ON SAMPLES FROM DISTRIBUTIONS CHOSEN FROM A DIRICHLET INVARIANT PROCESS

Yamato, Hajime
Department of Mathematics, Faculty of Science, Kagoshima University

https://doi.org/10.5109/13387

出版情報：Bulletin of informatics and cybernetics. 22 (3/4), pp.199-207, 1987-03. Research Association of Statistical Sciences
バージョン：
権利関係：

Bulletin of Informatics and Cybernetics Vol. 22, No. 3~4, 1987

ON SAMPLES FROM DISTRIBUTIONS CHOSEN FROM A DIRICHLET INVARIANT PROCESS

By

Hajime YAMATO*

Abstract

For a sample from a distribution chosen from a Dirichlet G-invariant process with non-atomic parameter, we present their joint distribution, and conditional distributions given the matching among the observations and given the number of the distinct observations. We evaluate the expectation of function of the sample by using the conditional distribution.

1. Introduction

The Dirichlet process was introduced by Ferguson [1] for Bayesian nonparametric inference. Ferguson [1] and Yamato [8], [9] used the Dirichlet process to estimate estimable parameter. It is well-known that a distribution chosen from a Dirichlet process is discrete with probability one. Therefore there is a possibility of duplication in the sample from a distribution chosen from a Dirichlet process. For a sample from a distribution chosen from a Dirichlet process, Yamato [10] obtained their joint distribution, and conditional distributions given the matching among the observations and given the number of the distinct observations.

Dalal [2] introduced the Dirichlet G-invariant process, which permits us to include additional beliefs in a prior distribution. When a symmetry about zero of the underlying distribution is known, a distribution chosen from the Dirichlet invariant process is symmetric about zero with probability one by taking parameter symmetric about zero. When a symmetry about the line $y=x$ on $\boldsymbol{R}^2$ is known for the underlying distribution on the 2-dimensional Euclidean space $\boldsymbol{R}^2$, a distribution chosen from the Dirichlet invariant process is symmetric about the line $y=x$ with probability one by taking parameter symmetric about the line $y=x$ on $\boldsymbol{R}^2$. At this point, the Dirichlet invariant process differs from the Dirichlet process. Dalal [2], [3], [4] used the Dirichlet invariant process to estimate the symmetric distribution and the unknown center of symmetry. Yamato [11] utilized the Dirichlet invariant process to estimate estimable parameter of degree 2 for the underlying distribution which is invariant under a group of tranformations.

Similarly to the Dirichlet process, a distribution chosen from a Dirichlet invariant process is discrete with probability one. Therefore there is a possibility of duplication in the sample from a distribution chosen from a Dirichlet invariant process. So we

* Department of Mathematics, Faculty of Science, Kagoshima University, Kagoshima 890, Japan.

shall investigate properties of the sample from a distribution chosen from a Dirichlet invariant process.

The purpose of this paper is to obtain, for a sample from a distribution chosen from a Dirichlet G-invariant process, their joint distribution and conditional distributions given the matching among the observations and given the number of the distinct observations. Furthermore we shall evaluate the expectation of function of the sample.

For the Dirichlet process with non-atomic parameter, Yamato [10] obtained the simple evaluation of the joint distribution of a sample by viewing the sample sequentially. Similarly, we shall evaluate the joint distribution of a sample from a distribution chosen from a Dirichlet G-invariant process with non-atomic parameter, by viewing the sample sequentially. Another evaluations of the joint distribution are given by Corollary 2.10 of Hannum [5] and Proposition 3.4.1 of Tiwari [7]. But, our evaluation is simple. Furthermore it yields that the distinct observations are independent and identically distributed given the matching among the observations. Also the distinct observations are independent and identically distributed given the number of distinct observations. We present these properties in Section 3.

In Section 4, we evaluate the expectation of function of the sample from a distribution chosen from a Dirichlet invariant process.

For preparation, we shall state the definition of the Dirichlet G-invariant process and quote its properties in Section 2.

2. Dirichlet G-invariant processes

We consider the Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$, where $\boldsymbol{X}$ is the d-dimensional Euclidean space and $\boldsymbol{A}$ is the σ-field of Borel sets of $\boldsymbol{X}$.

Let $G=\{g_1, g_2, \cdots, g_k\}$ be any finite group of measurable transformations $\boldsymbol{X}\to\boldsymbol{X}$. A measure μ is said to be invariant under the group of transformations G if $\mu(B)=\mu(g_iB)$ for any $B\in\boldsymbol{A}$ and $i=1, 2, \cdots, k$. A measurable partition $B_1, \cdots, B_m$ of $\boldsymbol{X}$ is said to be G-invariant if $B_j=g_iB_j$ for $j=1, \cdots, m$ and $i=1, \cdots, k$.

DEFINITION 1 (Dalal [2]). Let α be a finite, non-negative measure on $(\boldsymbol{X}, \boldsymbol{A})$ and be invariant under G. $\boldsymbol{P}$ is a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with parameter α, if $\boldsymbol{P}$ is invariant under G a.s. and if for any G-invariant measurable partition $(B_1, \cdots, B_m)$ of $\boldsymbol{X}$, the joint distribution of $(\boldsymbol{P}(B_1), \cdots, \boldsymbol{P}(B_m))$ is Dirichlet with parameter $(\alpha(B_1), \cdots, \alpha(B_m))$. We then write $\boldsymbol{P}\in\boldsymbol{DG}(\alpha)$.

DEFINITION 2 (Dalal [2]). Let $\boldsymbol{P}\in\boldsymbol{DG}(\alpha)$. We say $X_1, \cdots, X_n$ is a sample of size n from $\boldsymbol{P}$ if for any $m=1, 2, \cdots$ and measurable sets $A_1, \cdots, A_m, C_1, \cdots, C_n$,

$$P\{X_1\in C_1, \cdots, X_n\in C_n \mid \boldsymbol{P}(A_1), \cdots, \boldsymbol{P}(A_m), \boldsymbol{P}(C_1), \cdots, \boldsymbol{P}(C_n)\}=\textstyle\prod_{i=1}^n \boldsymbol{P}(C_i) \quad \text{a.s.}$$

PROPOSITION 3 (Hannum [5]). *Let $\boldsymbol{P}\in\boldsymbol{DG}(\alpha)$ and let $X_1, \cdots, X_n$ be a sample from $\boldsymbol{P}$. The joint distribution of $(X_1, \cdots, X_n)$ is exchangeable in the coordinate.*

PROPOSITION 4 (Dalal [2]). *Let $\boldsymbol{P}\in\boldsymbol{DG}(\alpha)$ and X be a sample of size 1 from $\boldsymbol{P}$. Then for any g in $\boldsymbol{G}$, X and gX have the same distribution Q, where $Q(\cdot)=\alpha(\cdot)/M$ and $M=\alpha(\boldsymbol{X})$.*

PROPOSITION 5 (Dalal [2]). *Let $\boldsymbol{P}\in\boldsymbol{DG}(\alpha)$, and $X_1, \cdots, X_n$ be a sample of size n*

from $\boldsymbol{P}$. *Then the conditional distribution of* $\boldsymbol{P}$ *given* $X_1, \cdots, X_n$ *is a Dirichlet G-invariant process with parameter* $\alpha+\sum_{i=1}^{n}\delta_{X_i}^{g}$, *where* $\delta_{X_i}^{g}=\sum_{j=1}^{k}\delta_{g_jX_i}/k$ $(i=, \cdots, n)$ *and* δ_x *is a measure degenerate at* x.

Let $X_1, \cdots, X_n$ be a sample of size n from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$. Then, from Proposition 5, the sample $X_1, X_2, \cdots, X_n$ can be regarded to be obtained sequentially such that at first a sample of size 1, X_1, is obtained from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$, secondly X_2 is obtained from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha+\delta_{X_1}^{g})$, thirdly X_3 is obtained from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha+\delta_{X_1}^{g}+\delta_{X_2}^{g})$, $\cdots$, and at last X_n is obtained from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha+\delta_{X_1}^{g}+\cdots+\delta_{X_{n-1}}^{g})$. Thus from Proposition 4 we have the following

LEMMA 6. *Let* $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$ *and* $X_1, \cdots, X_n$ *be a sample of size n from* $\boldsymbol{P}$. *Then we can view as follows*: X_1 *has the distribution Q and for* $j=1, \cdots, n-1$, *the conditional distribution of* X_{j+1} *given* $X_1, \cdots, X_j$ *is the distribution* $(MQ(\cdot)+\sum_{i=1}^{j}\delta_{X_i}^{g}(\cdot))/(M+j)$.

3. Properties of Samples

Let $G=\{g_1, \cdots, g_k\}$ be any finite group of measurable transformations $\boldsymbol{X}\to\boldsymbol{X}$. For any $x\in\boldsymbol{X}$, $G(x)$ denotes the set $\{g_1x, \cdots, g_kx\}$. Let $X_1, \cdots, X_n$ be a sample of size n from a distribution $\boldsymbol{P}$ chosen from a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with parameter α, where α is a finite, non-negative measure on $(\boldsymbol{X}, \boldsymbol{A})$ and invariant under G. We assume that α is non-atomic. We denote $\alpha(\boldsymbol{X})$ by M and $\alpha(\cdot)/M$ by $Q(\cdot)$. We can consider that the sample $X_1, \cdots, X_n$ is obtained sequentially, as stated in Section 2. We regard that two observations X_i, X_j are equivalent if there exists a $g\in G$ such that $X_i=gX_j$ and denote it by $X_i\equiv X_j$. Otherwise, we regard the two observations are distinct.

For n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{i=1}^{n}im(i)=n$, let $(X_1, \cdots, X_n)\in C_g(m(1), \cdots, m(n))$ be the event that there are $m(1)$ distinct values of X that occur only once, $m(2)$ distinct values that occur exactly twice, $\cdots$, $m(n)$ distinct values that occur exactly n times, where "distict" is taken as stated in the above. Note that $m(n)$ can take only 0 or 1 and if $m(n)=1$ then $m(1)=\cdots=m(n-1)=0$.

For the sample $X_1, \cdots, X_n$ with $(X_1, \cdots, X_n)\in C_g(m(1), \cdots, m(n))$, we denote the distinct observations by $X_{11}, \cdots, X_{1m(1)}, X_{21}, X_{22}, \cdots, X_{2m(2)}, \cdots, X_{n1}$, where for $i=1, 2, \cdots, n$ and $j=1, \cdots, m(i)$ X_{ij}, including its G-transformation, appears exactly i-times in the sample. If $m(j)=m$ $(1<j<n)$ and $X_{s(1)}\equiv\cdots\equiv X_{t(1)}$, $X_{s(2)}\equiv\cdots\equiv X_{t(2)}, \cdots$ $X_{s(m)}\equiv\cdots\equiv X_{t(m)}$ (the number of each equal X's is j) with $s(1)<s(2)<\cdots<s(m)$ and $s(i)=\min(s(i), \cdots, t(i))$ $(i=1, \cdots, m)$, then $X_{j1}=X_{s(1)}$, $X_{j2}=X_{s(2)}, \cdots, X_{jm(j)}=X_{s(m)}$.

For n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{1}^{n}im(i)=n$, let $(X_n, X_{n-1}, \cdots, X_1)\in C_{g,0}(m(1), \cdots, m(n))$ be the event that $X_n, \cdots, X_{n-m(1)+1}$ occur only once in the sample and their G-transformations do not occutr; that $X_{n-m(1)}, X_{n-m(1)-1}, \cdots, X_{n-(m(1)+2m(2)-1)}$ occur twice each in the order $X_{n-m(1)}\equiv X_{n-m(1)-1}, \cdots, X_{n-(m(1)+2m(2)-2)}\equiv X_{n-(m(1)+2m(2)-1)}$ and so on. Note that we regard two observations X_i, X_j are equivalent if there exists a $g\in G$ such that $X_i=gX_j$. For $(X_n, X_{n-1}, \cdots, X_1)\in C_{g,0}(m(1), \cdots, m(n))$, we denote $X_n, X_{n-1}, \cdots, X_{n-m(1)+1}, X_{n-m(1)}, X_{n-m(1)-2}, \cdots, X_{n-(m(1)+2m(2)-2)}, X_{n-(m(1)+2m(2))}, \cdots$ by $Y_{11}, Y_{12}, \cdots, Y_{1m(1)}, Y_{21}, Y_{22}, \cdots, Y_{2m(2)}, Y_{31}, \cdots$.

For example, we consider the case of $n=2$. Two non-negative integers $(m(1), m(2))$

with $m(1)+2m(2)=2$ are $(2,0)$ and $(0,1)$.

For $(X_2, X_1)\in C_{g,0}(2,0)$ we have $Y_{11}=X_2$, $Y_{12}=X_1$. Then from Lemma 6 we have, for $A_1, A_2\in \boldsymbol{A}$,

$$\begin{aligned} p_1&=P(Y_{11}\in A_2,\ Y_{12}\in A_1,\ (X_2, X_1)\in C_{g,0}(2,0)) \\ &=P(X_2\in A_2,\ X_1\in A_1,\ X_2\notin G(X_1)) \\ &=\int_A P(X_2\in A_2-G(x_1)\mid x_1)dQ(x_1) \end{aligned}$$

Since from Lemma 6, given $X_1=x_1$, X_2 has the distribution $(MQ(\cdot)+\delta^g_{x_1}(\cdot))/(M+1)$ and α is non-atomic,

$$\begin{aligned} p_1&=\int_A [MQ(A_2)/(M+1)]dQ(x_1) \\ &=MQ(A_1)Q(A_2)/(M+1) \\ &=M^{m(1)}Q(A_1)Q(A_2)/M^{(n)} \quad \text{with } n=2,\ m(1)=2 \text{ and } m(2)=0. \end{aligned}$$

For $(X_2, X_1)\in C_{g,0}(0,1)$, we have $Y_{21}=X_2$ and $X_1\equiv X_2$. Then from Proposition 3 and Lemma 6 we have, for any $A\in\boldsymbol{A}$,

$$\begin{aligned} p_2&=P(X_2\in A,\ X_1\in G(X_2)) \\ &=P(X_1\in A,\ X_2\in G(X_1)) \\ &=\int_A P(X_2\in G(x_1)\mid x_1)dQ(x_1) \\ &=\int_A [(MQ+\delta^g_{x_1})(G(x_1))/(M+1)]dQ(x_1) \\ &=\int_A [1/(M+1)]dQ(x_1) \\ &=Q(A)/(M+1) \\ &=M^{m(2)}Q(A)/M^{(n)} \quad \text{with } n=2,\ m(1)=0 \text{ and } m(2)=1. \end{aligned}$$

In general we have the following

LEMMA 7. *Let $X_1, \cdots, X_n$ be a sample of size n from a distribution chosen from a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with non-atomic parameter α. Then for n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{i=1}^n im(i)=n$ and any $A_{ij}\in\boldsymbol{A}$ $(i=1, \cdots, n,\ j=1, \cdots, m(i))$, we have*

$$P\{Y_{ij}\in A_{ij}\ (i=1, \cdots n,\ j=1, \cdots, m(i)),\ (X_n, \cdots, X_1)\in C_{g,0}(m(1), \cdots, m(n))\}$$
$$=\prod_{i=1}^n((i-1)!\,M)^{m(i)}\prod_{i=1}^n\prod_{j=1}^{m(i)}Q(A_{ij})/M^{(n)}.$$

PROOF. We saw above that Lemma 7 holds for $n=2$. Then we assume that Lemma 7 holds for $n(\geqq 2)$ and show that it holds for $n+1$. The method of proof is similar to the proof of Lemma 4 of Yamato [10], except that the matching of the observations is considered under the group of transformations G.

We denote the sample $X_{n+1}, X_n, \cdots, X_1$ with $(X_{n+1}, X_n, \cdots, X_1)\in C_{g,0}(m'(1), \cdots, m'(n+1))$ and $\sum_{i=1}^{n+1}im'(i)=n+1$ by $Y'_{11}, \cdots, Y'_{1m'(1)}, Y'_{21}, Y'_{21}, \cdots, Y'_{2m'(2)}, Y'_{2m'(2)}, \cdots$.

For the sample of size $n+1$ we have two cases: The one is that X_{n+1} and its G-transformation do not occur in the observations $X_n, \cdots, X_1$ and the other is that X_{n+1} and/or its G-transformation occur in the observations $X_n, \cdots, X_1$.

For the first case, we have $m'(1)\geqq 1$, $m'(n+1)=0$ and for $A_{ij}\in \boldsymbol{A}$ $(i=1, \cdots, n, j=1, \cdots, m(i))$

$$q=P\{Y'_{ij}\in A_{ij}\ (i=1, \cdots, n,\ j=1, \cdots, m'(i)),\ (X_{n+1}, \cdots, X_1)\in C_{g,0}(m'(1), \cdots, m'(n+1))\}$$
$$=\int_D P\{X_{n+1}\in A_{11}-\cup_{j=1}^{n} G(x_j)\,|\,x_1, \cdots, x_n\}\,dH(x_1, \cdots, x_n),$$

where $H(x_1, \cdots, x_n)$ is the joint distribution of $X_1, \cdots, X_n$ and

$$\begin{aligned} D=\{(x_1, \cdots, x_n)\,|\,&(x_n, x_{n-1}, \cdots, x_1)\in C_{g,0}(m(1), \cdots, m(n)),\\ &m(1)=m'(1)-1,\ m(i)=m'(i)\ (i=2, \cdots, n),\\ &y_{1,j-1}\in A_{1j}\ (j=2, \cdots, m'(1)),\\ &y_{ij}\in A_{ij}\ (i=2, \cdots, n,\ j=1, \cdots, m'(i))\}. \end{aligned}$$

Now $(x_n, x_{n-1}, \cdots, x_1)\in C_{g,0}(m(1), \cdots, m(n))$ denotes that each $x_n, \cdots, x_{n-m(1)+1}$ appear only once in $x_n, \cdots, x_1$ and their G-transformations do not appear; $x_{n-m(1)}, x_{n-m(1)-1}, \cdots, x_{n-m(1)-2m(2)+1}$ appear exactly twice in the order $x_{n-m(1)}\equiv x_{n-m(1)-1}, \cdots, x_{n-m(1)-2m(2)+2}\equiv x_{n-m(1)-2m(2)+1}$ and so on, where similarly to the sample we regard that two coordinates x_i, x_j are equivalant if there exists a $g\in G$ such that $x_i=g(x_j)$ and otherwise thay are distinct. For $(x_n, \cdots, x_1)\in C_{g,0}(m(1), \cdots, m(n))$, we denote $x_n, x_{n-1}, \cdots, x_{n-m(1)+1}, x_{n-m(1)}, x_{n-m(1)-2}, \cdots, x_{n-m(1)-2m(2)+2}, x_{n-m(1)-2m(2)}, \cdots$ by $y_{11}, y_{12}, \cdots, y_{1m(1)}, y_{21}, y_{22}, \cdots, y_{2m(2)}, y_{31}, \cdots$.

Since from Lemma 6, given $X_1, \cdots, X_n$, X_{n+1} has the distribution $(MQ+\sum_{i=1}^{n}\delta_{X_i}^{g})/(M+n)$ and α is non-atomic,

$$\begin{aligned} q&=\int_D [MQ(A_{11})/(M+n)]\,dH(x_1, \cdots, x_n)\\ &=MQ(A_{11})P\{(X_1, \cdots, X_n)\in D\}/(M+n)\\ &=[MQ(A_{11})/(M+n)]P\{Y_{1,j-1}\in A_{1j}\ (j=2, \cdots, m'(1)),\\ &\qquad Y_{ij}\in A_{ij}\ (i=2, \cdots, n,\ j=1, \cdots, m'(i)),\\ &\qquad (X_n, \cdots, X_1)\in C_{g,0}(m'(1)-1, m'(2), \cdots, m'(n))\} \end{aligned}$$

Since we assume that the lemma holds for n and $m'(n+1)=0$,

$$\begin{aligned} q&=[MQ(A_{11})/(M+n)]M^{m'(1)-1}\textstyle\prod_{i=2}^{n}((i-1)!\,M)^{m'(n)}\prod_{j=2}^{m'(1)}Q(A_{1j})\prod_{i=2}^{n}\prod_{j=1}^{m'(i)}Q(A_{ij})M^{(n)}\\ &=\textstyle\prod_{i=1}^{n+1}((i-1)!\,M)^{m'(i)}\prod_{i=1}^{n+1}\prod_{j=1}^{m'(i)}Q(A_{ij})/(M+n)^{(n+1)}. \end{aligned}$$

Similarly, we can prove the case that X_{n+1} and/or its G-transformation occur in the previous observations, by the method of the proof of Lemma 4 of Yamato [10] except that the matching of observations is taken under the transformation G. So we omit the proof for the case that X_{n+1} and/or its G-transformation occur in the previous observations. Thus we know that Lemma 7 holds for $n+1$ and proved it by induction.

By Proposition 3, Lemma 7 holds also for $(X_{i_1}, \cdots, X_{i_n})\in C_{g,0}(m(1), \cdots, m(n))$ with any permutation $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$. The number of ways that n observations $X_1, \cdots, X_n$ are permuted differently with $(X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))$ and $\sum_{i=1}^n im(i) = n$ is $n!/\prod_{i=1}^n [m(i)!(i!)^{m(i)}]$. To multiply the righthand side of the equation of Lemma 7 by this value yields the following.

PROPOSITION 8. *Let $X_1, \cdots, X_n$ be a sample of size n from a distribution chosen from a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with nonatomic parameter α. Then for n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{i=1}^n im(i)=n$ and any $A_{ij} \in \boldsymbol{A}$ $(i=1, \cdots, n, j=1, \cdots, m(i))$,*

$$P\{X_{ij} \in A_{ij}\ (i=1, \cdots, n,\ j=1, \cdots, m(i)),\ (X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))\}$$
$$= n!\, M^{\sum_1^n m(i)} \prod_{i=1}^n \prod_{j=1}^{m(i)} Q(A_{ij}) / M^{(n)} \prod_{i=1}^n (m(i)!\, i^{m(i)}).$$

The above evaluation is similar to the one for the Dirichlet process, which is given by Lemma 4 of Yamato [10]. Thus we have the followings similar to Lemma 5 and Theorem 1 of Yamato [10].

PROPOSITION 9. *Let $X_1, \cdots, X_n$ be a sample of size n from a distribution chosen from a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with nonatomic parameter α. Then for n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{i=1}^n im(i)=n$,*

$$P\{(X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))\} = n!\, M^{\sum_1^n m(i)} / M^{(n)} \prod_{i=1}^n (m(i)!\, i^{m(i)}).$$

THEOREM 10. *Let $X_1, \cdots, X_n$ be a sample of size n from a distribution chosen from a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with nonatomic parameter α. Let $m(1), \cdots, m(n)$ be n non-negative integers satisfying $\sum_{i=1}^n im(i)=n$. Then given $(X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))$, $X_{11}, X_{12}, \cdots, X_{1m(1)}, X_{21}, X_{22}, \cdots, X_{2m(2)}, \cdots, X_{n1}$ are independent and identically distributed with the distribution Q. That is, for any $A_{ij} \in \boldsymbol{A}$ $(i=1, \cdots, n, j=1, \cdots, m(i))$*

$$P\{X_{ij} \in A_{ij}\ (i=1, \cdots, n,\ j=1, \cdots, m(i)) \mid (X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))\}$$
$$= \prod_{i=1}^n \prod_{j=1}^{m(i)} Q(A_{ij}).$$

When $\sum_{i=1}^n m(i)=u$, i.e., the number of the distinct observations in the sample of size n is u, we denote the distinct observations by $W_1, \cdots, W_u$, where we consider two observations are distinct if the one can not be obtained from the other by the transformation G. For example, in case of $n=4$ and $u=2$, we have $W_1, W_2 = X_{11}, X_{21}$ for $(m(1), m(2), m(3), m(4)) = (1, 0, 1, 0)$ and $W_1, W_2 = X_{21}, X_{22}$ for $(m(1), m(2), m(3), m(4)) = (0, 2, 0, 0)$.

Given $(X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))$, $W_1, \cdots, W_u$ are equal to $X_{11}, X_{12}, \cdots, X_{1m(1)}, X_{21}, X_{22}, \cdots, X_{n1}$, respectively, where $u = \sum_{i=1}^n m(i)$. From Proposition 10 we have for any $A_i \in \boldsymbol{A}$ $(i=1, \cdots, u)$ with $u = \sum_{i=1}^n m(i)$,

$$P\{W_i \in A_i\ (i=1, \cdots, u) \mid (X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))\} = \prod_{i=1}^u Q(A_i).$$

The above conditional probability depends on only u and does not depend on each sequence $m(1), \cdots, m(n)$ satisfying $\sum_{i=1}^n m(i)=u$. Thus we have

$$P\{W_i \in A_i\ (i=1, \cdots, u) \mid \cup^*[(X_1, \cdots, X_n) \in C_g(m(1), \cdots, m(n))]\} = \prod_{i=1}^u Q(A_i),$$

where $\cup^*$ is the union over all n non-negative integers $m(1), \cdots, m(n)$ satisfying $\sum_{i=1}^n im(i)=n$ and $\sum_{i=1}^n m(i)=u$ with fixed n, u. The event $\cup^*[(X_1, \cdots, X_n) \in C_g(m(1), \cdots,$

$m(n))]$ is equal to that the number of distinct observations in the sample $X_1, \cdots, X_n$ is u. Therefore we have the following proposition, which is similar to case of the Dirichlet process by Korwar and Hollander [6].

THEOREM 11. *For a sample from a distribution chosen from a Dirichlet G-invariant process on $(\boldsymbol{X}, \boldsymbol{A})$ with nonatomic parameter α, given the number of the distinct observations in the sample, u, the distinct observations $W_1, \cdots, W_u$ are independent and identically distributed with the distribution Q, where two observations are considered distinct if the one can not be obtained from the other by the transformation G.*

4. Expectations of Functions of Samples

For n non-negative integers $m(1), \cdots, m(n)$ with $\sum_1^n im(i)=n$ and transformations $f_1, f_2, \cdots, f_{n-\Sigma m(i)}$ $(\in G)$, let $(X_n, X_{n-1}, \cdots, X_1)\in C_{f,0}(m(1), \cdots, m(n))$ be the event that $X_n, \cdots X_{n-m(1)+1}$ occur only once in the sample and their f-transformations do not occur; that $X_{n-m(1)}, X_{n-m(1)-1}, \cdots, X_{n-(m(1)+2m(2)-1)}$ occur twice each in the manner $X_{n-m(1)-1}=f_1X_{n-m(1)}, \cdots, X_{n-(m(1)+m(2)-1)}=f_{2m(2)-1}X_{n-(m(1)+2m(2)-2)}$ and so on. For the above event, we denote $X_n, X_{n-1}, \cdots, X_{n-m(1)+1}, X_{n-m(1)}, X_{n-m(1)-2}, \cdots, X_{n-(m(1)+2m(2)-2)}, X_{n-(m(1)+2m(2))}, \cdots$ by $U_{11}, U_{12}, \cdots, U_{1m(1)}, U_{21}, U_{22}, \cdots, U_{2m(2)}, U_{31}, \cdots$, respectively. Then similar to Lemma 7, we have the following.

LEMMA 12. *Let $X_1, \cdots, X_n$ be a sample of size n from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$, where α is nonatomic. Then for n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{i=1}^n im(i)=n$, transformations $f_1, f_2, \cdots, f_{n-\Sigma m(i)}$ $(\in G)$ and any $A_{ij}\in \boldsymbol{A}$ $(i=1, \cdots, n,\ j=1, \cdots, m(i))$, we have*

$$P\{U_{ij}\in A_{ij}\ (i=1, \cdots, n,\ j=1, \cdots, m(i)),\ (X_n, \cdots, X_1)\in C_{f,0}(m(1), \cdots, m(n))\}$$
$$=\prod_{i=1}^n((i-1)!\,M)^{m(i)}\prod_{i=1}^n\prod_{j=1}^{m(i)}Q(A_{ij})/[M^{(n)}k^{n-\Sigma m(i)}].$$

For n non-negative integers $m(1), \cdots, m(n)$ with $\sum_1^n im(i)=n$ and transformations $f_1, \cdots, f_{n-\Sigma m(i)}$ $(\in G)$, let $(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))$ be the event that there are $m(1)$ distinct values of X that occur only once and their f-transformations do not occur in the sample, $m(2)$ distinct values of X that occur only once and their each $f_1, \cdots, f_{m(2)}$ transformations occur only once, respectively, $m(3)$ distinct values that occur only once and their each $f_{m(2)+1}, f_{m(2)+2}; f_{m(2)+3}, f_{m(2)+4}; \cdots; f_{m(2)+2m(3)-1}, f_{m(2)+2m(3)}$ transformations occur only once and so on. For the above event, we denote the sample $X_1, \cdots, X_n$ by $V_{11}, \cdots, V_{1m(1)}, V_{21}, f_1V_{21}, V_{22}, f_2V_{22}, \cdots, V_{2m(2)}, f_{m(2)}V_{2m(2)}, V_{31}, f_{m(2)+1}V_{31}, f_{m(2)+2}V_{31}, V_{32}, f_{m(2)+3}V_{32}, f_{m(2)+4}V_{32}, V_{33}, \cdots$. Then the following proposition holds similarly to Proposition 8.

PROPOSITION 13. *Let $X_1, \cdots, X_n$ be a sample from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$, where α is non-atomic. Then for n non-negative integers $m(1), \cdots, m(n)$ with $\sum_1^n im(i)=n$, transformations $f_1, f_2, \cdots, f_{n-\Sigma m(i)}$ $(\in G)$ and any $A_{ij}\in \boldsymbol{A}$ $(i=1, \cdots, n,\ j=1, \cdots, m(i))$, we have*

$$P\{V_{ij}\in A_{ij}\ (i=1, \cdots, n,\ j=1, \cdots, m(i)),\ (X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))\}$$
$$=n!\,M^{\Sigma m(i)}\prod_{i=1}^n\prod_{j=1}^{m(i)}Q(A_{ij})/[M^{(n)}\prod_{i=1}^n(m(i)!\,i^{m(i)})k^{n-\Sigma m(i)}].$$

Thus we have the following.

PROPOSITION 14. *Let $X_1, \cdots, X_n$ be a sample from $\boldsymbol{P}$ with $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$, where α is nonatomic. Then for n non-negative integers $m(1), \cdots, m(n)$ with $\sum_{i=1}^n im(i)=n$ and trans-*

formations $f_1, \cdots, f_{n-\Sigma m(i)}$ $(\in G)$,

$$P\{(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))\} = n!\, M^{\Sigma m(i)}/[M^{(n)}\prod_{i=1}^n(m(i)!\, i^{m(i)})k^{n-\Sigma m(i)}].$$

Futhermore, for any $A_{ij}\in \boldsymbol{A}$ $(i=1, \cdots, n,\ j=1, \cdots, m(i))$

$$P\{V_{ij}\in A_{ij}\ (i=1, \cdots, n,\ j=1, \cdots, m(i))\,|\,(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))\} = \prod_{i=1}^n\prod_{j=1}^{m(i)}Q(A_{ij}).$$

THEOREM 15. *Let* $\boldsymbol{P}\in \boldsymbol{DG}(\alpha)$ *with nonatomic parameter* α *and let* $h(x_1, \cdots, x_n)$ *be a measurable real-valued function defined on* $(\boldsymbol{X}^n, \boldsymbol{A}^n)$ *and symmetric in* $x_1, \cdots, x_n$. $(\boldsymbol{X}^n, \boldsymbol{A}^n)$ *denotes the product of measurable space* $(\boldsymbol{X}, \boldsymbol{A})$. *Then for a samle of size n from* $\boldsymbol{P}$, $X_1, \cdots, X_n$, *we have*

$$Eh(X_1, \cdots, X_n)=\Sigma^*\Sigma_f \frac{n!\, M^{\Sigma m(i)}}{k^{n-\Sigma m(i)}M^{(n)}\prod_{i=1}^n(m(i)!\, i^{m(i)})} \times\int_{X^{\Sigma m(i)}} h(x_{11}, \cdots, x_{1m(1)}, x_{21}, f_1x_{21}, x_{22}, f_2x_{22}, \cdots, x_{n1}, \cdots, f_{n-\Sigma m(i)}x_{n1})\prod_{i=1}^n\prod_{j=1}^{m(i)}dQ(x_{ij}),$$

provided all integrals of the right-hand side exist, where Σ^* *denotes the summation over all n non-negatibe integers* $m(1), \cdots, m(n)$ *satisfying* $\Sigma_1^n im(i)=n$ *and* Σ_f *denotes the summation over all* $n-\Sigma m(i)$ *transformations* $f_1, \cdots, f_{n-\Sigma m(i)}$ *which belong to G.*

PROOF. From Proposition 14 and symmetry of h, we have

$$\begin{aligned} &E[h(X_1, \cdots, X_n)\,|\,(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))] \\ &=E[h(V_{11}, \cdots, V_{1m(1)}, V_{21}, f_1V_{21}, V_{22}, f_2V_{22}, \cdots)\,|\,(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))] \\ &=\int_{X^{\Sigma m(i)}} h(x_{11}, \cdots, x_{1m(1)}, x_{21}, f_1x_{21}, x_{22}, f_2x_{22}, \cdots)\prod_{i=1}^n\prod_{j=1}^{m(i)}dQ(x_{ii}). \end{aligned}$$

By proposition 9 and 14, we have

$$P\{(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))\,|\,(X_1, \cdots, X_n)\in C_g(m(1), \cdots, m(n))\}=1/k^{n-\Sigma m(i)},$$

for transformations $f_1, \cdots, f_{n-\Sigma m(i)}$ $(\in G)$. Therefore, we have

$$\begin{aligned} &E[h(X_1, \cdots, X_n)\,|\,(X_1, \cdots, X_n)\in C_g(m(1), \cdots, m(n))] \\ &=E[E[h(X_1, \cdots, X_n)\,|\,(X_1, \cdots, X_n)\in C_f(m(1), \cdots, m(n))] \\ &\qquad |\,(X_1, \cdots, X_n)\in C_g(m(1), \cdots, m(n))] \\ &=\Sigma_f \frac{1}{k^{n-\Sigma m(i)}}\int_{X^{\Sigma m(i)}} h(x_{11}, \cdots, x_{1m(1)}, x_{21}, f_1x_{21}, x_{22}, f_2x_{22}, \cdots)\prod_{i=1}^n\prod_{j=1}^{m(i)}dQ(x_{ij}). \end{aligned}$$

Applying Proposition 9 to the above equation, we have the desired evaluation.

Acknowledgements

The author would like to express his hearty thanks to Professor S. Kano for his kind comments.

References

[1] Ferguson, T. S.: *A Bayesian Analysis of Some Nonparametric Problem,* Ann. Statist., **1** (1973), 209-230.

[2] Dalal, S. R.: *Nonparametric and Robust Bayes Estimation of Location,* Proc. International Symposium on Optimizing Methods in Statistics (J. S. Rustagi Ed.), (1979). New York: Academic Press.

[3] Dalal, S. R.: *Dirichlet Invariant Processes and Applications to Nonparametric Estimation of Symmetric Distribution Function,* Stochastic process, Appl., **9** (1979), 99-107.

[4] Dalal, S. R.: *Non-parametric Bayes Decision Theory,* Bayesian Statistics (J. M. Bernardo, M. H. Degroot, D. V. Lindley & A. F. M. Smith, Eds.), (1980). Valencia, Spain: University Press.

[5] Hannum, R. C.: *Contributions to Bayesian Nonparametric Statistics,* Ph. D. dissertation (1979), Florida State University.

[6] Korwar, R. M. and Hollander, M.: *Contributions to the Theory of Dirichlet Processes,* Ann. Probability, **1** (1973), 705-711.

[7] Tiwari, R. C.: *A Mathematical Study of the Dirichlet Process,* Ph. D. dissertation (1981), Florida State University.

[8] Yamato, H.: *Relations between Limiting Bayes Estimates and the U-statistics for Estimable Parameters of Degree 2 and 3,* Commu. Statist.-Theor. Meth., (1977), **A6**, 55-66.

[9] Yamato, H.: *Relations between Limiting Bayes Estimates and U-statistics for Estimable Parameters,* J. Japan Statist. Soc., **7** (1977), 57-66.

[10] Yamato, H.: *Properties of Samples from Distributions Chosen from a Dirichlet Process,* Bull. Inform. Cybernetics, **21** (1984), 77-83.

[11] Yamato, H.: *Bayes Estimates of Estimable Parameters with a Dirichlet Invariant Process,* Commu. Statist.-Theor. Meth., (1986), **A15**, 2383-2390.

Communicated by S. Kanō
Received January 14, 1986