prod_{j=1}^i dx_j \right],$$

and power is shown the sum of these probabilities up to N stage. The ASN is given the product of the probability for accept H_0 or H_1 and sample size when the process terminates, and AC is sum of ASN multiplied by sample cost and average stage by set up cost.

The numerical evaluation of these values are very complicated so that Aroian evaluated only the power and ASN for $\alpha = \beta = 0.09975$, $N = 7$, $\mu_0 = -\mu_1 = 0.5$, shown in left side of Table 1. The extended results including standard deviation of ASN are shown in right side by PRTP command in knowledge function with good precision. Note that this process is a one by one sampling, so that the average cost is simply $(K_1 + K_2)$ multiplied by ASN, where K_1 and K_2 are sample and setup costs, respectively.

Table 1 Properties of closed SPRT for normal mean test.

μ	L. Aroian (1968)		PRTP Command (NISAN)			
	O.C.	A.S.N.	O.C.	A.S.N.	S.N.D.	Cost*
0.75	0.971	3.652	0.972	3.662	1.789	14.648
0.50 (μ_0)	0.898	4.391	0.903	4.395	1.981	17.580
0.25	0.739	5.012	0.734	5.010	1.986	20.040
0.00	0.506	5.266	0.498	5.231	1.956	20.924
- 0.25	0.262	5.012	0.266	5.009	1.992	20.036
- 0.50 (μ_1)	0.102	4.391	0.101	4.378	1.986	17.512
- 0.75	0.029	3.652	0.031	3.629	1.803	14.516

* $K_1 = 1$, $K_2 = 3$

For the extension of such kinds of investigation, let a process be considered for a mixed distribution $G(x)=(1-p)F(x|\mu_0, \sigma^2)+pF(x|\mu_1, \sigma^2)$, which has the same null, $p=0$ and alternative, $p=1$, hypotheses of Procedure 1. The distribution of mean for sample size n is not normal with mean, $E(X)=(1-p)\mu_0+p\mu_1$, and variance, $V(X)=[(\mu_0-\mu_1)^2p(1-p)+\sigma^2]/n$. The power, ASN and AC for this process are given in the following way.

THEOREM 3. Let T be an N -stage one sample normal mean test from mixed distribution $G(x)=(1-p)F(x|\mu_0, \sigma^2)+pF(x|\mu_1, \sigma^2)$, σ^2 : known, based on one by one sampling SPRT for $H_0: p=0$ against $H_1: p=1$. Then, the power, ASN and AC are given by

$$L(p)=\sum_{i=1}^N C^i \left[\int_{-\infty}^{L_i-Y_{i-1}} \dots \int_{L_1}^{U_1} \prod_{j=1}^i g_j(p) \prod_{j=1}^i dx_j \right], \quad (3.4)$$

$$E_1(p)=\sum_{i=1}^N i C^i \left[\int_{U_i-Y_{i-1}}^{\infty} + \int_{-\infty}^{L_i-Y_{i-1}} \right] \int_{L_{i-1}-Y_{i-2}}^{U_{i-1}-Y_{i-2}} \dots \int_{L_1}^{U_1} \prod_{j=1}^i g_j(p) \prod_{j=1}^i dx_j, \quad (3.5)$$

$$E_2(p)=K_1 \sum_{i=1}^{N-1} C^i \left[\int_{L_i-Y_{i-1}}^{U_i-Y_{i-1}} \dots \int_{L_1}^{U_1} \prod_{j=1}^i g_j(p) \prod_{j=1}^i dx_j \right] + K_2 E_1(p), \quad (3.6)$$

respectively, where

$$g_i(p)=(1-p)\exp[-(x_j-\mu_0)^2/(2\sigma^2)]+p\exp[-(x_j-\mu_1)^2/(2\sigma^2)],$$

$$\int_{L_0}^{U_0} g_0(p) dx_0=1, \quad C=(\sqrt{2\pi}\sigma)^{-1}, \quad Y_i=\sum_{j=1}^i x_j,$$

$$U_i=X_{i1}, \quad L_i=X_{i0}, \quad X_{ij}=\sigma^2/(\mu_1-\mu_0) \log A_j + i(\mu_1-\mu_0)/2,$$

$$A_0=\beta/(1-\alpha), \quad A_1=(1-\beta)/\alpha \quad (i=1, 2, \dots, N), \quad (j=0, 1).$$

PROOF. The probability of accept H_1 at i -th stage is given by the conditional probability of the mixed distribution

$$C^i \left[\int_{-\infty}^{L_i-Y_{i-1}} \dots \int_{L_1}^{U_1} \prod_{j=1}^i g_j(p) \prod_{j=1}^i dx_j \right],$$

and power, ASN and AC are given similarly by the proof of Theorem 2.

The numerical evaluations for this process are shown in Table 2 with ASN's stan-

Table 2 Power and ASN for extended mixed alternatives.

p	OC	ASN	SD
0.0 *	0.898 *	4.391 *	1.981
0.1	0.834	4.541	2.016
0.2	0.754	4.685	2.051
0.3	0.671	4.763	2.060
0.4	0.592	4.892	2.074
0.5	0.491	4.927	2.080
0.6	0.412	4.851	2.067
0.7	0.324	4.783	2.072
0.8	0.243	4.707	2.049
0.9	0.162	4.559	2.011
1.0 *	0.102 *	4.391 *	1.986

dard deviation by MIXRAN and PRTP commands, where MIXRAN is to give the mixed distribution.

3.2 Direct method for binomial case

Colton-McPherson [4] proposed a two stage group sampling process from binomial distribution $B(n, p)$, and gave the significance level, power and ASN considering the loss, where p is the probability of success defined in a single trial. Suppose that an extended process T be an N stage group sampling test process of Type 2 for binomial distribution $B(n_i, p)$, $i=1, 2, \dots, N$, and a hypothesis is $H_0: p=p_0$ against $H_1: p=p_1$. Then, the process T is shown in the following way.

PROCEDURE 2. At i -th stage, $1 \leq i \leq N-1$,

accept H_0 , if $\sum_{j=1}^i r_j > C_N$,
 accept H_1 , if $\sum_{j=1}^i r_j \leq C_i$,
 continue sampling, otherwise,

the final stage, i.e. $i=N$

accept H_0 , if $\sum_{j=1}^i r_j > C_N$,
 accept H_1 , otherwise,

where C_i : critical value of i -th stage, r_i : number of successes, $i=1, 2, \dots, N$.

THEOREM 4. Let T be an N stage test process which has Procedure 2. Then, the power, ASN and AC are given by

$$L(p) = 1 - \sum_{i=1}^N \sum_{j=C_{i-1}+1}^{C_i} h_{i-1}(k_{i-1}) G(n_i, C_{i-j}, p), \quad (3.7)$$

$$E_1(p) = \sum_{i=1}^N \left\{ n_i \left[\sum_{k_i=C_{i-1}+1}^{C_N} h_{i-1}(k_i) \right] \right\}, \quad (3.8)$$

$$E_2(p) = K_1 \sum_{i=0}^{N-1} \sum_{k_i=C_i+1}^{C_N} h_i(k_i) + K_2 E_1(p), \quad (3.9)$$

where

$$g(n, j, p) = \begin{cases} \binom{n}{j} p^j (1-p)^{n-j} & 0 \leq j \leq n \\ 0 & \text{otherwise,} \end{cases}$$

$$G(n, c, p) = \begin{cases} \sum_{j=0}^c g(n, j, p) & 0 \leq j \leq n \\ 0 & \text{otherwise,} \end{cases}$$

$$h_i(k_i) = \begin{cases} \sum_{k_{i-1}=C_{i-1}+1}^{k_i} h_{i-1}(k_{i-1}) g(n_i, k_i - k_{i-1}, p) & i > 1 \\ 1 & i = 1. \end{cases}$$

PROOF. The probability of continuing sampling at i -th stage is sum of conditional probabilities

$$\sum_{k_i=C_{i-1}+1}^{C_N} h_{i-1}(k_i),$$

based on cumulative binomial probability. Therefore, the power, ASN and AC are given by its functions.

3.3 Repeated significance test

Pocock [13] proposed an N stage group repeated significance test of Type 1 with sample size n for two sample normal mean test from normal populations $N(\mu_i, \sigma^2)$, $i=1, 2, \dots, N$, σ^2 : known. He obtained equal nominal significance level α_i , $i=1, 2, \dots, N$, for overall significance level $\alpha=0.050$ in testing the hypothesis $H_0: \mu_1=\mu_2$ against $H_1: \mu_1 \neq \mu_2$. The extended process adding different sample size n_m at m -th stage, $m=1, 2, \dots, N$, is shown in the following way.

PROCEDURE 3. At m -th stage, $1 \leq m \leq N$,

continue sampling, $p_m > \alpha_m$ and $m < N$,

accept H_0 , if $p_m > \alpha_m$ and $m = N$,

accept H_1 , if $p_m \leq \alpha_m$,

where

$$p_m = 2[1 - \Phi\{\sqrt{M_m}d_m/\sqrt{2}\sigma\}], \quad M_m = \sum_{j=1}^m n_j, \quad d_m = \left\{ \sum_{j=1}^m (n_j(\bar{x}_{1j} + \bar{x}_{2j})) \right\} / M_m,$$

$$\bar{x}_{ij} = \left[\sum_{k=1}^{n_j} x_{ijk} \right] / n_j, \quad \Phi(x) = \int_{-\infty}^x \phi(t) dt, \quad \phi(t) = \exp[-t^2/2]/(\sqrt{2\pi}),$$

α_m : nominal significance level of m -th stage, ($i=1, 2$), ($j=1, 2, \dots, N$).

Then, the power, ASN and AC are given by the following Theorem 5.

THEOREM 5. Let T be an N stage group repeated significance test, which has Procedure 3. Then, the power, ASN and AC are given by

$$L(\mu) = 1 - \sum_{i=1}^N \int_{u(\alpha_{i/2})}^{\infty} [f_i(s) + f_i(-s)] ds, \quad (3.10)$$

$$E_1(\mu) = 2 \left\{ \sum_{i=1}^N M_i \int_{u(\alpha_{i/2})}^{\infty} [f_i(s) + f_i(-s)] ds + M_N \int_{-u(\alpha_{N/2})}^{u(\alpha_{N/2})} f_N(s) ds \right\}, \quad (3.11)$$

$$E_2(\mu) = 2K_1 \sum_{i=1}^{N-1} \int_0^{u(\alpha_{i/2})} f_i(s) ds + K_2 E_1(\mu), \quad (3.12)$$

respectively, where

$$f_i(s) = \begin{cases} \phi(s - \delta_i) & (i=1) \\ \int_0^{u(\alpha_{i/2})} \phi(s - t - \delta_i) f_{i-1}(t) dt & (i=2, 3, \dots, N), \end{cases}$$

$$\int_0^{u(\alpha_0)} f_0(s) ds = 1, \quad \delta_i = \sqrt{n_i} \mu / \sigma, \quad \mu = \mu_1 - \mu_2.$$

PROOF. The probability density function at i -th stage is given by the convolution

$$\int_0^{u(\alpha_{i/2})} \phi(s - t - \delta_i) f_{i-1}(t) dt.$$

Therefore the power, ASN and AC are given in use of this function, recursively.

COROLLARY 2. Suppose that T has Procedure 3, then the overall significance level does not depend on sample size n_i , $i=1, 2, \dots, N$.

PROOF. Under null hypothesis, $\mu_1=\mu_2$, $L(\mu)$ does not include δ_i , i.e., sample size n_i .

Pocock obtained equal nominal significance levels α_i table for some stages. That is, the nominal significance levels are 0.029 and 0.016 for 2 and 5 stages when overall significance level is 0.050, respectively. He recommended that these values are asymptotically adapted for normal mean test for unknown variance, i.e. t test. For small sample size, however, t test depends on sample size n . PRDIST command is to give easily such nominal probabilities. Table 3 shows a value of the final α_N in case of every $\alpha_i=0.029$ or 0.016 as Pocock suggested. Table 4 shows values of the final α_N in case of every $\alpha_i=0.010$ for overall $\alpha=0.050$. This shows that it is not so wrong to use the values of normal mean test for t test. If sample size n intends to infinity, final α_N equals to α_i .

Gould-Pecore [7] has investigated about N stage group repeated significance test of Type 2, with sample size n for two sample normal mean test. For extended process adding different sample size n_m at m -th stage, the following theorem is derived.

THEOREM 6. Let T be an N stage group repeated significance test of Type 2. Then, the power, ASN and AC are given by

$$L(\mu)=1-\sum_{i=1}^N \int_{u(\alpha_i/2)}^{\infty} [g_i(s)+g_i(-s)]ds, \quad (3.13)$$

$$E_1(\mu)=2\left\{\sum_{i=1}^N M_i \left[\int_{u(\alpha_i/2)}^{\infty} [g_i(s)+g_i(-s)]ds + \int_{-u(\alpha_{iA}/2)}^{u(\alpha_{iA}/2)} g_i(s)ds \right] \right. \\ \left. + M_N \int_{-u(\alpha_{NA}/2)}^{u(\alpha_{NA}/2)} [g_N(s)+g_N(-s)]ds \right\}, \quad (3.14)$$

$$E_2(\mu)=2K_1 \sum_{i=1}^{N-1} \int_{u(\alpha_{iA}/2)}^{u(\alpha_i/2)} g_i(s)ds + K_2 E_1(\mu), \quad (3.15)$$

where α_{mA} : nominal acceptance probability at m -th stage,

$$g_i(s)=\begin{cases} \phi(s-\delta_i) & (i=1) \\ \int_0^{u(\alpha_i/2)} \phi(s-t-\delta_i)g_{i-1}(t)dt & (i=2, 3, \dots, N) \end{cases}$$

$$\int_0^{u(\alpha_0)} g_0(s)ds=1, \quad \delta_i=\sqrt{n_i}\mu/\sigma, \quad \mu=\mu_1-\mu_2$$

PROOF. The probability density function at i -th stage is given

$$\int_0^{u(\alpha_i/2)} \phi(s-t-\delta_i)g_{i-1}(t)dt$$

Therefore the power, ASN and AC are given similarly by Theorem 5.

Table 3 The final α_N in case of every sig. level is $\alpha_i=2.9\%$ or 1.6% ($i=1, 2, \dots, N-1$) for t test.

$N \backslash n$	5	20	100
2	0.027	0.029	0.029
5	0.015	0.016	0.016

Table 4 The final α_N in case of every sig. level is $\alpha_i=1.0\%$ ($i=1, 2, \dots, N-1$) for t test.

$N \backslash n$	5	20
3	0.040	0.041
5	0.033	0.034
7	0.025	0.027

Table 5 The final α_N in case of every nominal significance level is used $\alpha_i=3.1\%$ or 2.4% ($i=1, 2, \dots, N-1$) for t test for $\alpha_{iA}=50\%$.

$N \backslash n$	5	20	100
2	0.028	0.031	0.031
3	0.019	0.023	0.024

Gould-Pecore obtained equal nominal significance levels $\alpha_i=0.031$ and 0.024 for 2 and 3 stages, respectively. Table 5 shows the final values α_N corrected for overall $\alpha=0.050$ for the probability of acceptance $\alpha_{iA}=0.500$.

4. Inference Processes via Different Tests with a Similar Purpose

In view of practical and reasonable situations, there are some inference processes which are composed of two or three different nonparametric and parametric tests with a similar purpose. Elder-Muse [6] discussed a two stage process, including both tests by attributes with unit-limit and variables with average-limit. That is, let T be a two stage test process with sample size n from normal population $N(\mu, \sigma^2)$. Let μ_0 denote the average limit, and let $\mu_0 - c\sigma$ denote the unit limit and k represent the maximum number of unit limit violations, where c is an arbitrary positive constant.

THEOREM 7. Elder-Muse [6] *Let T be a two stage test process of Type 1, composed both tests by attributes with unit limit and variables with average limit. Then, the power is given by*

$$L(\mu) = 1 - \sum_{i=0}^k \binom{n}{i} [\Phi(c-t)]^{n-i} [\Phi(t-c)]^i \{1 - \Phi[(\mu_0 - a(t))/\sqrt{b(t)}]\}, \quad (4.1)$$

where

$$a(t) = \mu + \sigma g(v)(n-i)/n - \sigma g(-v)i/n,$$

$$b(t) = (\sigma^2/2)(1 - g(v)(v + g(v))(n-i)/n - g(-v)(-v + g(-v))i/n),$$

$$g(v) = \phi(v)/\Phi(v), \quad v = c - t, \quad t = (\mu_0 - \mu)/\sigma.$$

PROOF. The probability of acceptance is given by the product of conditional probability derived using truncated normal probability density functions and the probability of i -th limit violations.

For another process of Type 1 for location, median, Wilcoxon and t tests in order are applied. That is, three stage group sampling process of Type 1, composed median, Wilcoxon and t tests adding sample n_i , $i=1, 2, 3$, are to be considered. In these cases, PRTP command is applicable to study statistical properties of the processes. For instance, the overall significance level is shown 0.0910, that is, 0.0226 for median test, 0.0394 for Wilcoxon test, 0.0290 for t test, to an exponential distribution with mean zero and variance unity, where $n_1=n_2=n_3=10$ and $\alpha_1=\alpha_2=\alpha_3=0.050$.

5. Inference Processes via Different Tests with Different Purposes

The inference processes in this section are mainly used for composite hypothesis and series of hypotheses, in which some different statistical tests are applied by not adding samples, for example, location and dispersion tests for the type of distribution and multiple test. The former process is used for testing a hypothesis $H_0: F(x)=F_0(x)$ against $H_1: F(x)=F_1(x)$. The power of this process becomes complicated, and alternative distribution may be skewed or unclear types of distribution. The illustrative hypothesis is $H_0: \mu=\mu_0, \sigma^2=\sigma_0^2$ against $H_1: \mu \neq \mu_0, \sigma^2 \neq \sigma_0^2$. The latter process is used for testing the series of hypotheses $H_{01}, H_{02}, \dots, H_{0N}$ against hypotheses $H_{11}, H_{12}, \dots, H_{1N}$. The simple process may be two stage process with chi-square test for dispersion and u -test for mean. The first stage is whether the variance is unity, i.e., $\sigma^2=1$ or not, if accepted, and secondly to test whether mean is zero or not under the condition $\sigma^2=1$.

THEOREM 8. *Let T be two stage process of Type 1 with chi-square test for dispersion and u -test for mean. Then, the mathematical formulation for power is shown by*

$$L(\mu) = C \int_0^{\chi_{n-1}^2(\alpha_1)/\lambda} \int_{-\infty}^{u(\alpha_2) - \sqrt{n}\mu} x^{(n-3)/2} \exp[-(x+u^2)/2] du dx, \quad (5.1)$$

where

$$C = 2^{-n/2} \{\sqrt{\pi} \Gamma[(n-1)/2]\}^{-1}.$$

PROOF. The joint distribution of χ_{n-1}^2 and u distributions is shown by

$$2^{-n/2} \{\sqrt{\pi} \Gamma[(n-1)/2]\}^{-1} x^{(n-3)/2} \exp[-(x+u^2)/2].$$

Therefore, the power function is given by (5.1).

The illustrative nonparametric two stage process of Type 1 is Ansari-Bradley test for dispersion and Wilcoxon test for location in order, which are independent tests each other. This process is examined to calculate nominal α_i , $i=1, 2$, for fixed overall α by PRDIST command. Given $\alpha_1=0.050$ and overall $\alpha=0.100$, the percent point for α_2 is given by about 1.94 for both sides test. This value is corresponding to 0.0524 for both side probabilities under normal distribution, where theoretical value calculated from Theorem 1 is 0.0526. Note that the properties of this test process are different from

those of Lepage test, whose statistic is shown by the sum of square of these of these two statistics.

For extended another process, an inference process which consists of test for the type of distribution and the local most powerful test for the prespecified distribution is to be considered. The test for normal distribution and t test, the test for logistic distribution and Wilcoxon test and the test for exponential distribution and Savage test, in order, are illustrative test processes.

6. Extension of Relative Efficiency to Inference Process and Discussion

Comparing two techniques for the inference, efficiency and deficiency are usually used. These values are calculated by the ratio of samples or variances of two techniques which achieve the prefixed design. In case of two tests, the relative efficiency is obtained by the ratio of sample sizes of two tests with just equal power for the same design. The asymptotic relative efficiency also obtained by the limiting ratio of sample sizes required to the limiting power. That is, the relative efficiency and asymptotic relative efficiency between two tests are defined in the following way. Consider samples from distribution $F(x|\theta)$ and let T_1 and T_2 be two tests with given level of significance levels α for the hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: \theta = \theta_1$. Let $\beta_N(\theta)$ and $\beta_N^*(\theta)$ denote the power functions of T_1 and T_2 based on N observations.

DEFINITION 2. Relative efficiency (Lehmann [9]) Let $\beta_{N1}(\theta)$ be a specified power given θ and $N1$ with $\alpha < \beta < 1$. Then, if there exists $N2$ such that $\beta_{N2}^*(\theta) = \beta_{N1}(\theta)$,

$$\rho(T_1, T_2) = N1/N2 \quad (6.1)$$

is defined to be the relative efficiency of the test T_2 with respect to the test T_1 .

DEFINITION 3. Asymptotic relative efficiency (Pitman [12]) Let $\beta_N(\theta)$ be a specified power given θ and N with $\alpha < \beta < 1$. Consider a sequence of alternatives θ_N such that $\beta_N(\theta_N) \rightarrow \beta_0$ (as $N \rightarrow \infty$), and a sequence $M = h(N)$ such that $\beta_M^*(\theta_N) \rightarrow \beta_0$ (as $N \rightarrow \infty$), where β_0 is a constant value which belongs to $(0, 1)$. Then if

$$e(T_1, T_2) = \lim_{N \rightarrow \infty} N/M \quad (6.2)$$

exists and is independent of α and β , $e(T_1, T_2)$ is defined to be the asymptotic relative efficiency of the test T_2 with respect to the test T_1 .

For the relative efficiency, the following theorem is derived.

THEOREM 9. Let $\rho(T_i, T_j)$ be the relative efficiency of test T_j with respect to the test T_i , $i, j = 1, 2, 3$, $i \neq j$. For a specified power given θ ,

$$\rho(T_i, T_j) = 1/\rho(T_j, T_i), \quad (6.3)$$

$$\rho(T_i, T_k) = \rho(T_i, T_j)\rho(T_j, T_k), \quad (6.4)$$

where $i, j, k = 1, 2, 3$, $i \neq j$, $j \neq k$, $i \neq k$.

PROOF. Proof is omitted.

The extended relative efficiency is defined to the test process T_2 with respect to the test T_1 .

Table 6 The relative efficiency of two stage t test with respect to t test.

mean	0.2	0.4	0.6	0.8	1.0
efficiency	0.810	0.897	0.912	0.966	1.123

DEFINITION 4. Let $\beta_N(\theta)$ be a specified power given θ and N with $\alpha < \beta < 1$. Then if there exists $ASN(T_2)$ such that $\beta_{ASN(T_2)}(\theta) = \beta_N(\theta)$,

$$\rho(T_1, T_2) = N / ASN(T_2) \quad (6.5)$$

is defined to be the relative efficiency of the test process T_2 with respect to the test T_1 .

The numerical evaluation is applied for relative efficiency of two stage t test process T_2 , which has same additional sample numbers, with respect to ordinary t test T_1 . Table 6 shows the values of relative efficiency on normal distribution with mean zero and variance unity for the first population, and the second population is shifted with the same variance and distribution type, where sample sizes are $n_1 = n_2 = 15$ for the test process T_2 with respect to the test T_1 .

7. Conclusion

Although there are difficulties and complications to make clear properties numerically by the theoretical formulae under variety of conditions except only the simple case like in statistical text book, there exist much more possibilities in view of statistical computing to evaluate statistical properties under various situations. Actually, the knowledge commands in NISAN system provides useful knowledge, which makes clear the properties numerically under variety of conditions in statistical data analysis.

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