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Abstract

Finite tree automata on infinite trees which move from the root infinitely are investigated through six types of acceptance conditions. Each condition is an extension of the one used for finite automata on ω-words. We relate the six classes of infinite tree languages with the corresponding classes of ω-languages defined by finite automata, and the relationship among these six classes is established.

1. Introduction

Rabin [6-8] introduced finite tree automata on infinite trees to study the problems of definability in monadic second-order theories. As by-products, he obtained several decidability results. He considered the acceptance conditions which are extensions of the ones studied in [1, 3]. For ω-languages, a series of acceptances by finite automata have been introduced and investigated [1-4, 8, 10-12]. These acceptance definitions for ω-languages can be also naturally extended for the infinite tree acceptance by finite tree automata and we can define the classes of infinite tree languages defined by these finite tree automata. In this paper we deal with six types of acceptance conditions for finite tree automata on infinite trees which move from the root infinitely. Each condition uses a family of final sets for acceptance as was done in [3] for finite automata on ω-words. The purpose of this paper is to establish the relationship among these infinite tree language classes. First it is proved that a single final set is sufficient for four types of acceptance definitions. We also prove an embedding theorem which relates the infinite tree language classes with the ω-language classes. With the aid of this embedding theorem and a result due to Rabin [7], we classify the infinite tree language classes. A summary of the results is figured out in Fig. 1 in Section 5.

It is known that the nondeterministic infinite tree language class is larger than the deterministic one for any type of the acceptances [9]. To refine this result, we consider the nondeterministic degree which is defined to be the maximum number of possible transitions of a finite tree automaton. Then the determinism is regarded as the nondeterministic degree one. We show that the nondeterministic degree gives a hierarchy between the nondeterministic class and the deterministic one for each acceptance definition. This is a property special to (infinite) tree languages. In addition to this observation, the results in this paper show that the class of infinite tree languages accepted by

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nondeterministic finite tree automata which accept in the manner of Rabin [6] is the most general one and this class may be considered as a natural extension of the class of ω-regular languages [3], where deterministic finite automata are used as basis.

2. Preliminaries

Let $\Sigma_k = \{0, 1, \cdots, k-1\}$ be an alphabet consisting of k symbols. The *infinite k-ary tree* is the set $T_k = \Sigma_k^*$. The empty string ε is the roof of T_k. For $x \in T_k$, $x0, x1, \cdots,$ $x(k-1)$ are the sons of x arranged from left to right. Note that $T_k \subset T_{k+1}$ for $k \geqq 1$. Let Σ be a finite alphabet. An *infinite k-ary tree over Σ* (a *k-ary Σ-tree*) is a function $t : T_k \to \Sigma$. The set of all k-ary Σ-trees is denoted by Σ^{T_k}. For x, y in T_k, we say that x is a prefix of y denoted by $x \leqq y$ if $y = xz$ for some z in Σ_k^*. The prefix relation gives a partial ordering on T_k. A *path* in T_k is a subset $\pi \subseteq T_k$ satisfying (i) $\varepsilon \in \pi$, (ii) for $x \in \pi$, exactly one of k sons $x0, x1, \cdots, x(k-1)$ belongs to π, (iii) π is the smallest subset of T_k satisfying (i) and (ii).

For a set S, $|S|$ denotes the cardinality of S and $P(S)$ is the set of all subsets of S. For a function $f : X \to Y$ and a subset X' of X, $f|X'$ denotes the restriction of f to X'.

DEFINITION. A (*nondeterministic*) *finite k-ary tree automation* is a quintuple $M = \langle S, \Sigma, \Delta, s_0, \mathscr{F} \rangle$, where

(1) S is a finite set of states,

(2) Δ is a transition function $\Delta : S \times \Sigma \to P(S^k) - \{\phi\}$, where S^k is the k-fold product of S,

(3) s_0 is a state in S called the *initial state*,

(4) $\mathscr{F}$ is a family of subsets of S. F in $\mathscr{F}$ is called a *final set* and states in F are called *final states*.

M is said to be *deterministic* if $|\Delta(s, \sigma)| = 1$ for each s in S and each σ in Σ.

Let $M = \langle S, \Sigma, \Delta, s_0, \mathscr{F} \rangle$ be a finite k-ary tree automaton and let $t : T_k \to \Sigma$ be a k-ary Σ-tree. A *run* of M on t is a function $r : T_k \to S$ such that $r(\varepsilon) = s_0$ and for each x in T_k, $(r(x0), r(x1), \cdots, r(x(k-1))) \in \Delta(r(x), t(x))$. Let π be a path in T_k. For a run $r : T_k \to S$ we define

$$In(r|\pi) = \{s \in S \mid s = r(x) \text{ for infinitely many } x \text{ in } \pi\}.$$

DEFINITION. A finite k-ary tree automaton $M = \langle S, \Sigma, \Delta, s_0, \mathscr{F} \rangle$ is said to *accept* a k-ary Σ-tree t in the sense of $C_i (i=1, \cdots, 6)$ if there exists a run r of M on t such that for each path $\pi \subseteq T_k$ there is a final set F in $\mathscr{F}$ which satisfies the condition C_i described below. A run r satisfying the condition C_i is called a C_i-*accepting run* of M on t.

(C_1) $In(r|\pi) \cap F \neq \varnothing$.

(C_2) $In(r|\pi) \subseteq F$.

(C_3) $r(\pi) \cap F \neq \varnothing$.

(C_4) $r(\pi) \subseteq F$.

(C_5) $In(r|\pi) = F$.

(C_6) $r(\pi) = F$.

The acceptance defined by C_1 condition is sometimes called the acceptance in the

sense of Büchi and the C_5-acceptance is called the acceptance in the sense of Muller.

DEFINITION. For $i=1, \cdots, 6$, the set $L_i(M)$ of k-ary Σ-trees is defined as

$$L_i(M)=\{t \in \Sigma^{T_k} \mid t \text{ is accepted by } M \text{ in the sencse of } C_i\}.$$

We say that a set $L \subseteq \Sigma^{T_k}$ is *accepted* by M in the sense of C_i if $L=L_i(M)$.

DEFINITION. For $i=1, \cdots, 6$, we define

(1) $\mathfrak{N}_i^{(k)}=\{L_i(M) \mid M \text{ is a nondeterministic finite } k\text{-ary tree automaton}\}$,

(2) $\mathfrak{D}_i^{(k)}=\{L_i(M) \mid M \text{ is a deterministic finite } k\text{-ary tree automaton}\}$.

In this paper we are concerned only with the classes for $k=1, 2$. We denote $\mathfrak{N}_i^{(1)}$ (resp. $\mathfrak{D}_i^{(1)}$) by $\mathfrak{N}_i$ (resp. $\mathfrak{D}_i$). The classes $\mathfrak{N}_i^{(2)}$ and $\mathfrak{D}_i^{(2)}$ are specially denoted by $\mathfrak{N}\mathfrak{T}_i$ and $\mathfrak{D}\mathfrak{T}_i$, respectively. A binary Σ-tree is simply called a Σ-tree and a finite binary tree automaton is called a finite tree automaton. For $k=1$, this is just the case of finite automata on ω-words [1-4, 8, 10-12]. In fact, $T_1=\{0\}^*$ and Σ^{T_1} is the same as the set Σ^ω of ω-words. A subset of Σ^{T_1} is called an ω-language. We call a finite unary tree automaton by a finite automaton. The reader is referred to [2, 11] for the terminologies on ω-words and ω-languages.

Finite automata on ω-words have been extensively studied and the classes $\mathfrak{N}_i$ and $\mathfrak{D}_i(i=1, \cdots, 6)$ of ω-languages have been completely determined [2, 10-12]. Table 1 summarizes the results on $\mathfrak{N}_i$ and $\mathfrak{D}_i(i=1, \cdots 6)$. Here R denotes the class of ω-regular languages [3]. With respect to the product topology on Σ^ω based on the discrete topology on Σ, the classes in Table 1 are characterized as $G^R=G \cap R$, $F^R=F \cap R$, $G_\delta^R=G_\delta \cap R$ and $F_\sigma^R=F_\sigma \cap R$, where G(resp. F) denotes the class of open (resp. closed) sets and G_δ(resp. F_δ) denotes the class of countable intersections (resp. unions) of open (resp. closed) sets. The inclusion relations among these classes are $G^R \subsetneqq G_\delta^R \subsetneqq R$, $F^R \subsetneqq F_\sigma^R \subsetneqq R$ and $G^R \cup F^R \subsetneqq G_\delta^R \cap R_\sigma^R$. We refer the reader to [4, 10-12] for more details on these classes.

Table 1

	C_1	C_2	C_3	C_4	C_5	C_6
$\mathfrak{D}_i$	G_δ^R	F_σ^R	G^R	F^R	R	$G_\delta^R \cap F_\sigma^R$
$\mathfrak{N}_i$	R	F_σ^R	G^R	F^R	R	F_σ^R

3. Finite Tree Automata with a Single Final Set

In this section we consider finite tree automata which have just one final set. In the literatures [4, 10], it has been known that a single final set is sufficient for the ω-word acceptances by C_i conditions for $i=1, \cdots 4$. The purpose of this section is to show that an analogue also holds for infinite tree acceptances.

A finite tree automaton $M=\langle S, \Sigma, \Delta, s_0, \{F\}\rangle$ with a final set F is also denoted by $M=\langle S, \Sigma, \Delta, s_0, F\rangle$.

THEOREM 3.1. *Let* $M=\langle S, \Sigma, \Delta, s_0, \mathfrak{F}\rangle$ *be a finite tree automaton. Then for each* $i=1, \cdots, 4$, *there exists a finite tree automaton* $\tilde{M}=\langle \tilde{S}, \Sigma, \tilde{\Delta}, \tilde{s}_0, \tilde{F}\rangle$ *with a final set* $\tilde{F}$ *such that* $L_i(M)=L_i(\tilde{M})$. *Furthermore, if* M *is deterministic, so is* $\tilde{M}$.

PROOF. The idea of the proof is due to [4, Theorem 4.1; 10, Lemma 7].

CASE 1. $i=1, 3$: $\tilde{M}$ is simply defined by $\tilde{M}=\langle S, \Sigma, \Delta, s_0, \tilde{F}\rangle$, where $\tilde{F}=\{s \mid s\in F$ for some F in $\mathcal{F}\}$.

CASE 2. $i=2$: $\tilde{M}$ is defined as follows:

(1) $\tilde{S}=\{(s, \tilde{\mathcal{F}}) \mid \tilde{\mathcal{F}}\subseteq\mathcal{F}$ and for each F in $\tilde{\mathcal{F}}$, s is in $F\}$.

(2) $\tilde{s}_0=(s_0, \varnothing)$.

(3) If (s_1, s_2) is in $\Delta(s, \sigma)$, then for all $\varnothing\neq\tilde{\mathcal{F}}\subseteq\mathcal{F}$, $\tilde{\Delta}((s, \tilde{\mathcal{F}}), \sigma)$ contains $((s_1, \{F \mid F\in\tilde{\mathcal{F}}, s_1\in F\}), (s_2, \{F \mid F\in\tilde{\mathcal{F}}, s_2\in F\}))$ and $\tilde{\Delta}((s, \varnothing), \sigma)$ contains $((s_1, \{F \mid F\in\mathcal{F}, s_1\in F\}), (s_2, \{F \mid F\in\mathcal{F}, s_2\in F\}))$.

(4) $\tilde{F}=\{(s, \tilde{\mathcal{F}}) \mid (s, \tilde{\mathcal{F}})\in\tilde{S}$ with $\tilde{\mathcal{F}}\neq\varnothing\}$.

Let $t: T_2\to\Sigma$ be a Σ-tree. First we establish a one-to-one correspondence between the runs of M on t and the runs of $\tilde{M}$ on t. Let $\tilde{r}: T_2\to\tilde{S}$ be an arbitrary run of $\tilde{M}$ on t. Then we associate it with a run $r: T_2\to S$ such that $r(x)=s$, where $\tilde{r}(x)=(s, \tilde{\mathcal{F}})$ for x in T_2. By (3), we can see that r is a run of M on t. Conversely for a run $r: T_2\to S$ of M on t, we associate it with a run $\tilde{r}: T_2\to\tilde{S}$ of $\tilde{M}$ on t which is defined as follows: $\tilde{r}$ is inductively defined. First we define $\tilde{r}(\varepsilon)=(s_0, \varnothing)$. Suppose that $\tilde{r}(x)=(s, \tilde{\mathcal{F}})$ is defined. Then note that $r(x)=s$. Assume that in the run r a transition $(s_1, s_2)\in$ $\Delta(s, t(x))$ is taken at the node x. Then if $\tilde{\mathcal{F}}=\varnothing$, the transition $((s_1, \{F \mid F\in\mathcal{F}, s_1\in F\})$, $(s_2, \{F \mid F\in\mathcal{F}, s_2\in F\}))$ is taken at x in the run $\tilde{r}$. Namely we define $\tilde{r}(x0)=(s_1, \{F \mid F\in\mathcal{F}, s_1\in F\})$ and $\tilde{r}(x1)=(s_2, \{F \mid F\in\mathcal{F}, s_2\in F\})$. If $\tilde{\mathcal{F}}\neq\varnothing$, then we put $\tilde{r}(x0)=(s_1, \{F \mid F\in\tilde{\mathcal{F}}, s_1\in F\})$ and $\tilde{r}(x1)=(s_2, \{F \mid F\in\tilde{\mathcal{F}}, s_2\in F\})$. Again by (3) we see that $\tilde{r}$ is a run of $\tilde{M}$ on t. By this correspondence we assume that r and $\tilde{r}$ denote the runs corresponding to each other. To prove $L_2(M)=L_2(\tilde{M})$, it suffices to show that the following two statements are equivalent for each path $\pi\subseteq T_2$.

(5) There exists F in $\mathcal{F}$ such that $In(r \mid \pi)\subseteq F$.

(6) $In(\tilde{r} \mid \pi)\subseteq\tilde{F}$.

For a path $\pi=\{x_0<x_1<\cdots x_n<\cdots\}$, let $\tilde{r}(x_n)=(s_n, \mathcal{F}_n)$ for $n\geq 0$. Then there exists an integer n_0 such that $In(r \mid \pi)=\{s_n \mid n\geq n_0\}$ and $In(\tilde{r} \mid \pi)=\{(s_n, \mathcal{F}_n) \mid n\geq n_0\}$. Suppose that $In(r \mid \pi)\subseteq F$ for some F in $\mathcal{F}$. Then by the choice of n_0 we see that s_n is in F for all $n\geq n_0$. If $\mathcal{F}_{n_1}=\varnothing$ for some $n_1\geq n_0$, then by the choice of n_0 we see that $\mathcal{F}_n=\varnothing$ for infinitely many $n\geq n_1$. As we observed above, s_{n_1+1} is in F. Therefore by (3) we see that F is in $\mathcal{F}_{n_1+1}$. Moreover, since s_n is in F for all $n\geq n_1$, it follows from (3) that F is in $\mathcal{F}_n$ for all $n>n_1$. Hence $\mathcal{F}_n\neq\varnothing$ for all $n>n_1$. This is a contradiction. Thus $\mathcal{F}_n\neq\varnothing$ for all $n\geq n_0$. Therefore $In(\tilde{r} \mid \pi)\subseteq\tilde{F}$. Conversely suppose that $In(\tilde{r} \mid \pi)\subseteq\tilde{F}$. Then it follows from (3) and (4) that $\mathcal{F}_n\supseteq\mathcal{F}_{n+1}$ and $\mathcal{F}_n\neq\varnothing$ for all $n\geq n_0$. Since $\mathcal{F}_n$ is finite and $\mathcal{F}_n\neq\varnothing$, there exists F in $\mathcal{F}$ such that F is in $\mathcal{F}_n$ for all $n\geq n_0$. Therefore s_n is in F for all $n\geq n_0$. Thus $In(r \mid \pi)\subseteq F$.

CASE 3. $i=4$: $\tilde{M}$ is defined as follows:

(1) $\tilde{S}=\{(s, \tilde{\mathcal{F}}) \mid \tilde{\mathcal{F}}\subseteq\mathcal{F}$ and for each F in $\tilde{\mathcal{F}}$, s is in $F\}$.

(2) $\tilde{s}_0=(s_0, \mathcal{F}_0)$, where $\mathcal{F}_0=\{F \mid F\in\mathcal{F}, s_0\in F\}$.

(3) If (s_1, s_2) is in $\Delta(s, \sigma)$, then for all $\tilde{\mathcal{F}}\subseteq\mathcal{F}$, $\Delta((s, \tilde{\mathcal{F}}), \sigma)$ contains $((s_1, \{F \mid F\in\tilde{\mathcal{F}}, s_1\in F\}), (s_2, \{F \mid F\in\tilde{\mathcal{F}}, s_2\in F\}))$.

(4) $\tilde{F}=\{(s, \tilde{\mathcal{F}}) \mid (s, \tilde{\mathcal{F}})\in\tilde{S}$ with $\tilde{\mathcal{F}}\neq\varnothing\}$.

Let $t: T_2\to\Sigma$ be a Σ-tree. As in Case 2, by the one-to-one correspondence between the runs of M on t and the runs of $\tilde{M}$ on t, we denote by r and $\tilde{r}$ the runs correspond-

ing to each other. Let $\pi=\{x_0<x_1<\cdots x_n<\cdots\}$ be an arbitrary path in T_2 and let $\tilde{r}(x_n)=(s_n, \mathcal{F}_n)$ for $n\geq0$. Then we show the equivalence of the following two statements.

(5) There exists F in $\mathcal{F}$ such that $r(\pi)\subseteq F$.

(6) $\tilde{r}(\pi)\subseteq\tilde{F}$.

Suppose that $r(\pi)\subseteq F$ for some F in $\mathcal{F}$. Since $r(\varepsilon)=s_0$ is in F, we see that F is in $\mathcal{F}_0$. By induction we can see that F is in $\mathcal{F}_n$ for all $n\geq0$. Hence $\mathcal{F}_n\neq\emptyset$ for all $n\geq0$. Therefore $\tilde{r}(\pi)\subseteq\tilde{F}$. Conversely, if $\tilde{r}(\pi)\subseteq\tilde{F}$, then $\mathcal{F}_n\neq\emptyset$ for all $n\geq0$. Since $\mathcal{F}_n\supseteq\mathcal{F}_{n+1}$ and $\mathcal{F}_n\neq\emptyset$ for $n\geq0$, there exists F such that F is in $\mathcal{F}_n$ for all $n\geq0$. Since $\tilde{r}(x_n)=(r(x_n), \mathcal{F}_n)$, it follows from (3) that $r(x_n)$ is in F for all $n\geq0$. Hence $r(\pi)\subseteq F$.

In Cases 1-3, the construction of $\tilde{M}$ shows that if M is deterministic then so is $\tilde{M}$.

By Theorem 3.1, we hereafter consider only finite tree automata with a single final set for C_i-acceptances for $i=1, \cdots, 4$.

For the acceptance conditions C_5 and C_6, the number of final sets affects the definability. We will discuss this matter in Section 6.

4. Embedding Theorem

Let $\Sigma=\{a, b\}$. We define $\iota: \Sigma^{T_1}\rightarrow\Sigma^{T_2}$ and $\rho: \Sigma^{T_2}\rightarrow\Sigma^{T_1}$ as follows: For w in Σ^{T_1}, $\iota(w)$ is defined by $\iota(w)(x)=w(x)$ for x in T_1 and $\iota(w)(x)=a$ for x in T_2-T_1. For t in Σ^{T_2}, we define $\rho(t)=t|T_1$. Namely, $\iota(w)$ is a Σ-tree whose leftmost path is the same as w and other nodes are labelled with a. $\rho(t)$ is the ω-word obtained by spelling out the labels of the leftmost path of t.

THEOREM 4.1. For $i=1, \cdots, 6$, if L is in $\mathcal{NT}_i$ (resp. $\mathcal{DT}_i$), then $\rho(L)$ is in $\mathcal{N}_i$ (resp. $\mathcal{D}_i$).

PROOF. CASE 1. $i=1$: Given a finite tree automaton $M=\langle S, \Sigma, \Delta, s_0, F\rangle$ with $L=L_1(M)$, we construct a finite automaton $\tilde{M}=\langle\tilde{S}, \Sigma, \tilde{\Delta}, \tilde{s}_0, \tilde{F}\rangle$ as follows: For s in S, let $M_s=\langle S, \Sigma, \Delta, s, F\rangle$ and let $S_1=\{s\in S|L_1(M_s)\neq\emptyset\}$. Then we define as follows:

(1) $\tilde{S}=S\cup\{\bot\}$, where $\bot\notin S$.

(2) $\tilde{s}_0=s_0$.

(3) For each s in S and σ in Σ, $\tilde{\Delta}(s, \sigma)$ is equal to the set $\{p|\text{There is }(p, q)\in\Delta(s, \sigma)\text{ with }q\in S_1\}$ if this set is not empty else $\{\bot\}$. And $\tilde{\Delta}(\bot, \sigma)=\{\bot\}$.

(4) $\tilde{F}=F$.

If t is in L, then there exists a run r of M on t such that for any path π in T_2, $In(r|\pi)\cap F\neq\emptyset$. Therefor $r(0^n1)$ is in S_1 for all $n\geq0$. Hence $\tilde{r}$, defined by $\tilde{r}(x)=r(x)$ for x in T_1, is a run of $\tilde{M}$ on $\rho(t)$. Moreover $\tilde{r}$ is a C_1-accepting run since $In(r|\pi_0)\cap F\neq\emptyset$ for the leftmost path π_0 in T_2. Thus $\rho(L)\subseteq L_1(\tilde{M})$. Conversely if w is in $L_1(\tilde{M})$, then it follows from the definitions of S_1 and $\tilde{\Delta}$ that there exist a Σ-tree t and a C_1-accepting run of M on t such that $w=\rho(t)$. Therefore $L_1(\tilde{M})\subseteq\rho(L)$. Thus $\rho(L)=L_1(\tilde{M})$.

CASE 2. $i=2$: This case can be proved in the same way as Case 1.

CASE 3. $i=3$: Let $M=\langle S, \Sigma, \Delta, s_0, F\rangle$ be a finite tree automaton with $L=L_3(M)$. Without loss of generality, we may assume that s_0 is not in F. We construct a finite automaton $\tilde{M}=\langle\tilde{S}, \Sigma, \tilde{\Delta}, \tilde{s}_0, \tilde{F}\rangle$ satisfying $\rho(L)=L_3(\tilde{M})$ in the following way: Let $S_3=\{s\in S|L_3(M_s)\neq\emptyset\}$. Notice that if s is in F then $L_3(M_s)=\Sigma^{T_2}$. Hence $F\subseteq S_3$. For each s in S and σ in Σ, let $S_{(s, \sigma)}=\{p|\text{There is }(p, q)\in\Delta(s, \sigma)\text{ with }q\in S_3\}$.

(1) $\tilde{S} = S \cup \{\bot, \top\}$, where $\bot, \top \notin S$.

(2) $\tilde{s}_0 = s_0$.

(3) For each s in S and σ in Σ, $\tilde{\Delta}(s, \sigma)$ is defined by

$$\tilde{\Delta}(s, \sigma) = \begin{cases} \bot & \text{if } S_{(s, \sigma)} = \varnothing \\ \top & \text{if } S_{(s, \sigma)} \cap F \neq \varnothing \\ S_{(s, \sigma)} & \text{otherwise} \end{cases}$$

and $\tilde{\Delta}(\bot, \sigma) = \{\bot\}$, $\tilde{\Delta}(\top, \sigma) = \{\top\}$.

(4) $\tilde{F} = \{\top\}$.

Let t be in L and let r be a C_3-accepting run of M on t. Then we inductively define $\tilde{r}: T_1 \to \tilde{S}$ as follows: Let $\tilde{r}(\varepsilon) = r(\varepsilon) = s_0$. Assume that $\tilde{r}(0^n)$ is defined. If $\tilde{r}(0^n)$ is in S, then two cases arise. First if $r(0^{n+1})$ is in F, then $\tilde{r}(0^{n+1}) = \top$. If $r(0^{n+1})$ is $S - F$, then $\tilde{r}(0^{n+1}) = r(0^{n+1})$. If $\tilde{r}(0^n) = \top$, then $\tilde{r}(0^{n+1}) = \top$. Since $r(\pi_0) \cap F \neq \varnothing$ for the leftmost path π_0, there exists an integer $n_0 \geq 0$ such that $\tilde{r}(0^n)$ is in $S - F$ for $0 \leq n \leq n_0$ and $\tilde{r}(0^n) = \top$ for all $n > n_0$. Since r is a C_3-accepting run of M on t and since $r(0^n)$ is in $S - F$ for $0 \leq n \leq n_0$, $r(0^{n+1}) \in S_{(r(0^n), t(0^n))}$ for $0 \leq n \leq n_0$ and $r(0^{n_0+1}) \in S_{(r(0^m), t(0^m))} \cap F$, where $m = n_0$. Therefore $\tilde{r}$ is a C_3-accepting run of $\tilde{M}$ on $\rho(t)$. Hence $\rho(L) \subseteq L_3(\tilde{M})$. Conversely, let $\tilde{r}$ be a C_3-accepting run of $\tilde{M}$ on w in $L_3(\tilde{M})$. Then by (3) and (4), there exists an integer $n_0 \geq 0$ such that $\tilde{r}(0^n)$ is in $S - F$ for $0 \leq n \leq n_0$ and $\tilde{r}(0^n) = \top$ for all $n > n_0$. Furthermore for each n, $0 \leq n \leq n_0$, there is q_n in S_3 such that $(\tilde{r}(0^{n+1}), q_n) \in \Delta(\tilde{r}(0^n), w(0^n))$ and $\tilde{r}(0^{n_0+1}) \in F \subseteq S_3$. Therefore by the definition of S_3, we can see that there exist a Σ-tree t with $\rho(t) = w$ and a C_3-accepting run of M on t.

CASE 4. $i = 4$: This case can be proved in a manner similar to Case 1.

CASE 5. $i = 5$: For a finite tree automaton $M = \langle S, \Sigma, \Delta, s_0, \mathscr{F} \rangle$ with $L = L_5(M)$, we construct a finite automaton $\tilde{M} = \langle \tilde{S}, \Sigma, \tilde{\Delta}, \tilde{s}_0, \tilde{\mathscr{F}} \rangle$ as follows: Let $S_5 = \{s \in S \mid L_5(M_s) \neq \varnothing\}$. Then we define as follows:

(1) $\tilde{S} = S \cup \{\bot\}$, where $\bot \notin S$.

(2) $\tilde{s}_0 = s_0$.

(3) For each s in S and σ in Σ, $\tilde{\Delta}(s, \sigma)$ is equal to the set $\{p \mid \text{There is } (p, q) \in \Delta(s, \sigma) \text{ with } q \in S_5\}$ if this set is not empty else $\{\bot\}$. And $\tilde{\Delta}(\bot, \sigma) = \{\bot\}$.

(4) $\tilde{\mathscr{F}} = \mathscr{F}$.

Then by an argument similar to Case 1, we can prove $\rho(L) = L_5(\tilde{M})$.

CASE 6. $i = 6$: Let $M = \langle S, \Sigma, \Delta, s_0, \mathscr{F} \rangle$ be a finite tree automaton satisfying $L = L_6(M)$. For $\varnothing \neq Q \subseteq S$ and s in S, let $\mathscr{F}_Q = \{F \subseteq S \mid F \cup Q \in \mathscr{F}\}$ and let $M_{(Q, s)} = \langle S, \Sigma, \Delta, s, \mathscr{F}_Q \rangle$. Then let $S_Q = \{s \in Q \mid L_6(M_{(Q, s)}) \neq \varnothing\}$. Without loss of generality, we may assume that every F in $\mathscr{F}$ contains the initial state s_0. For F in $\mathscr{F}$, an F-chain is a set $\{S_i \mid 0 \leq i \leq n\}$ of subsets of F such that $S_0 = \{s_0\}$, $S_n = F$, $S_i \subsetneqq S_{i+1}$ and $|S_i| = i + 1$ for $0 \leq i \leq n$. Then let $\mathscr{S}_F$ be the set of all F-chains. We define a finite automaton $\tilde{M} = \langle \tilde{S}, \Sigma, \tilde{\Delta}, \tilde{s}_0, \tilde{\mathscr{F}} \rangle$ as follows:

(1) $\tilde{S} = S \times P(S) \cup \{\bot\}$.

(2) $\tilde{s}_0 = (s_0, \{s_0\})$.

(3) For each s in S, σ in Σ and $Q \subseteq S$, $\tilde{\Delta}((s, Q), \sigma)$ is equal to the set $\{(p, Q \cup \{p\}) \mid \text{There is } (p, q) \in \Delta(s, \sigma) \text{ with } q \in S_{Q \cup \{q\}}\}$ if this set is not empty else $\{\bot\}$. And

$\tilde{\Delta}(\perp, \sigma) = \{\perp\}$.

(4) $\tilde{\mathcal{F}} = \{G \subseteq S \times P(S) \mid p_1(G) = F$ and $p_2(G) \in \mathcal{S}_F$ for some F in $\mathcal{F}\}$, where p_1 and p_2 are the projections to the first and second components, respectively.

For t in $L_6(M)$, let r be a C_6-accepting run of M on t. Let $s_n = r(0^n)$ and $F_n = \{s_i \mid 0 \leq i \leq n\}$ for $n \geq 0$. Then we define $\tilde{r}: T_1 \to \tilde{S}$ by $\tilde{r}(0^n) = (s_n, F_n)$ for $n \geq 0$. We show that $\tilde{r}$ is a C_6-accepting run of $\tilde{M}$ on $\rho(t)$. First note that $\tilde{r}(\varepsilon) = (s_0, \{s_0\})$. For each $n \geq 0$, consider the transition $(r(0^{n+1}), r(0^n 1)) \in \Delta(r(0^n), t(0^n))$. Since r is a C_6-accepting run of M on t, $r(0^n 1)$ is in $S_{F_n \cup \{r(0^n 1)\}}$. Therefore $\tilde{r}$ is a run of $\tilde{M}$ on $\rho(t)$. Since $r(\pi_0)$ is in $\mathcal{F}$ for the leftmost path π_0 in T_2, $F = \{s_n \mid n \geq 0\}$ is in $\mathcal{F}$. Since $F_0 = \{s_0\}$, $F_n \subseteq F_{n+1}$ for all $n \geq 0$ and $F = \bigcup_{n=0}^{\infty} F_n$, the set $\{(s_n, F_n) \mid n \geq 0\}$ is in $\tilde{\mathcal{F}}$. Hence $\tilde{r}$ is a C_6-accepting run of $\tilde{M}$ on $\rho(t)$. Thus $\rho(L) \subseteq L_6(\tilde{M})$. Conversely, for a C_6-accepting run $\tilde{r}$ of $\tilde{M}$ on w in $L_6(\tilde{M})$, we can find a Σ-tree t with $\rho(t) = w$ and a C_6-accepting run of M on t by using the definitions of S_Q and $\tilde{\Delta}$. Thus $L_6(\tilde{M}) \subseteq \rho(L)$.

In Cases 1-6, it is obvious that if M is deterministic then so is $\tilde{M}$.

REMARK. For a finite tree automaton $M = \langle S, \Sigma, \Delta, s_0, \mathcal{F} \rangle$, we define a finite automaton $M' = \langle S, \Sigma, \Delta', s_0, \mathcal{F} \rangle$ by setting $\Delta'(s, \sigma) = \{p \mid (p, q) \in \Delta(s, \sigma)\}$ for s in S and σ in Σ. Then M' does not necessarily accept $\rho(L_i(M))$ in the sense of C_i for $i = 1, \cdots, 6$. Namely, the definitions of $\tilde{M}$ in the proof of Theorem 4.1 are essential. For example, let $M_i = \langle S, \Sigma, \Delta_i, s_0, F_i \rangle$ be

$$S = \{s_0, s_1\},$$

$$\Sigma = \{a\},$$

$$\Delta_i(s_0, a) = \begin{cases} (s_1, s_0) & i = 3 \\ (s_0, s_1) & i \neq 3. \end{cases}$$

$$\Delta_i(s_1, a) = (s_1, s_1) \quad \text{for} \quad i = 1, \cdots, 6.$$

$$F_i = \begin{cases} \{s_1\} & i = 3 \\ \{s_0\} & i \neq 3. \end{cases}$$

Then $L_i(M_i) = \varnothing$ for each $i = 1, \cdots, 6$. However, $L_i(M_i') \neq \varnothing$ for each $i = 1, \cdots, 6$.

REMARK. In the proof of Theorem 4.1, $|\tilde{\mathcal{F}}| = |\mathcal{F}|$ for $i = 1, \cdots, 5$. However, for the C_6-acceptance, the size of $\tilde{\mathcal{F}}$ is increased in the construction of $\tilde{M}$.

The following embedding theorem establishes the relationship between $\mathfrak{N}_i$(resp. $\mathfrak{D}_i$) and $\mathfrak{N}\mathfrak{I}_i$(resp. $\mathfrak{D}\mathfrak{I}_i$) for $i = 1, \cdots, 6$.

THEOREM 4.2. For $i = 1, \cdots, 6$ and $L \subseteq \Sigma^{T_1}$, L is in $\mathfrak{N}_i$(resp. $\mathfrak{D}_i$) if and only if $\iota(L)$ is in $\mathfrak{N}\mathfrak{I}_i$(resp. $\mathfrak{D}\mathfrak{I}_i$).

PROOF. For $L \subseteq \Sigma^{T_1}$, let $M = \langle S, \Sigma, \Delta, s_0, \mathcal{F} \rangle$ be a finite automaton such that $L = L_i(M)$. Then a finite tree automaton $\tilde{M} = \langle \tilde{S}, \Sigma, \tilde{\Delta}, \tilde{s}_0, \tilde{\mathcal{F}} \rangle$ which accepts $\iota(L)$ in the sense of C_i is constructed as follows:

(1) $\tilde{S} = S \cup \{\perp, \top\}$, where $\perp, \top \notin S$.

(2) $\tilde{s}_0 = s_0$.

(3) For each s in S,

$$\tilde{\Delta}(s, \sigma) = \{(p, \top) \mid p \in \Delta(s, \sigma)\} \qquad \text{for } \sigma \text{ in } \Sigma,$$

$$\tilde{\Delta}(\top, a) = \{(\top, \top)\}, \ \Delta(\top, b) = \{(\bot, \bot)\},$$

$$\tilde{\Delta}(\bot, \sigma) = \{(\bot, \bot)\} \qquad \text{for } \sigma \text{ in } \Sigma.$$

$$(4) \quad \tilde{\mathscr{F}} = \begin{cases} \mathscr{F} \cup \{\{\top\}\} & \text{for } i = 1, 2, 3, 5 \\ \mathscr{F} \cup \{\{s_0, \top\}\} & \text{for } i = 4, 6. \end{cases}$$

Notice that if M is deterministic then so is $\tilde{M}$.

The converse direction follows from Theorem 4.1.

5. Comparison of the Classes Concerned

This section relates the classes $\mathscr{N}\mathscr{T}_i$ and $\mathscr{D}\mathscr{T}_i$ for $i = 1, \cdots, 6$. The results are summarized in the lattice diagram in Fig. 1, where an unbroken line indicates a proper containment and a broken line means a containment whose properness is unknown. One can also draw a similar diagram for the deterministic classes.
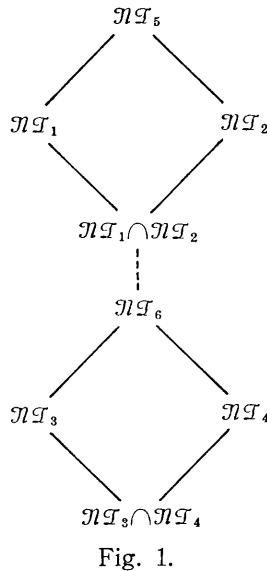

Fig. 1.

NOTATION. Let t be in Σ^{T_2}. For a path $\pi = \{x_0 < x_1 < \cdots x_n < \cdots\}$ in T_2, we denote by $[t \mid \pi] \in \Sigma^{T_1}$ a function defined by $[t \mid \pi](0^n) = t(x_n)$ for $n \geq 0$.

PROPOSITION 5.1. (1) $\mathscr{N}\mathscr{T}_1$ and $\mathscr{N}\mathscr{T}_2$ are incomparable.

(2) $\mathscr{D}\mathscr{T}_1$ and $\mathscr{D}\mathscr{T}_2$ are incomparable.

PROOF. Let $\Sigma = \{a, b\}$ and let

$$L_1 = t(L'), \text{ where } L' = (ba^*)^\omega \subseteq \Sigma^{T_1},$$

$$L_2 = \{t \in \Sigma^{T_2} \mid [t \mid \pi] \in \Sigma^* a^\omega \subseteq \Sigma^{T_1} \text{ for every path } \pi \text{ in } T_2\}.$$

We can easily construct a deterministic finite tree automaton which accepts L_i in the

sense of C_i for $i=1, 2$. Since L' is known not to be in F_σ^R[2, 11] and since $\mathfrak{N}_2=F_\sigma^R$ from Table 1, it follows from Theorem 4.2 that L_1 is not in $\mathfrak{N}\mathfrak{T}_2$. On the other hand, Rabin [7] has proved that L_2 is not in $\mathfrak{N}\mathfrak{T}_1$. Hence L_1 is in $\mathfrak{D}\mathfrak{T}_1-\mathfrak{N}\mathfrak{T}_2$ and L_2 is in $\mathfrak{D}\mathfrak{T}_2-\mathfrak{N}\mathfrak{T}_1$.

PROPOSITION 5.2. (1) $\mathfrak{N}\mathfrak{T}_3$ and $\mathfrak{N}\mathfrak{T}_4$ are incomparable.

(2) $\mathfrak{D}\mathfrak{T}_3$ and $\mathfrak{D}\mathfrak{T}_4$ are incomparable.

PROOF. These results follow from Theorem 4.2 and the fact that G^R and F^R in Table 1 are incomparable.

THEOREM 5.3. (1) $\mathfrak{N}\mathfrak{T}_3\cup\mathfrak{N}\mathfrak{T}_4\subsetneqq\mathfrak{N}\mathfrak{T}_6$.

(2) $\mathfrak{D}\mathfrak{T}_3\cup\mathfrak{D}\mathfrak{T}_4\subsetneqq\mathfrak{D}\mathfrak{T}_6$.

PROOF. By Theorem 3.1, we deal with a finite tree automaton $M=\langle S, \Sigma, \Delta, s_0, F\rangle$ with just one final set. Then we consider finite tree automata $M_0=\langle S, \Sigma, \Delta, s_0, \mathfrak{F}_0\rangle$ and $M_1=\langle S, \Sigma, \Delta, s_0, \mathfrak{F}_1\rangle$, where $\mathfrak{F}_0=\{G\,|\,G\cap F\neq\emptyset, G\subseteq S\}$ and $\mathfrak{F}_1=\{G\,|\,\emptyset\neq G\subseteq F\}$. Then it is not difficult to see that $L_3(M)=L_6(M_0)$ and $L_4(M)=L_6(M_1)$. By the definition, it is obvious that if M is deterministic then so are M_0 and M_1. The properness of the inclusions follows from Theorem 4.2 and the fact that $G^R\cup F^R\subsetneqq G_\delta^R\cap F_\sigma^R$ in Table 1.

THEOREM 5.4 [5]. (1) $\mathfrak{N}\mathfrak{T}_6\subseteq\mathfrak{N}\mathfrak{T}_1\cap\mathfrak{N}\mathfrak{T}_2$.

(2) $\mathfrak{D}\mathfrak{T}_6\subseteq\mathfrak{D}\mathfrak{T}_1\cap\mathfrak{D}\mathfrak{T}_2$.

PROOF. Let $M=\langle S, \Sigma, \Delta, s_0, \mathfrak{F}\rangle$ be a finite tree automaton. We construct a finite tree automaton $M'=\langle S', \Sigma, \Delta', s_0', F'\rangle$ such that $T_6(M)=T_1(M')=T_2(M')$ as follows:

(1) $S'=S\times P(S)$.

(2) $s_0'=(s_0, \{s_0\})$.

(3) For each $s\in S$, $Q\subseteq S$ and $\sigma\in\Sigma$, $\Delta'((s, Q), \sigma)=\{((p, Q_1), (q, Q_2))\,|\,(p, q)\in\Delta(s, \sigma), Q_1=Q\cup\{p\}, Q_2=Q\cup\{q\}\}$.

(4) $F'=\{(s, F)\,|\,s\in S, F\in\mathfrak{F}\}$.

As simulating M, M' stores the states occurring on the path from the root to the current node in the second component of its state. Let t be in $L_1(M')$ and let r' be a C_1-accepting run of M' on t. Then we define r by $r(x)=p_1(r'(x))$, where p_1 is the projection to the first component. By (3), r is a run of M on t. We show that r is a C_6-accepting run of M on t. For an arbitrary path $\pi=\{x_0<x_1<\cdots x_n<\cdots\}$ in T_2, let $r'(x_n)=(s_n, F_n)$ for $n\geq0$. Then by definition, we see that $F_n\subseteq F_{n+1}$ for $n\geq0$. Therefore there exists an integer n_0 such that $F_{n_0}=F_n$ for all $n\geq n_0$. Note that $F_{n_0}=\{s_n\,|\,n\geq0\}$. Since r' is a C_1-accepting run, $\mathrm{In}(r'\,|\,\pi)\cap F'\neq\emptyset$. Therefore there is (s, F) in F' such that $(s, F)=(s_n, F_n)$ for infinitely many $n\geq0$. Since $F_{n_0}=F_n$ for all $n\geq n_0$, $F=F_{n_0}$. Since $F_{n_0}=r(\pi)$, $r(\pi)$ is in $\mathfrak{F}$. Hence r is a C_6-accepting run of M on t. Thus $L_1(M')\subseteq L_6(M)$.

Conversely, let t be in $L_6(M)$ and let r be a C_6-accepting run of M on t. We define r' inductively. Let $r'(\varepsilon)=(s_0, \{s_0\})$. If $r'(x)=(s, Q)$ is defined, then let $r'(x0)=(r(x0), Q\cup\{r(x0)\})$ and $r'(x1)=(r(x1), Q\cup\{r(x1)\})$. Then r' is a run of M' on t since r is a run of M on t. Let $\pi=\{x_0<x_1<\cdots x_n<\cdots\}$ be an arbitrary path in T_2. Let $r'(x_n)=(s_n, F_n)$ for $n\geq0$. Since r is a C_6-accepting run, $r(\pi)=F$ for some F in $\mathfrak{F}$. Since $F_n\subseteq F_{n+1}$ and $F_n=\{s_i\,|\,0\leq i\leq n\}$, there exists an integer n_0 such that $F=F_n$ for all $n\geq n_0$. Since $\{s_n\,|\,n\geq n_0\}$ is finite, there exists s in S such that $(s, F)=(s_n, F_n)$ for

infinitely many $n \geq 0$. Therefore $In(r' | \pi) \cap F' \neq \emptyset$. Thus r' is a C_1-accepting run of M' on t. Hence $L_6(M) \subseteq L_1(M')$. Thus $L_6(M) = L_1(M')$.

We now prove $L_2(M') = L_6(M)$. We use the same definitions as those in the proof of $L_1(M') = L_6(M)$. Suppose that r' be a C_2-accepting run of M' on t. Then for every path π, $In(r' | \pi) \subseteq F'$. Since $In(r' | \pi) \neq \emptyset$, there exists (s, F) in $\mathcal{F}'$ such that $(s, F) = (s_n, F_n)$ for infinitely many $n \geq 0$. Since $F_n \subseteq F_{n+1}$ for $n \geq 0$, there exists an integer n_0 such that $F = F_n$ for all $n \geq n_0$. Therefore the run r defined by $r(x) = p_1(r'(x))$ for x in T_2 has the property that $r(\pi) = F$. Thus r is a C_6-accepting run of M on t. Hence $L_2(M') \subseteq L_6(M)$. To see the converse inclusion, it may be sufficient to note that in the proof of $L_6(M) \subseteq L_1(M')$ we have $(s_n, F_n) = (s_n, F)$ for all $n \geq n_0$, where F is in $\mathcal{F}$. Thus $In(r' | \pi) \subseteq F'$. Therefore r' is also a C_2-accepting run of M' on t. Hence $L_6(M) \subseteq L_2(M')$.

By the construction, it is also obvious that if M is deterministic then so is M'.

REMARK. It is an open problem whether the inclusion is proper in Theorem 5.4.

THEOREM 5.5.   (1) $\mathfrak{NT}_1 \cup \mathfrak{NT}_2 \subseteq \mathfrak{NT}_5$

(2) $\mathfrak{DT}_1 \cup \mathfrak{DT}_2 \subsetneqq \mathfrak{DT}_5$.

PROOF. Given a finite tree automaton $M = \langle S, \Sigma, \Delta, s_0, F \rangle$, we define in the same way as Theorem 5.3 finite tree automata $M_0 = \langle S, \Sigma, \Delta, s_0, \mathcal{F}_0 \rangle$ and $M_1 = \langle S, \Sigma, \Delta, s_0, \mathcal{F}_1 \rangle$, where $\mathcal{F}_0 = \{G \mid G \cap F \neq \emptyset, G \subseteq S\}$ and $\mathcal{F}_1 = \{G \mid \emptyset \neq G \subseteq F\}$. Then we can see that $L_1(M) = L_5(M_0)$ and $L_2(M) = L_5(M_1)$.

We now prove the properness of the inclusions. Let $\Sigma = \{a, b\}$ and let

$$L_1 = \iota(L'), \text{ where } L' = (ba^*)^\omega \subseteq \Sigma^{T_1},$$

$$L_2 = \{t \in \Sigma^{T_2} \mid [t | \pi] \in a\Sigma^* a^\omega \text{ for every path } \pi \subseteq T_2\}.$$

We will show that $L = L_1 \cup L_2$ is not in $\mathfrak{NT}_1 \cup \mathfrak{NT}_2$. Assume that L is in $\mathfrak{NT}_1$. Let $M = \langle S, \Sigma, \Delta, s_0, F \rangle$ be a finite tree automaton accepting L in the sense of C_1. One can modify M to get a finite tree automaton M_2 which accepts L_2 in the sense of C_1 since for each $t \in L_1$, $t(\varepsilon) = b$ and on the other hand for each $t \in L_2$, $t(\varepsilon) = a$. By the result due to Rabin [7], L_2 is not in $\mathfrak{NT}_1$. Hence L is not in $\mathfrak{NT}_1$. Now assume that L is in $\mathfrak{NT}_2$. Then by the same consideration, one can get a finite tree automaton which accepts L_1 in the sense of C_2. In the same way as Proposition 5.1, this yields a contradiction. Hence L is not in $\mathfrak{NT}_2$. Therefore L in not in $\mathfrak{NT}_1 \cup \mathfrak{NT}_2$.

We can easily construct a deterministic finite tree automaton which accepts L in the sense of C_5.

6. Structural Measures for Finite Tree Automata

In this section we introduce two structural measures for finite tree automata to give further observations on $\mathfrak{NT}_i$ for $i = 1, \cdots, 6$. Let $M = \langle S, \Sigma, \Delta, s_0, \mathcal{F} \rangle$ be a finite tree automaton. The first measure is the *nondeterministic degree* of M, denoted by $ndeg(M)$, which is defined to be $\max\{|\Delta(s, \sigma)| \mid s \in S, \sigma \in \Sigma\}$. The second measure is the number $|\mathcal{F}|$ of final sets. As shown in Section 3, the size of $\mathcal{F}$ does not affect the C_i-acceptances for $i = 1, \cdots, 4$.

DEFINITION. For $i = 1, \cdots, 6$ and an integer $k \geq 1$, a set L in $\mathfrak{NT}_i$ is said to be of

nondeterministic degree k if $k=\min\{ndeg\ (M)\,|\,L=L_i(M)\}$. Then we define

$$k-\mathcal{N}\mathcal{T}_i=\{L\in\mathcal{N}\mathcal{T}_i\,|\,L\ \text{is of nondeterministic degree}\ l\ \text{with}\ 1\leqq l\leqq k\}\,.$$

A deterministic finite automaton is of nondeterministic degree one. The following theorem refines the result in [9] which separates the nondeterminsitic classes from the deterministic classes.

THEOREM 6.1. *For* $i=1,\ \cdots,\ 6$ *and for* $k\geqq1$, $k-\mathcal{N}\mathcal{T}_i\subsetneqq(k+1)-\mathcal{N}\mathcal{T}_i$.

PROOF. Let $\Sigma=\{a,\ b\}$. For $n\geqq1$, let t_n be the Σ-tree defined by

$$t_n(x)=\begin{cases} b & \text{if}\ |x|=n \\ a & \text{otherwise} \end{cases}$$

Let $L_k=\{t_1,\ \cdots,\ t_k\}$. We show that L_{k+1} is in $(k+1)-\mathcal{N}\mathcal{T}_i$ but not in $k-\mathcal{N}\mathcal{T}_i$ for $i=1,\ \cdots,\ 6$. One can easily construct a nondeterministic finite tree automaton of nondeterministic degree $k+1$ which accepts L_{k+1} in the sense of C_i. Suppose that L_{k+1} is accepted in the sence of C_i by a nondeterministic finite tree automaton $M=\langle S,\ \Sigma,\ \Delta,\ s_0,\ \mathcal{F}\rangle$ with $ndeg(M)\leqq k$. Let r_n be a C_i-accepting run of M on t_n for $n=1,\ \cdots,\ k+1$. Since $|\Delta(s_0,\ a)|\leqq k$, there exist $m,\ n,\ 1\leqq m<n\leqq k+1$, such that $r_m(\varepsilon)=r_n(\varepsilon)=s_0$ and $(r_m(0),\ r_m(1))=(r_n(0),\ r_n(1))$. Let t be a Σ-tree defined by

$$t(x)=\begin{cases} b & \text{if either}\ |x|=m\ \text{and}\ 0\leqq x\ \text{or}\ |x|=n\ \text{and}\ 1\leqq x \\ a & \text{otherwise} \end{cases}$$

Then from r_m and r_n we can construct a C_i-accepting run of M on t in the following way. We prune the subtree at the node 1 from r_m and instead of it we graft the subtree of r_n at the node 1. Then the resulting run is also a C_i-accepting run of M on t. However, t does not belong to L_{k+1}, a contradiction. Hence L_{k+1} is not in $k-\mathcal{N}\mathcal{T}_i$.

DEFINITION. The *Muller degree* of a language L in $\mathcal{N}\mathcal{T}_5$, denoted by $mdeg(L)$, is the minimum number of final sets sufficient for its acceptance in the sense of C_5, i.e.

$$mdeg(L)=\min\{k\,|\,L=L_5(M)\ \text{for}\ M=\langle S,\ \Sigma,\ \Delta,\ s_0,\ \mathcal{F}\rangle\ \text{with}\ k=|\mathcal{F}|\}\,.$$

THEOREM 6.2. *For each* $k\geqq0$, *there exists a language* L_k *in* $\mathcal{N}\mathcal{T}_5$ *such that* $k=mdeg(L_k)$.

PROOF. For $n\geqq1$, let t_n be the Σ-tree such that $[t_n|\pi]=(a^n b)^\omega$ for each path π in T_2. Let $L_0=\varnothing$ and $L_k=\{t_1,\ \cdots,\ t_k\}$ for $k\geqq1$. It is clear that $mdeg(L_0)=0$. We will prove that $mdeg(L_k)=k$ for $k\geqq1$. One can easily construct a deterministic finite tree automaton $M=\langle S,\ \Sigma,\ \Delta,\ s_0,\ \mathcal{F}\rangle$ such that $k=|\mathcal{F}|$ and $L_k=L_5(M)$. Thus $mdeg(L_k)\leqq k$. Assume that $mdeg(L_k)=m<k$. Then there exists a nondeterministic finite tree automaton $M=\langle S,\ \Sigma,\ \Delta,\ s_0,\ \mathcal{F}\rangle$ such that $m=|\mathcal{F}|$ and $L_k=L_5(M)$. Then by Theorem 4.1, there exists a finite automaton $\tilde{M}=\langle\tilde{S},\ \Sigma,\ \tilde{\Delta},\ \tilde{s}_0,\ \tilde{\mathcal{F}}\rangle$ such that $m=|\tilde{\mathcal{F}}|$ and $\rho(L_k)=L_5(\tilde{M})$ because the construction of $\tilde{M}$ from M preserves the number of final sets. Since $\rho(L_k)=(ab)^\omega\cup(a^2b)^\omega\cup\cdots\cup(a^kb)^\omega$ and $m<k$, there exist $i\neq j$ such that both $(a^ib)^\omega$ and $(a^jb)^\omega$ can be accepted with the same final set F in $\tilde{\mathcal{F}}$. Then there is $w\in\Sigma^*$ such that $a^ibw(a^jb)^\omega$ can be accepted by $\tilde{M}$ with F in the sense of C_5. This is a contradiction. Hence $mdeg(L_k)=k$.

Remark. Theorem 6.2 still holds when we restrict the case to the deterministic C_5-acceptance. By the results in Section 3, we can also define a similar degree for a language in $\mathfrak{N}\mathcal{T}_6$. Although the language L_k defined in the proof of Theorem 6.2 also seems to be a candidate which has exactly degree k in this case, we have not proved this yet.

7. Concluding Remarks

By the results in Sections 4-6, the class $\mathfrak{N}\mathcal{T}_5$ that is studied in [6] is the largest one among the classes defined by finite tree automata. This class is closed under Boolean operations [6] and for each L in $\mathfrak{N}\mathcal{T}_5$, $\rho(L)$ is an ω-regular language. Hence the class $\mathfrak{N}\mathcal{T}_5$ may be considered to be a natural extension of the class of ω-regular languages to infinite tree languages.

In [5], Moriya has characterized $\mathfrak{N}\mathcal{T}_3$ and $\mathfrak{N}\mathcal{T}_4$ in terms of general topology on Σ^{T_2}. We have not considered this approach in detail.

We have ignored the classes $\mathfrak{N}_i^{(k)}$ for $k>2$. However, the arguments in this paper are easily applicable to $\mathfrak{N}_i^{(k)}$ and we can get similar results for $\mathfrak{N}_i^{(k)}$ for $k>2$. For instance, Theorems 4.1 and 4.2 can be generalized as follows: Let $\iota_k: \Sigma^{T_k} \to \Sigma^{T_{k+1}}$ be a function defined by $\iota_k(t)(x)=t(x)$ if x is in T_k and $\iota_k(t)(x)=a$ if x is in $T_{k+1}-T_k$. Let $\rho_k: \Sigma^{T_{k+1}} \to \Sigma^{T_k}$ be a function defined by $\rho_k(t)=t|T_k$ for t in T_{k+1}. Then we have the following generalibations:

Theorem 4.1'. If L is in $\mathfrak{N}_i^{(k+1)}$, then $\rho_k(L)$ is in $\mathfrak{N}_i^{(k)}$ for $i=1, \cdots, 6$.

Theorem 4.2'. L is in $\mathfrak{N}_i^{(k)}$ if ond only if $\iota_k(L)$ is in $\mathfrak{N}_i^{(k+1)}$ for $i=1, \cdots, 6$.

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