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EDGEWORTH EXPANSION FOR ONE-SAMPLE U-STATISTICS

By

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Abstract

Under some regularity conditions on kernel, an asymptotic expansion with remainder term $o(N^{-1})$ is established for one-sample U-statistics with kernel of arbitrary degree. This is an extension of the result by Callaert, Janssen and Veraverbeke [1].

1. Introduction

Let $X_1, X_2, \cdots, X_N$, be independently and identically distributed random variables with common distribution function F. Let h be symmetric function of its arguments, satisfying $Eh(X_1, X_2, \cdots, X_r)=0$ with $r\leqq N$, h is called a kernel and r is called its degree. We shall define a one-sample U-statistic with a kernel of degree r, h, by

$$U_N=\binom{N}{r}^{-1}\sum_{1\leq i_1<i_2<\cdots<i_r\leq N}h(X_{i_1}, X_{i_2}, \cdots, X_{i_r}).$$

In the case of degree two, Callaert, Janssen and Veraverbeke [1] have obtained the asymptotic expansion of the distribution of U_N with the remainder term $o(N^{-1})$. In this paper, using the forward martingale characterization of U_N, we obtain an asymptotic expansion of U_N with a kernel of arbitrary degree r. In Section 2 we obtain a representation for U_N in terms of forward martingales, and get the bounds of absolute moments of martingales. We state the main theorem in Section 3 and prove it in Section 4.

2. Preliminaries

We shall represent U_N in terms of forward martingales. This representation is due to Hoeffding [5] (cf. Serfling [7] p. 178). Under the assumption $E|h(X_1, X_2, \cdots, X_r)|<\infty$, let us define the following notations: for $1\leqq k\leqq r$

$$w_k(x_1, x_2, \cdots, x_k)=E\{h(X_1, X_2, \cdots, X_r)|X_1=x_1, X_2=x_2, \cdots, X_k=x_k\},$$

$$g_1(x_1)=w_1(x_1),\ g_2(x_1, x_2)=w_2(x_1, x_2)-\sum_{i=1}^{2}g_1(x_i)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$g_r(x_1, x_2, \cdots, x_r)=w_r(x_1, x_2, \cdots, x_r)-\sum_{1\leq i_1<\cdots<i_{r-1}\leq r}g_{r-1}(x_{i_1}, \cdots, x_{i_{r-1}})$$
$$-\sum_{1\leq i_1<\cdots<i_{r-2}\leq r}g_{r-2}(x_{i_1}, \cdots, x_{i_{r-2}})-\cdots-\sum_{i=1}^{r}g_1(x_i),$$

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and

$$A_{k,N}=\sum_{1\le i_1<i_2<\cdots<i_k\le N} g_k(X_{i_1}, X_{i_2}, \cdots, X_{i_k}), \quad \text{for } 1\leqq k\leqq r.$$

Then U_N can be rewritten as

$$U_N=\binom{N}{r}^{-1}\sum_{k=1}^{r}\binom{N-k}{r-k}A_{k,N}.$$

It is shown in the proof of Lemma 2 that $\{A_{k,N}\}_{N\ge k}$ is a forward martingale for each $k=1, 2, \cdots, r$.

By the definition of g_k, it is easily shown that if one of $\{i_1, i_2, \cdots, i_k\}$ is not contained in $\{j_1, j_2, \cdots, j_s\}$, then

$$E\{g_k(X_{i_1}, X_{i_2}, \cdots, X_{i_k})|X_{j_1}, X_{j_2}, \cdots, X_{j_s}\}=0. \tag{2.1}$$

Using this property, we can prove the useful two lemmas.

LEMMA 1. *If* $E|g_k(X_1, X_2, \cdots, X_k)|<\infty$ *and one of* $\{i_1, i_2, \cdots, i_k\}$ *is not contained in* $\{j_1, j_2, \cdots, j_s\}$, *then for* f *satisfying* $E|fg_k|<\infty$,

$$E\{f(X_{j_1}, X_{j_2}, \cdots, X_{j_s})g_k(X_{i_1}, X_{i_2}, \cdots, X_{i_k})\}=0.$$

PROOF. Taking the conditional expectation, we have the desired result from (2.1).

Before describing the next lemma, we prepare the notations. For $1\leqq N_1<N_2<\cdots<N_k\leqq N$ and $1\leqq k\leqq r$, let us define

$$B_k(N_1, N_2, \cdots, N_k)=\sum_{i_1=1}^{N_1}\sum_{i_2=i_1+1}^{N_2}\cdots \sum_{i_k=i_{k-1}+1}^{N_k} g_k(X_{i_1}, X_{i_2}, \cdots, X_{i_k}).$$

Then we have the upper bound of the pth absolute moment of B_k.

LEMMA 2. *Given the existence of the pth* ($p\geqq 2$) *absolute moment of kernel h, there exist a positive constant C such that*

$$E|B_k(N_1, N_2, \cdots, N_k)|^p\leqq C(\textstyle\prod_{i=1}^{k}N_i)^{p/2}. \tag{2.2}$$

If the second moment of kernel h is finite, the inequality (2.2) *holds also with* $p=1$.

PROOF. The latter part of the lemma immediately follows from the former. Therefore we consider the case $p\geqq 2$. By induction on s we prove the following inequality for $1\leqq s\leqq k$ and $1\leqq m_1<m_2<\cdots<m_s<i_{s+1}, \cdots, i_k$,

$$\begin{aligned} &E|\textstyle\sum_{i_1=1}^{m_1}\sum_{i_2=i_1+1}^{m_2}\cdots\sum_{i_s=i_{s-1}+1}^{m_s} g_k(X_{i_1}, X_{i_2}, \cdots, X_{i_s}, \cdots, X_{i_k})|^p \\ &\leqq (C_p)^s E|g_k(X_1, X_2, \cdots, X_k)|^p(\textstyle\prod_{i=1}^{s}m_i)^{p/2} \end{aligned} \tag{2.3}$$

where $C_p=\{8(p-1)\max(1, 2^{p-3})\}^p$.

When $s=1$, let $Y_j=\sum_{i_1=1}^{j} g_k(X_{i_1}, X_{i_2}, \cdots, X_{i_k})$ for $j=1, 2, \cdots, m_1$. Then for $j=1, 2, \cdots, m_1$, we have $Y_j-Y_{j-1}=g_k(X_j, X_{i_2}, \cdots, X_{i_k})$ and $j<i_2, i_3, \cdots, i_k$, where $Y_0=0$. Since $Y_1, Y_2, \cdots, Y_{j-1}$ are functions of $X_1, X_2, \cdots, X_{j-1}, X_{i_2}, \cdots, X_{i_k}$, we find from (2.1) that

$$\begin{aligned} &E\{Y_j-Y_{j-1}|Y_1, Y_2, \cdots, Y_{j-1}\} \\ &=E(E\{g_k(X_j, X_{i_2}, \cdots, X_{i_k})|X_1, X_2, \cdots, X_{j-1}, X_{i_2}, \cdots, X_{i_k}\}|Y_1, Y_2, \cdots, Y_{j-1})=0. \end{aligned}$$

Therefore $\{Y_j\}_{0\le j\le m_1}$ is a forward martingale. Applying an upper bound for moments of martingales obtained by Dharmadhikari, Fabian and Jogdeo [2], we have the inequality (2.3) when $s=1$.

Assume that (2.3) holds for 1, 2, $\cdots$, $s-1$ instead of s. For $s\leqq j\leqq m_s$, put $\tilde{m}_i(j)=\min(m_i, j-s+i)$ (for $i=1, 2, \cdots, s-1$) and

$$Z_j=\sum_{i_1=1}^{\tilde{m}_1(j)}\cdots\sum_{i_{s-1}=i_{s-2}+1}^{\tilde{m}_{s-1}(j)}\sum_{i_s=i_{s-1}+1}^{j}g_k(X_{i_1}, \cdots, X_{i_{s-1}}, X_{i_s}, \cdots, X_{i_k}).$$

Then in the same way of $s=1$, $\{Z_j\}_{0\le j\le m_s}$ is a forward martingale, where $Z_j=0$ for $0\leqq j<s$. Hence using the result of Dharmadhikari et al. [2], we get the inequality (2.3) for s. Thus we have the desired result.

3. Main Theorem

Before we state the main theorem, we define the following notations:

$$\xi_k^2=Eg_k^2(X_1, X_2, \cdots, X_k) \quad \text{for} \quad 1\leqq k\leqq r,$$

$$\sigma_N^2=E\{U_N\}^2=\frac{r^2}{N}\xi_1^2+\frac{\{r(r-1)\}^2}{2N(N-1)}\xi_2^2+\cdots+\frac{r!}{N(N-1)\cdots(N-r+1)}\xi_r^2,$$

$$\eta(t)=E(\exp\{itg_1(X_1)\}),$$

$$\zeta(x, y)=w_2(x, y)-\frac{r-2}{r-1}\{g_1(x)+g_1(y)\} \quad \text{for} \quad r\geqq 2,$$

$$\kappa_3=\xi_1^{-3}(Eg_1^3(X_1)+3(r-1)E\{g_1(X_1)g_1(X_2)g_2(X_1, X_2)\}),$$

$$\begin{aligned}\kappa_4=\xi_1^{-4}(&Eg_1^4(X_1)-3\xi_1^4+12(r-1)E\{g_1^2(X_1)g_1(X_2)g_2(X_1, X_2)\}\\&+12(r-1)^2E\{g_1(X_2)g_1(X_3)g_2(X_1, X_2)g_2(X_1, X_3)\}\\&+4(r-1)(r-2)E\{g_1(X_1)g_1(X_2)g_1(X_3)g_3(X_1, X_2, X_3)\}),\end{aligned}$$

and

$$Q_N(x)=\Phi(x)-\phi(x)\left\{\frac{\kappa_3}{6N^{1/2}}(x^2-1)+\frac{\kappa_4}{24N}(x^3-3x)+\frac{\kappa_3^2}{72N}(x^5-10x^3+15x)\right\}$$

where $\Phi(x)$ and $\phi(x)$ denote the distribution function and the density of the standard normal distribution.

THEOREM. *If the following conditions are satisfied (when $r=1$, the condition (C) must be omitted)*

(A) $E|h(X_1, X_2, \cdots, X_r)|^5<\infty$

(B) $\limsup_{|t|\to\infty}|\eta(t)|<1$

(C) *there exist positive constants $c<1$ and $\alpha<1/8$ such that for $m=[N^\alpha]$,*

$$P\left(\left|E\left(\exp\left\{it\sigma_N^{-1}\frac{r(r-1)}{N(N-1)}\sum_{j=m+1}^N\zeta(X_1, X_j)\right\}\right|X_{m+1}, \cdots, X_N)\right|\leqq c\right)$$

$$\geqq 1-o\left(\frac{1}{N\log N}\right)$$

uniformly for all $t\in[N^{3/4}/\log N, N\log N]$ then

$$\sup_x|P(\sigma_N^{-1}U_N\leqq x)-Q_N(x)|=o(N^{-1}).$$

REMARK 1. Instead of condition (A), the asymptotic expansion is valid under the existence of a fourth moment of kernel h. Lin [6] has proved it in the case of one-sample U-statistics with kernel of degree two. For arbitrary degree, we can similarly prove it by the way of Lin [6].

REMARK 2. The asymptotic expansion with remainder term $o(N^{-1/2})$ is valid without condition (C).

4. Proof of the Theorem

For $r=1$, U_N is a sum of independently and identically distributed random variables and the expansion of U_N has been obtained already (cf. Gnedenko and Kolmogorov [4]). Then we consider the case $r\geqq 2$.

Let

$$\Psi_N(t)=E\{\exp(it\sigma_N^{-1}U_N)\}$$

and for $s=1, 2, 3,$

$$_s\Psi_N(t)=E\{\exp(it\textstyle\sum_{k=1}^{6-s}d_{k,N}A_{k,N})\}$$

where

$$d_{k,N}=\sigma_N^{-1}\binom{N}{r}^{-1}\binom{N-k}{r-k}.$$

Then for $_2\Psi_N(t)$, we have the following lemma.

LEMMA 3. *If (A) is satisfied, then there exist positive constants* $K_s(s=1, 2, \cdots, 6)$ *such that for all* $t(-\infty<t<\infty)$, *all integers* N *and* m *with* $6<m<N-2$

$$\begin{aligned}|_2\Psi_N(t)|\leqq&\left|\eta\left(\frac{rt}{N\sigma_N}\right)\right|^{m-6}(K_1\textstyle\sum_{s=0}^{3}|t|^s d_{2,N}^s m^s N^s+K_2|t|d_{3,N}mN^2)\\&+K_3t^4d_{2,N}^4m^2N^2+K_4t^2d_{2,N}d_{3,N}mN^{3/2}\\&+K_5t^2d_{3,N}^2mN^2+K_6|t|d_{4,N}m^{1/2}N^{3/2}.\end{aligned}\tag{4.1}$$

Note that if $r=2$, *the terms which include* $d_{3,N}$ *or* $d_{4,N}$ *are omitted. Similarly we omit* $K_6|t|d_{4,N}m^{1/2}N^{3/2}$, *if* $r=3$.

PROOF. See Appendix 1.

Furthermore, for $_1\Psi_N(t)$ we have Lemma 4.

LEMMA 4. *If (A) is satisfied, then there exist positive constants* $M_s(s=1, 2, \cdots, 12)$ *such that for all* $t(-\infty<t<\infty)$, *all integers* N *and* m *with* $8<m<N-3$

$$\begin{aligned}|_1\Psi_N(t)|\leqq&E(|E(\exp\{itd_{2,N}\textstyle\sum_{j=m+1}^{N}\zeta(X_1, X_j)\}|X_{m+1}, \cdots, X_N)|^{m-8})\\&\times(M_1\textstyle\sum_{s=0}^{1}|t|^{s+1}d_{2,N}^s d_{4,N}m^{2s+1}N^3+M_2\sum_{s=0}^{1}|t|^{s+2}d_{2,N}^s d_{3,N}^2 m^{2(s+1)}N^4\\&+M_3\textstyle\sum_{s=0}^{2}|t|^{s+1}d_{2,N}^s d_{3,N}m^{2s+1}N^2+M_4\sum_{s=0}^{3}|t|^s d_{2,N}^s m^{2s})\\&+M_5|t|^3d_{2,N}^2d_{4,N}m^{9/2}N^{3/2}+M_6t^2d_{3,N}d_{4,N}mN^{5/2}\\&+M_7t^4d_{2,N}^2d_{3,N}^2m^5N^2+M_8t^4d_{2,N}^3d_{3,N}m^{13/2}N\\&+M_9t^4d_{2,N}^4m^8+M_{10}|t|^3d_{3,N}^3m^{3/2}N^3\\&+M_{11}t^2d_{4,N}^2mN^3+M_{12}|t|d_{5,N}m^{1/2}N^2.\end{aligned}\tag{4.2}$$

Note that if $r=2$, *the terms which include* $d_{3,N}$, $d_{4,N}$ *or* $d_{5,N}$ *are omitted. Similarly if*

$r=3$, *we omit the terms which include* $d_{4,N}$ *or* $d_{5,N}$, *and if* $r=4$, *omit* $M_{12}|t|d_{5,N}m^{1/2}N^2$.

PROOF. See Appendix 2.

Now applying Lemma 3, 4 and Esseen's smoothing lemma [3] we shall prove the Theorem. In the sequel, we consider the proof for the case $r\geqq 5$. When $r=2, 3, 4$, we can prove the Theorem more easily.

Let

$$\tilde{\Psi}_N(t)=\int_{-\infty}^{\infty}\exp(itx)dQ_N(x)$$

$$=\exp\left(-\frac{t^2}{2}\right)\left\{1+\frac{\kappa_3}{6N^{1/2}}(it)^3+\frac{\kappa_4}{24N}(it)^4+\frac{\kappa_3^2}{72N}(it)^6\right\}.$$

From smoothing lemma, we have

$$\sup_x|P(\sigma_N^{-1}U_N\leqq x)-Q_N(x)|\leqq\frac{1}{\pi}\int_{-N\log N}^{N\log N}|t|^{-1}|\Psi_N(t)-\tilde{\Psi}_N(t)|dt+o(N^{-1}).$$

Since the proof for the negative part of t is similar to that for the positive one, we shall show that

$$\int_0^{N\log N}t^{-1}|\Psi_N(t)-\tilde{\Psi}_N(t)|dt=o(N^{-1}).$$

Since $d_{k,N}=O(N^{1/2-k})$ and $E|A_{k,N}|\leqq O(N^{k/2})$ from (2.2) in Lemma 2,

$$\int_0^{N\log N}t^{-1}|\Psi_N(t)-{}_1\Psi_N(t)|dt\leqq\sum_{k=6}^{r}d_{k,N}E|A_{k,N}|\int_0^{N\log N}dt$$

$$=o(N^{-1}).$$

Similary we have

$$\int_0^{N^{3/4}/\log N}t^{-1}|{}_1\Psi_N(t)-{}_2\Psi_N(t)|dt=o(N^{-1})$$

and

$$\int_0^{N^{1/4}/\log N}t^{-1}|{}_2\Psi_N(t)-{}_3\Psi_N(t)|dt=o(N^{-1}).$$

Then, putting

$$(\mathrm{I})=\int_0^{N^{1/4}/\log N}t^{-1}|{}_3\Psi_N(t)-\tilde{\Psi}_N(t)|dt,$$

$$(\mathrm{II})=\int_{N^{1/4}/\log N}^{N^{3/4}/\log N}t^{-1}|{}_2\Psi_N(t)|dt,$$

$$(\mathrm{III})=\int_{N^{3/4}/\log N}^{N\log N}t^{-1}|{}_1\Psi_N(t)|dt,$$

and

$$(\mathrm{IV})=\int_{\log N}^{\infty}t^{-1}|\tilde{\Psi}_N(t)|dt,$$

we have

$$\int_0^{N\log N}t^{-1}|\Psi_N(t)-\tilde{\Psi}_N(t)|dt\leqq(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+o(N^{-1}). \tag{4.3}$$

It immediately follows from condition (A) that $(\mathrm{IV})=o(N^{-1})$. Next, we shall prove that (I), (II) and (III) are $o(N^{-1})$.

Order of (I): Let us define

$$\Psi_N^*(t)=E\Big(\exp(itd_{1,N}A_{1,N})\Big\{1+itd_{2,N}A_{2,N}+\frac{(it)^2}{2}d_{2,N}^2A_{2,N}^2\Big\}\Big)$$
$$+E(itd_{3,N}A_{3,N}\exp(itd_{1,N}A_{1,N})).$$

Then,

$$\int_0^{N^{1/4}/\log N}t^{-1}|\Psi_N^*(t)-{}_3\Psi_N(t)|\,dt$$
$$\leqq\int_0^{N^{1/4}/\log N}t^{-1}\Big|E\Big(\exp(itd_{1,N}A_{1,N})\Big\{\exp(itd_{2,N}A_{2,N})-\Big(1+itd_{2,N}A_{2,N}$$
$$+\frac{(it)^2}{2}d_{2,N}^2A_{2,N}^2\Big)\Big\}\Big)\Big|\,dt+\int_0^{N^{1/4}/\log N}t^{-1}|E(itd_{3,N}A_{3,N}\exp(itd_{1,N}A_{1,N})$$
$$\times\{\exp(itd_{2,N}A_{2,N})-1\})|\,dt+o(N^{-1}).$$

Using the similar way which has been described in Callaert et al. [1] pp 304-306, we can establish that the first term is $o(N^{-1})$. Applying Schwartz's inequality and (2.2) in Lemma 2, we can easily obtain the order of the second term.

Now we evaluate the difference of $\Psi_N^*(t)$ and $\tilde{\Psi}_N(t)$. From Lemma 1 $\Psi_N^*(t)$ can be rewritten as

$$\Psi_N^*(t)=\eta^N(d_{1,N}t)+it\eta^{N-2}(d_{1,N}t)d_{2,N}\frac{N(N-1)}{2}E(\exp\{itd_{1,N}(g_1(X_1)+g_1(X_2))\}g_2(X_1,X_2))$$
$$+\frac{(it)^2}{2}\eta^{N-2}(d_{1,N}t)d_{2,N}^2\frac{N(N-1)}{2}E(\exp\{itd_{1,N}(g_1(X_1)+g_1(X_2))\}g_2^2(X_1,X_2))$$
$$+\frac{(it)^2}{2}\eta^{N-3}(d_{1,N}t)d_{2,N}^2N(N-1)(N-2)E(\exp\{itd_{1,N}(g_1(X_1)+g_1(X_2)$$
$$+g_1(X_3))\}g_2(X_1,X_2)g_2(X_1,X_3))+\frac{(it)^2}{2}\eta^{N-4}(d_{1,N}t)d_{2,N}^2\frac{N(N-1)(N-2)(N-3)}{4}$$
$$\times\{E(\exp\{itd_{1,N}(g_1(X_1)+g_1(X_2))\}g_2(X_1,X_2))\}^2+it\eta^{N-3}(d_{1,N}t)d_{3,N}\frac{N(N-1)(N-2)}{6}$$
$$\times E(\exp\{itd_{1,N}(g_1(X_1)+g_1(X_2)+g_1(X_3))\}g_3(X_1,X_2,X_3)).$$

Let us denote $\Psi_N^*(t)$ by

$$\Psi_N^*(t)=I_0^*+itd_{2,N}\frac{N(N-1)}{2}I_2^*E_1^*+\frac{(it)^2}{2}d_{2,N}^2\frac{N(N-1)}{2}I_2^*E_2^*$$
$$+\frac{(it)^2}{2}d_{2,N}^2N(N-1)(N-2)I_3^*E_3^*+\frac{(it)^2}{2}d_{2,N}^2\frac{N(N-1)(N-2)(N-3)}{4}I_4^*E_4^*$$
$$+itd_{3,N}\frac{N(N-1)(N-2)}{6}I_3^*E_5^*.$$

Then approximations of I_k^* and E_k^* are given as follows:

$$I_k=\exp\Big(-\frac{t^2}{2}\Big)\Big(1-\frac{(it)^2}{2N}\Big\{\frac{(r-1)^2\xi_2^2}{2\xi_1^2}+k\Big\}+\frac{(it)^3}{6N^{1/2}\xi_1^3}Eg_1^3(X_1)$$
$$+\frac{(it)^4}{24N\xi_1^4}\{Eg_1^4(X_1)-3\xi_1^4\}+\frac{(it)^6}{72N\xi_1^6}\{Eg_1^3(X_1)\}^2\Big),$$

$$E_1=\frac{(it)^2}{N\xi_1^2}E\{g_1(X_1)g_1(X_2)g_2(X_1, X_2)\}$$

$$+\frac{(it)^3}{N^{3/2}\xi_1^3}E\{g_1^2(X_1)g_1(X_2)g_2(X_1, X_2)\},$$

$$E_2=Eg_2^2(X_1, X_2)$$

$$E_3=\frac{(it)^2}{N\xi_1^2}E\{g_1(X_2)g_1(X_3)g_2(X_1, X_2)g_2(X_1, X_3)\},$$

$$E_4=\Big(\frac{(it)^2}{N\xi_1^2}E\{g_1(X_1)g_1(X_2)g_2(X_1, X_2)\}\Big)^2,$$

$$E_5=\frac{(it)^3}{N^{3/2}\xi_1^3}E\{g_1(X_1)g_1(X_2)g_1(X_3)g_3(X_1, X_2, X_3)\}.$$

By the same way of Lemma 2 as Callaert et al. [1], there exist positive constants ε and a such that for $0\leqq t\leqq\varepsilon N^{1/2}$,

$$|I_k^*-I_k|\leqq o(N^{-1})P(t)\exp(-at^2)$$

where $P(t)$ is a polynomial in t.

Furthermore, from condition (A), and $\sigma_N^{-1}=N^{1/2}(\xi_1 r)^{-1}(1+O(N^{-1}))$, we have the following inequalities:

$$|E_1^*-E_1|\leqq t^4O(N^{-2})+t^2O(N^{-2})+|t|^3O(N^{-5/2}),$$

$$|E_2^*-E_2|\leqq|t|O(N^{-1/2}),$$

$$|E_3^*-E_3|\leqq|t|^3O(N^{-3/2})+t^2O(N^{-2}),$$

$$|E_4^*-E_4|\leqq|t|^5O(N^{-5/2})+t^6O(N^{-3})+t^4O(N^{-3}),$$

$$|E_5^*-E_5|\leqq t^4O(N^{-2})+|t|^3O(N^{-5/2}).$$

Define

$$\tilde{\Psi}_N^*(t)=I_0+itd_{2,N}\frac{N(N-1)}{2}I_2E_1+\frac{(it)^2}{2}d_{2,N}^2\frac{N(N-1)}{2}I_2E_2$$

$$+\frac{(it)^2}{2}d_{2,N}^2N(N-1)(N-2)I_3E_3+\frac{(it)^2}{2}d_{2,N}^2\frac{N(N-1)(N-2)(N-3)}{4}I_4E_4$$

$$+itd_{3,N}\frac{N(N-1)(N-2)}{6}I_3E_5.$$

Then,

$$\int_0^{N^{1/4}/\log N}t^{-1}|\Psi_N^*(t)-\tilde{\Psi}_N(t)|\,dt$$

$$\leqq\int_0^{N^{1/4}/\log N}t^{-1}|\Psi_N^*(t)-\tilde{\Psi}_N^*(t)|\,dt+\int_0^{N^{1/4}/\log N}t^{-1}|\tilde{\Psi}_N^*(t)-\tilde{\Psi}_N(t)|\,dt.$$

Since $\sigma_N^{-1}=N^{1/2}(\xi_1 r)^{-1}(1+O(N^{-1}))$ and for any k,

$$\int_0^\infty t^k\exp(-t^2/2)dt<\infty,$$

we get that the last term is $o(N^{-1})$.

For the first term, we have

$$|\Psi_N^*(t)-\tilde{\Psi}_N^*(t)|\leqq|I_0^*-I_0|+|t|\frac{r(r-1)}{2\sigma_N}\{|E_1^*(I_2^*-I_2)|+|I_2(E_1^*-E_1)|\}$$

$$+\frac{t^2}{2}\frac{\{r(r-1)\}^2}{2\sigma_N N(N-1)}\{|E_2^*(I_2^*-I_2)|+|I_2(E_2^*-E_2)|\}$$

$$+\frac{t^2}{2}\frac{\{r(r-1)\}^2(N-2)}{2\sigma_N N(N-1)}\{|E_3^*(I_3^*-I_3)|+|I_3(E_3^*-E_3)|\}$$

$$+\frac{t^2}{2}\frac{\{r(r-1)\}^2(N-2)(N-3)}{4\sigma_N N(N-1)}\{|E_4^*(I_4^*-I_4)|+|I_4(E_4^*-E_4)|\}$$

$$+|t|\frac{r(r-1)(r-2)}{6\sigma_N}\{|E_5^*(I_3^*-I_3)|+|I_3(E_5^*-E_5)|\}.$$

Therefore, using the previous discussions, we can establish that the first term is $o(N^{-1})$.

Order of (II): Applying (4.1) in Lemma 3, condition (B) and the same arguments which have been described in Callaert et al. [1] pp 308-309, we get the order of (II).

Order of (III): Combining (4.2) in Lemma 4 and condition (C), we can easily establish that the order of (III) is $o(N^{-1})$.

Thus we showed that (I), (II) and (III) are $o(N^{-1})$ and therefore by (4.3) we have the desired result.

Appendix

1. Proof of Lemma 3

We have

$$_2\Psi_N(t)=E(\exp\{itd_{1,N}B_1(m)\}\exp\{itd_{1,N}(A_{1,N}-B_1(m))\}\exp\{itd_{2,N}B_2(m,N)\}$$
$$\times\exp\{itd_{2,N}(A_{2,N}-B_2(m,N))\}\exp\{itd_{3,N}B_3(m,N-1,N)\}$$
$$\times\exp\{itd_{3,N}(A_{3,N}-B_3(m,N-1,N))\}\exp\{itd_{4,N}B_4(m,N-2,N-1,N)\}$$
$$\times\exp\{itd_{4,N}(A_{4,N}-B_4(m,N-2,N-1,N))\}).$$

Let us define $B_k^*=B_k(m,\cdots)$ and $R_k=\exp\{itd_{k,N}(A_{k,N}-B_k^*)\}$ $(k=1,2,3,4)$. Then expanding $\exp\{itd_{k,N}B_k^*\}$ $(k=3,4)$, we have

$$|_2\Psi_N(t)|\leqq|E(\exp\{itd_{1,N}B_1^*\}\exp\{itd_{2,N}B_2^*\}\textstyle\prod_{j=1}^4 R_j)|$$
$$+|t|d_{3,N}|E(\exp\{itd_{1,N}B_1^*\}\textstyle\prod_{j=1}^4 R_jB_3^*)|$$
$$+t^2d_{2,N}d_{3,N}E|B_2^*B_3^*|+t^2d_{3,N}^2E|B_3^*|^2+|t|d_{4,N}E|B_4^*|.$$

Therefore using (2.2) in Lemma 2, we can obtain (4.1) in the same way of Lemma 4 of Callaert et al. [1].

2. Proof of Lemma 4

We get

$$_1\Psi_N(t)=E(\exp\{itd_{2,N}\textstyle\sum_{1\leq i<j\leq N}\zeta(X_i,X_j)\}\exp\{itd_{3,N}B_3(m,N-1,N)\}$$
$$\times\exp\{itd_{3,N}(A_{3,N}-B_3(m,N-1,N))\}\exp\{itd_{4,N}B_4(m,N-2,N-1,N)\}$$

$$\times\exp\{itd_{4,N}(A_{4,N}-B_4(m, N-2, N-1, N))\}$$
$$\times\exp\{itd_{5,N}B_5(m, N-3, N-2, N-1, N)\}$$
$$\times\exp\{itd_{5,N}(A_{5,N}-B_5(m, N-3, N-2, N-1, N))\}).$$

Let us define $D=\exp\{itd_{2,N}\sum_{1\leq i<j\leq N}\zeta(X_i, X_j)\}$, $B_3^*=B_3(m, N-1, N)$, $B_4^*=B_4(m, N-2, N-1, N)$, $B_5^*=B_5(m, N-3, N-2, N-1, N)$ and $R_k=\exp\{itd_{k,N}(A_{k,N}-B_k^*)\}$ $(k=3, 4, 5)$. Then expanding $\exp\{itd_{k,N}B_k^*\}$ $(k=3, 4, 5)$, we have

$$|{}_1\Psi_N(t)|\leq\sum_{s=0}^{2}|t|^s d_{3,N}^s|E\{DB_3^{*s}\prod_{j=3}^{5}R_j\}|+|t|^3 d_{3,N}^3 E|B_3^*|^3$$
$$+|t|d_{4,N}|E\{DB_4^*\prod_{j=3}^{5}R_j\}|+t^2 d_{3,N}d_{4,N}E|B_3^*B_4^*|$$
$$+t^2 d_{4,N}^2 E(B_4^*)^2+|t|d_{5,N}E|B_5^*|$$

Using (2.2) in Lemma 2 and applying the factorization of D which has been discussed in Lemma 5 of Callaert et al. [1], we have the desired result.

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