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ON THE ASYMPTOTIC NORMALITY FOR NONPARAMETRIC SEQUENTIAL DENSITY ESTIMATION*

By

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Abstract

Let $f_n(x)$ be a recursive kernel estimator of a probability density function f at a point x. We show that if $N(t)$ is a sequence of positive integer-valued random variables and $\pi(t)$ a sequence of positive numbers with $N(t)/\pi(t) \to \theta$ in probability as $t \to \infty$, where θ is a positive discrete random variable, then $(N(t)h^p_{N(t)})^{1/2}(f_{N(t)}(x)-f(x))$ is asymptotically normally distributed under certain conditions.

1. Introduction

Let X be a p-dimensional random vector on a probability space $(\Omega, \mathcal{B}, P)$ having a probability density function (p.d.f.) f with respect to the Lebesgue measure on R^p. There is a vast literature on the problem of estimating the p.d.f. f (see Devroye and Györfi [3], and Prakasa Rao [9] for example). In particular, estimators have been proposed in some recursive manners by several authors on behalf of the following two advantages: data need not be stored, and the estimators are easily updated when new data become available. In this paper we consider the recursive kernel estimator proposed by the author [4].

On the other hand, in many practical situations the number of observations $N(t)$ which we observe in a time-interval $(0, t]$ is random. The problem of sequential estimation of the p.d.f. by using positive integer-valued random variables (i.e., stopping rules) were studied by Davies and Wegman [2], Carroll [1], Wegman and Davies [11] and the author [5], for example. Carroll [1] and the author [5] investigated the asymptotic normality of estimates of the p.d.f. under random sample sizes. In this paper we shall show that the asymptotic normality holds for a more general class of positive integer-valued random variables $N(t)$ than the classes of Carroll [1] and the author [5]. We note that the extension to this general class was motivated by the discussion in Rényi [10]. Throughout this paper we consider the estimator $f_{N(t)}(x)$ of the p.d.f. $f(x)$ based on $X_1, X_2, \cdots, X_{N(t)}$, which is defined by

$$f_{N(t)}(x)=\sum_{j=1}^{N(t)} a_j\beta_{jN(t)}K_j(x, X_j)+\beta_{0N(t)}K(x), \tag{1.1}$$

where $X_1, X_2, \cdots$ are independent observations of X,

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$$K_n(x, y)=h_n^{-p}K((x-y)/h_n) \qquad \text{for} \quad x, y\in R^p, \tag{1.2}$$

K is a bounded, integrable, real-valued Borel measurable function on R^p and $\{h_n\}$ with $h_0=h_1$ is a nonincreasing sequence of positive numbers converging to zero,

$$a_n=a/n \qquad \text{for any fixed} \quad a\in(0, 1], \tag{1.3}$$

and

$$\beta_{mn}=\begin{cases} \prod_{j=m+1}^{n}(1-a_j) & \text{if} \quad n>m\geqq 0 \\ 1 & \text{if} \quad n=m\geqq 0. \end{cases} \tag{1.4}$$

The aim of this paper is to show that under certain conditions $(N(t)h_{N(t)}^p)^{1/2}(f_{N(t)}(x)-f(x))$ is asymptotically normally distributed. In Section 2 we shall make some preparations and auxiliary results. In Section 3 we shall give our main theorem.

2. Auxiliary Results

In this section we shall make some preparations and auxiliary results. Set

$$\gamma_1=1 \quad \text{and} \quad \gamma_n=\sum_{j=2}^{n}(1-a_j) \qquad \text{for} \quad n\geqq 2.$$

where a_n is as defined in (1.3). Clearly,

$$\beta_{mn}=\gamma_n\gamma_m^{-1} \qquad \text{for} \quad n\geqq m\geqq 1. \tag{2.1}$$

It is known in [4] that

$$L_1n^{-a}\leqq\gamma_n\leqq L_2n^{-a} \qquad \text{for some constants } L_1,\ L_2>0 \text{ and all } n\geqq 1 \tag{2.2}$$

and

$$\beta_{mn}\sim m^a n^{-a} \qquad \text{as} \quad n\geqq m\to\infty, \tag{2.3}$$

where "$\sim$" means the asymptotic equivalence. For a real-valued functing g let $C(g)$ be the set of continuity points of g. Throughout this paper we assume the function K in Section 1 to satisfy

$$\int_{R^p}K(u)du=1, \qquad \int_{R^p}\|u\|_p^2|K(u)|du<\infty$$

$$\int_{R^p}u_iK(u)du=0 \qquad \text{for} \quad i=1, \cdots, p \quad \text{with} \quad u=(u_1, \cdots, u_p)$$

and

$$\|u\|_p^p|K(u)|\to 0 \qquad \text{as} \quad \|u\|_p\to\infty,$$

where $\|\cdot\|_p$ denotes the Euclidean norm on R^p. On the sequence $\{h_n\}$ in Section 1 we shall impose some or all of the following conditions: For a fixed $a\in(0, 1]$,

(H1) $\quad nh_n^p\uparrow\infty \quad$ as $\quad n\to\infty$,

(H2) $\quad n^{1-2a}h_n^p\to 0 \quad$ as $\quad n\to\infty$,

(H3) $\quad n^{1-2a}h_n^p\sum_{j=1}^{n}j^{2(a-1)}h_j^{-p}\to\beta \quad$ as $n\to\infty$ for some constant $\beta>0$, (H4) $n^{3/2-3a}h_n^{3p/2}\sum_{j=1}^{n} j^{3(a-1)}h_j^{-2p}\to 0$ as $n\to\infty$,

(H5) $(n^{1-2a}h_n^p)^{1/2}\sum_{j=1}^{n} j^{a-1}h_j^2\to 0$ as $n\to\infty$,

(H6) For any $\varepsilon>0$ there exists a positive constant $\delta=\delta(\varepsilon)$ such that $|n/m-1|<\delta$ implies $|h_n/h_m-1|<\varepsilon$.

EXAMPLE.
Let

$$h_n=n^{-r/p} \quad \text{with} \quad \max\{p/(p+4),\ 1-2a\}<r<1.$$

Then $\{h_n\}$ satisfies (H1)$\sim$(H6) with $\beta=(2a+r-1)^{-1}$. Throughout this paper $C, C_1, C_2, \cdots$ denote appropriate positive constants. The following lemma can be found in the author [6].

LEMMA 2.1. *Let $\{h_n\}$ be a sequence of positive numbers converging to zero. Suppose that k is a bounded, integrable, real-valued Borel measurable function on R^p satisfying*

$$\|u\|_p^p|k(u)|\to 0 \quad \text{as} \quad \|u\|_p\to\infty.$$

Let g be an integrable, real-valued Borel measurable function on R^p. Then for each point $x\in C(g)$,

$$\int_{R^p} h_n^{-p}k((x-u)/h_n)g(u)du\to g(x)\int_{R^p}k(u)du \quad \text{as} \quad n\to\infty$$

and

$$\sup_{n\geq 1}\int_{R^p} h_n^{-p}|k((x-u)/h_n)||g(u)|du\leqq C,$$

where C may depend on x.

LEMMA 2.2. *Let a constant $a\in(0, 1]$ be given. Suppose that a sequence of positive numbers $\{h_n\}$ converging to zero satisfies* (H2), (H3) *and* (H4). *Let $\{Z_n\}$ be a sequence of independent random variables with $EZ_n=0$. Assume that*

$$h_n^pEZ_n^2\to\xi \quad \text{as } n\to\infty \text{ for some constant } \xi>0$$

and

$$h_n^{2p}E|Z_n|^3\leqq C \quad \text{for all} \quad n\geqq 1.$$

Then,

$$(nh_n^p)^{1/2}\sum_{j=1}^{n}a_j\beta_{jn}Z_j \xrightarrow[L]{} N(0,\ B) \quad \text{as} \quad n\to\infty \text{ (in law)},$$

where $B=a^2\beta\xi$ (>0), and a_n and β_{mn} are as defined in (1.3) *and* (1.4), *respectively.*

PROOF. It was shown in Lemma 2.2 of [6] that

$$\sum_{j=1}^{n}(j^2\gamma_j^2h_j^p)^{-1}\sim\beta(nh_n^p\gamma_n^2)^{-1} \quad \text{as} \quad n\to\infty,$$

which, together with (2.1), (2.2), (H2), the assumption of EZ_n^2 and the Toeplitz lemma (see Loève [7], page 238), implies that

$$nh_n^p\sum_{j=1}^{n}j^{-2}\beta_{jn}^2EZ_j^2\to\beta\xi \quad \text{as} \quad n\to\infty. \tag{2.4}$$

Set

$$U_n=a_n\gamma_n^{-1}Z_n, \quad S_n=\sum_{j=1}^{n}U_j \quad \text{and} \quad s_n^2=\mathrm{Var}(S_n)=a^2\sum_{j=1}^{n}j^{-2}\gamma_j^{-2}EZ_j^2.$$

From (2.4)

$$s_n^2\sim B(nh_n^p\gamma_n^2)^{-1} \quad \text{as} \quad n\to\infty. \tag{2.5}$$

By the assumption of $E|Z_n|^3$ and (2.2) we get that

$$E|U_n|^3\leqq C_1n^{3(a-1)}h_n^{-2p} \quad \text{for all} \quad n\geqq 1,$$

which, together with (2.2), (2.5) and (H4), yields that

$$s_n^{-3}\sum_{j=1}^{n}E|U_j|^3\to 0 \quad \text{as} \quad n\to\infty.$$

Thus by the Liapounov theorem we have

$$s_n^{-1}S_n \xrightarrow[L]{} N(0, 1) \quad \text{as} \quad n\to\infty. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$(nh_n^p)^{1/2}\sum_{j=1}^{n}a_j\beta_{jn}Z_j=(nh_n^p\gamma_n^2s_n^2)^{1/2}s_n^{-1}S_n \xrightarrow[L]{} N(0, B) \quad \text{as} \quad n\to\infty.$$

This completes the proof.

We shall give a definition of the smoothness of a function g.

Definition. Let g be a real-valued function on R^p. We say that the function g belongs to the class $\mathcal{M}_p$ (abbreviated as $g\in\mathcal{M}_p$) if there exist bounded, continuous second partial derivatives $\partial^2 g(x)/\partial x_i\partial x_j$ on R^p for all $i, j=1, \cdots, p$.

Lemma 2.3. *Assume $g\in\mathcal{M}_p$. Suppose that k is a real-valued Borel measurable function on R^p satisfying*

$$\int_{R^p}u_ik(u)du=0 \quad \textit{for} \quad i=1, \cdots, p \quad \textit{with} \quad u=(u_1, \cdots, u_p)$$

and

$$\int_{R^p}\|u\|_p^2|k(u)|du<\infty.$$

Then there exists a positive constant C not depending on h such that

$$\sup_{x\in R^p}\left|\int_{R^p}k(u)\{g(x-hu)-g(x)\}du\right|\leqq Ch^2 \quad \textit{for all} \quad h>0.$$

The proof of this lemma is omitted because it is easily shown by the Taylor theorem.

The following proposition shows the asymptotic normality of $(nh_n^p)^{1/2}(f_n(x)-f(x))$.

Proposition 2.4. *Let $\{h_n\}$ satisfy* (H2)$\sim$(H5). *Assume $f\in\mathcal{M}_p$. Then for each point x with $f(x)>0$,*

$$(nh_n^p)^{1/2}(f_n(x)-f(x)) \longrightarrow N(0, \sigma^2(x)) \quad \textit{as} \quad n\to\infty,$$

where

$$\sigma^2(x)=a^2\beta f(x)\int_{R^p}K^2(u)du.$$

Proof. Let any x with $f(x)>0$ be fixed. Set

$$Z_n=K_n(x, X_n)-EK_n(x, X_n) \quad \text{and} \quad \delta_n=EK_n(x, X_n)-f(x).$$

Then, replacing $N(t)$ in (1.1) by n we get

$$(nh_n^p)^{1/2}(f_n(x)-f(x))$$
$$=(nh_n^p)^{1/2}\beta_{0n}(K(x)-f(x))+(nh_n^p)^{1/2}\sum_{j=1}^{n}a_j\beta_{jn}Z_j+(nh_n^p)^{1/2}\sum_{j=1}^{n}a_j\beta_{jn}\delta_j. \tag{2.7}$$

From (2.2) and (H2) the first term in the right hand side of (2.7) converges to zero as n tends to infinity. In view of Lemma 2.3, (2.1), (2.2) and (H5) the last term in the right hand side of (2.7) converges to zero as n tends to infinity. Thus the proposition will be proved if we show that

$$(nh_n^p)^{1/2}\sum_{j=1}^{n}a_j\beta_{jn}Z_j \longrightarrow N(0,\ \sigma^2(x)) \qquad \text{as} \quad n\to\infty. \tag{2.8}$$

From Lemma 2.1 and (1.2)

$$h_n^p EZ_n^2 \to f(x)\int_{R^p}K^2(u)du\ (>0) \qquad \text{as} \quad n\to\infty.$$

By the Hölder inequality and Lemma 2.1 we have that $h_n^{2p}E|Z_n|^3 \leqq C_1$ for all $n\geqq 1$. Since all the conditions of Lemma 2.2 are satisfied, the relation (2.8) holds. This completes the proof.

The next lemma was provided by Rényi [10].

LEMMA 2.5. *Let $\{Y_n\}$ be a sequence of independent random variables defined on a probability space $(\Omega,\ \mathcal{A},\ P)$ such that putting*

$$S_n=\frac{1}{B_n}\sum_{j=1}^{n}Y_j \qquad \textit{where} \quad B_n\to\infty$$

the random variable S_n converges in law to a random variable with the distribution function F. Then for any event $A\in\mathcal{A}$ with $P(A)>0$ the conditional probability $P\{S_n<x\,|\,A\}$ tends to $F(x)$ for every $x\in C(F)$.

3. Main Result

In this section we shall show the asymptotic normality of $(N(t)h_{N(t)}^p)^{1/2}(f_{N(t)}(x)-f(x))$. Let $[b]$ denote the largest integer not greater than b. For any fixed $x\in R^p$ set

$$U_n^{(1)}=K_n(x,\ X_n)-EK_n(x,\ X_n), \qquad U_n^{(2)}=EK_n(x,\ X_n)-f(x), \tag{3.1}$$
$$S_n=\sum_{j=1}^{n}a_j\beta_{jn}\{K_j(x,\ X_j)-f(x)\}, \qquad V_n=(nh_n^p)^{1/2}S_n \qquad \text{for} \quad n\geqq 1,$$

and $S_0=V_0=0$. It is clear from (2.7) that

$$(nh_n^p)^{1/2}(f_n(x)-f(x))=V_n+(nh_n^p)^{1/2}\beta_{0n}(K(x)-f(x)) \qquad \text{for} \quad n\geqq 1. \tag{3.2}$$

Now, we shall give the condition on $N(t)$. For any $t\in(0,\ \infty)$ let $N(t)$ be a positive integer-valued random variable defined on the probability space $(\Omega,\ \mathcal{B},\ P)$ given in Section 1.

DEFINITION. A sequence of positive integer-valued random variables $N(t)$ is said to satisfy Condition A if there exist a positive random variable θ defined on $(\Omega,\ \mathcal{B},\ P)$ having a discrete distribution and a sequence of positive numbers $\pi(t)$ with $\pi(t)\to\infty$ as $t\to\infty$ such that

$$N(t)/\pi(t) \xrightarrow[P]{} \theta \quad \text{as} \quad t\to\infty \text{ (in probability)}.$$

Here, by the positive random variable θ having a discrete distribution we mean that there exists a sequence of positive numbers l_k $(k=1, 2, \cdots)$ (k may be finite or infinite) such that

$$\sum_{k=1}^{\infty} p_k=1 \qquad \text{where} \quad p_k=P\{\theta=l_k\}>0. \tag{3.3}$$

Throughout this section $\pi(t)$ and θ are as given in the above definition.

REMARK. The stopping rules $N(t)$ treated by Carroll [1] and the author [5] satisfy Condition A with $P\{\theta=1\}=1$.

LEMMA 3.1. *Let* $\{h_n\}$ *be a nonincreasing sequence of positive numbers converging to zero and satisfy* (H1) *and* (H5). *Let* $\{\delta_n\}$ *be a sequence of real numbers satisfying* $|\delta_n| \leqq C_1 h_n^2$ *for all* $n\geqq 1$. *Suppose that* $\{Z_n\}$ *is a sequence of independent random varables satisfying*

$$EZ_n=0, \quad h_n^p EZ_n^2 \leqq C_2 \quad \textit{and} \quad nh_n^p \sum_{j=1}^{n} a_j^2\beta_{jn}^2 EZ_j^2 \leqq C_2 \qquad \textit{for all} \quad n\geqq 1.$$

Set

$$W_n=\sum_{j=1}^{n} a_j\beta_{jn}Z_j+\sum_{j=1}^{n} a_j\beta_{jn}\delta_j.$$

If $N(t)$ *satisfies Condition A then*

$$(N(t)h_{N(t)}^p)^{1/2}(W_{N(t)}-W_{[\theta\pi(t)]}) \xrightarrow[P]{} 0 \qquad \textit{as} \quad t\to\infty.$$

The proof of this lemma is deferred to Appendix. We shall now state our result.

THEOREM. *Assume* $f\in\mathcal{M}_p$. *Let* $\{h_n\}$ *satisfy* (H1)$\sim$(H6). *Suppose that* $N(t)$ *satisfies Condition A. Then for each point* x *with* $f(x)>0$,

$$(N(t)h_{N(t)}^p)^{1/2}(f_{N(t)}(x)-f(x)) \xrightarrow[L]{} N(0, \sigma^2(x)) \qquad \textit{as} \quad t\to\infty,$$

where

$$\sigma^2(x)=a^2\beta f(x)\int_{R^p} K^2(u)du.$$

PROOF. For simplicity put $N=N(t)$. Let any x with $f(x)>0$ be fixed. First we shall show that

$$V_{[\theta\pi(t)]} \xrightarrow[L]{} N(0, \sigma^2(x)) \qquad \text{as} \quad t\to\infty. \tag{3.4}$$

Since by (2.2) and (H2)

$$(nh_n^p)^{1/2}\beta_{0n}\to 0 \qquad \text{as} \quad n\to\infty, \tag{3.5}$$

it follows from Proposition 2.4 and (3.2) that

$$V_n \xrightarrow[L]{} N(0, \sigma^2(x)) \qquad \text{as} \quad n\to\infty.$$

Hence by Lemma 2.5 we get that for any fixed k

$$P\{V_n<y\,|\,\theta=l_k\}\to F(y) \qquad \text{as } n\to\infty \quad \text{for each } y\in R, \tag{3.6}$$

where F denotes the distribution function of $N(0, \sigma^2(x))$. Let any $\varepsilon>0$ be fixed. From

(3.3) there exists a positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} p_k < \varepsilon. \tag{3.7}$$

Fix any $y \in R$. By (3.3) and (3.7)

$$|P\{V_{[\theta\pi(t)]} < y\} - F(y)| < \sum_{k=1}^{k_0} |P\{V_{n(k,t)} < y \mid \theta = l_k\} - F(y)| + \varepsilon \quad \text{for any } t \in (0, \infty), \tag{3.8}$$

where $n(k, t) = [l_k \pi(t)]$. Hence, in view of (3.6) and (3.8) we obtain (3.4). From (3.2) it is clear that

$$(Nh_N^p)^{1/2}(f_N(x) - f(x)) = V_N + (Nh_N^p)^{1/2}\beta_{0N}(K(x) - f(x)). \tag{3.9}$$

Since Condition A implies that $N \xrightarrow[P]{} \infty$ as $t \to \infty$, by use of (3.5)

$$(Nh_N^p)^{1/2}\beta_{0N} \xrightarrow[P]{} 0 \quad \text{as} \quad t \to \infty. \tag{3.10}$$

Thus, in view of (3.9) and (3.10), in order to prove the theorem it suffices to show that

$$V_N \xrightarrow[L]{} N(0, \sigma^2(x)) \quad \text{as} \quad t \to \infty. \tag{3.11}$$

From (3.1)

$$V_N = V_{[\theta\pi(t)]} + (Nh_N^p)^{1/2}(S_N - S_{[\theta\pi(t)]}) + V_{[\theta\pi(t)]}\{(Nh_N^p/([\theta\pi(t)]h_{[\theta\pi(t)]}^p))^{1/2} - 1\}.$$

Hence, taking account of (3.4), in order to show (3.11) it suffices to prove that

$$(Nh_N^p)^{1/2}(S_N - S_{[\theta\pi(t)]}) \xrightarrow[P]{} 0 \quad \text{as} \quad t \to \infty \tag{3.12}$$

and

$$V_{[\theta\pi(t)]}\{(Nh_N^p/([\theta\pi(t)]h_{[\theta\pi(t)]}^p))^{1/2} - 1\} \xrightarrow[P]{} 0 \quad \text{as} \quad t \to \infty. \tag{3.13}$$

First we shall show (3.13). Condition A implies that

$$N/[\theta\pi(t)] \xrightarrow[P]{} 1 \quad \text{as} \quad t \to \infty,$$

which, together with (H6), yields that

$$Nh_N^p/([\theta\pi(t)]h_{[\theta\pi(t)]}^p) \xrightarrow[P]{} 1 \quad \text{as} \quad t \to \infty. \tag{3.14}$$

Thus, by virtue of (3.4) and (3.14) we obtain (3.13). Finally, we shall show (3.12). From (3.1) we get

$$S_n = \sum_{i=1}^{2}\sum_{j=1}^{n} a_j \beta_{jn} U_j^{(i)}. \tag{3.15}$$

By Lemma 2.3

$$|U_n^{(2)}| \leqq C_1 h_n^2 \quad \text{for all} \quad n \geqq 1. \tag{3.16}$$

From Lemma 2.1

$$h_n^p E\{(U_n^{(1)})^2\} \leqq h_n^p E K_n^2(x, X_n) \leqq C_2 \quad \text{for all} \quad n \geqq 1. \tag{3.17}$$

(H3) implies that

$$n^{1-2a} h_n^p \sum_{j=1}^{n} j^{2(a-1)} h_j^{-p} \leqq C_3 \quad \text{for all} \quad n \geqq 1,$$

which, together with (2.1), (2.2) and (3.17), yields that

$$nh_n^p \sum_{j=1}^{n} a_j^2 \beta_{jn}^2 E\{(U_j^{(1)})^2\} \leqq C_4 \qquad \text{for all} \quad n \geqq 1. \tag{3.18}$$

Thus, combining Lemma 3.1 and (3.15) to (3.18) we obtain (3.12). This completes the proof.

Appendix

PROOF OF LEMMA 3.1. Let any positive numbers ε and ξ be fixed. From (3.3) there exists a positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} p_k < \xi/4. \tag{A.1}$$

Fix a positive constant C_1, which will be chosen later. Choose $\rho(0<\rho<1/2)$ such that

$$C_1 \varepsilon^{-2}\{1-((1-\rho)/(1+\rho))^a\}^2 < \xi/(8k_0) \tag{A.2}$$

and

$$C_1 \varepsilon^{-2} \rho < \xi/(8k_0). \tag{A.3}$$

For each $t \in (0, \infty)$ let $n(t)$ be a nonnegative integer with $n(t) \to \infty$ as $t \to \infty$. Set

$$M_1 = [(1-\rho)n(t)] \quad \text{and} \quad M_2 = [(1+\rho)n(t)]. \tag{A.4}$$

By virtue of (2.3) it is easy to show that for all $t \geqq$ some t_0

$$1 \leqq M_1, \ 1 \leqq M_2 - M_1 < 2\rho M_2, \ M_2/M_1 < 3 \tag{A.5}$$

and

$$(1-\beta_{M_1 M_2})^2 < 2\{1-((1-\rho)/(1+\rho))^a\}^2. \tag{A.6}$$

By the assumption of δ_n and (A.5)

$$\begin{aligned}(M_2 h_{M_2}^p)^{1/2} \max_{M_1 \leq i \leq M_2} \left| \sum_{j=1}^{i} a_j \beta_{ji} \delta_j \right| &\leqq C_2 (M_2/M_1)^a M_2^{1/2-a} h_{M_2}^{p/2} \sum_{j=1}^{M_2} j^{a-1} h_j^2 \\ &\leqq C_3 M_2^{1/2-a} h_{M_2}^{p/2} \sum_{j=1}^{M_2} j^{a-1} h_j^2 \qquad \text{for} \quad t \geqq t_0. \end{aligned} \tag{A.7}$$

From (H5) there exists a positive integer n_0 such that

$$n^{1/2-a} h_n^{p/2} \sum_{j=1}^{n} j^{a-1} h_j^2 < \varepsilon/(8C_3) \qquad \text{for all} \quad n \geqq n_0. \tag{A.8}$$

As $M_2 \geqq n_0$ for all $t \geqq$ some $t_1 (\geqq t_0)$, (A.7) and (A.8) yield that

$$(M_2 h_{M_2}^p)^{1/2} \max_{M_1 \leq i \leq M_2} \left| \sum_{j=1}^{i} a_j \beta_{ji} \delta_j \right| < \varepsilon/8 \qquad \text{for all} \quad t \geqq t_1. \tag{A.9}$$

Set $n(k, t) = [l_k \pi(t)]$ for $k=1, 2, \cdots$. For simplicity put $N=N(t)$. It is clear that

$$P\{(Nh_N^p)^{1/2} |W_N - W_{[\theta\pi(t)]}| \geqq \varepsilon\} \leqq I_1(t) + I_2(t), \tag{A.10}$$

where

$$I_1(t) = \sum_{k=1}^{\infty} P\{(Nh_N^p)^{1/2} |W_N - W_{n(k,t)}| \geqq \varepsilon, \ |N - n(k, t)| < \rho n(k, t), \ \theta = l_k\}$$

and

$$I_2(t) = P\{|N - [\theta\pi(t)]| \geqq \rho[\theta\pi(t)]\}.$$

Condition A implies that

$$I_2(t)<\xi/2 \qquad \text{for all} \quad t \geqq \text{some } t_2. \tag{A.11}$$

From (A.1)

$$I_1(t)<\sum_{k=1}^{k_0} P\{(Nh_N^p)^{1/2}|W_N-W_{n(k,t)}|\geqq\varepsilon,\ |N-n(k,t)|<\rho n(k,t),\ \theta=l_k\}+\xi/4. \tag{A.12}$$

Fix k with $1\leqq k\leqq k_0$ and put $n(t)=n(k, t)$. Let M_i ($i=1, 2$) be as defined in (A.4) for this $n(t)$. Fix $t\geqq t_3(k)\equiv\max\{t_1(k), t_2\}$. Then, taking $M_1<n(t)\leqq M_2$ into consideration we get that

$$\begin{aligned}
J(t)&\equiv P\{(Nh_N^p)^{1/2}|W_N-W_{n(t)}|\geqq\varepsilon,\ |N-n(t)|<\rho n(t),\ \theta=l_k\}\\
&\leqq P\{(ih_i^p)^{1/2}|W_i-W_{n(t)}|\geqq\varepsilon \text{ for some } i \text{ with } M_1<i\leqq M_2\}\\
&\leqq P\{(\max_{M_1<i\leqq M_2}(ih_i^p)^{1/2})(\max_{M_1<i\leqq M_2}|W_i-W_{M_1}|)\geqq\varepsilon/2\}.
\end{aligned} \tag{A.13}$$

By use of (2.1) and the monotonicity of γ_n we have that for i with $M_1<i\leqq M_2$

$$\begin{aligned}
|W_i-W_{M_1}|&\leqq\left|\sum_{j=1}^{M_1}a_j(\beta_{ji}-\beta_{jM_1})Z_j\right|+\left|\sum_{j=M_1+1}^{i}a_j\beta_{ji}Z_j\right|+\left|\sum_{j=1}^{i}a_j\beta_{ji}\delta_j\right|+\left|\sum_{j=1}^{M_1}a_j\beta_{jM_1}\delta_j\right|\\
&\leqq(\gamma_{M_1}-\gamma_i)\left|\sum_{j=1}^{M_1}a_j\gamma_j^{-1}Z_j\right|+\gamma_i\left|\sum_{j=M_1+1}^{i}a_j\gamma_j^{-1}Z_j\right|+2\max_{M_1\leqq i\leqq M_2}\left|\sum_{j=1}^{i}a_j\beta_{ji}\delta_j\right|.
\end{aligned} \tag{A.14}$$

Hence from (A.9), (A.13), (A.14) and the monotonicity of nh_n^p, h_n^p and γ_n

$$\begin{aligned}
J(t)&\leqq P\left\{(M_2h_{M_1}^p)^{1/2}(\gamma_{M_1}-\gamma_{M_2})\left|\sum_{j=1}^{M_1}a_j\gamma_j^{-1}Z_j\right|+(M_2h_{M_2}^p)^{1/2}\max_{M_1<i\leqq M_2}\gamma_i\left|\sum_{j=M_1+1}^{i}a_j\gamma_j^{-1}Z_j\right|>\varepsilon/4\right\}\\
&\leqq J_1(t)+J_2(t),
\end{aligned} \tag{A.15}$$

where

$$J_1(t)=P\left\{(M_2h_{M_1}^p)^{1/2}(\gamma_{M_1}-\gamma_{M_2})\left|\sum_{j=1}^{M_1}a_j\gamma_j^{-1}Z_j\right|>\varepsilon/8\right\}$$

and

$$J_2(t)=P\left\{(M_2h_{M_2}^p)^{1/2}\max_{M_1<i\leqq M_2}\gamma_i\left|\sum_{j=M_1+1}^{i}a_j\gamma_j^{-1}Z_j\right|>\varepsilon/8\right\}.$$

First we shall estimate $J_1(t)$. By the Chebychev inequality, (A.5), (A.6) and the assumption of EZ_n^2 we get

$$\begin{aligned}
J_1(t)&\leqq C_4\varepsilon^{-2}(1-\beta_{M_1M_2})^2(M_2/M_1)M_1h_{M_1}^p\sum_{j=1}^{M_1}a_j^2\beta_{jM_1}^2EZ_j^2\\
&\leqq C_5\varepsilon^{-2}\{1-((1-\rho)/(1+\rho))^a\}^2.
\end{aligned} \tag{A.16}$$

Next we shall estimate $J_2(t)$. From the Hájek-Rényi inequality (see Petrov [8], page 51), (A.5) and the monotonicity of h_n we have

$$\begin{aligned}
J_2(t)&\leqq C_6\varepsilon^{-2}M_2h_{M_2}^p\sum_{j=M_1+1}^{M_2}a_j^2EZ_j^2\leqq C_7\varepsilon^{-2}M_2h_{M_2}^p\sum_{j=M_1+1}^{M_2}j^{-2}h_j^{-p}\\
&\leqq C_7\varepsilon^{-2}M_2M_1^{-2}(M_2-M_1)\leqq C_8\varepsilon^{-2}\rho.
\end{aligned} \tag{A.17}$$

Set $C_1=\max\{C_5, C_8\}$. Then by (A.2), (A.3), (A.16) and (A.17) we have

$$J_1(t)<\xi/(8k_0) \quad \text{and} \quad J_2(t)<\xi/(8k_0) \qquad \text{for all} \quad t\geqq t_3(k),$$

which, together with (A.15), implies that for $k(1\leqq k\leqq k_0)$

$$P\{(Nh_N^p)^{1/2}|W_N - W_{n(k,t)}| \geqq \varepsilon,\ |N-n(k,t)| < \rho n(k,t),\ \theta = l_k\} < \xi/(4k_0) \quad \text{for all} \quad t \geqq t_3(k). \tag{A.18}$$

From (A.12) and (A.18)

$$I_1(t) < \xi/2 \qquad \text{for large } t,$$

which, together with (A.10) and (A.11), yields that

$$P\{(Nh_N^p)^{1/2}|W_N - W_{[\theta\pi(t)]}| \geqq \varepsilon\} < \xi \qquad \text{for large } t.$$

Thus the proof of Lemma 3.1 was completed.

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