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CATEGORICAL RELATIONAL DATABASE MODELS

By

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Abstract

A category of relational models and its quotient category are defined and their basic properties are discussed. Moreover a categorical join dependency and a category of adjoint models, which dualize the notion of relational models, are studied.

1. Introduction

Theory of categories [1, 8] founded by S. Mac Lane and S. Eilenberg can be applied to various fields of mathematics and has the advantage that it serves a global point of view for logical structures of many mathematical objects. For instance category theory has been a useful tool in studying theories of automata, mathematical languages, systems, graphs and programmings [1, 2, 3, 4, 5] as mathematical foundations of computer science.

The purpose of this paper is to investigate a categorical aspect of relational detabase models. This categorical viewpoint for theory of detabase models was initiated by the work of A. Kato [6, 7]. Given an image factorization system $(\mathcal{E}, \mathcal{M})$ of a category $\mathcal{C}$ and a functor $X:\mathcal{X}\rightarrow\mathcal{C}$, we define relational models and their morphisms in section 2. Then we can naturally obtain a category $R(X)$ of relational models and morphisms between them. Also we show some basic properties of the category $R(X)$ of relational models and consider an equivalence relation on the class of all morphisms in $R(X)$ to obtain a quotient category $R[X]$ of $R(X)$. In section 3 a join dependency associated with a family of natural transformations $k_\alpha : X\dot{\rightarrow}X_\alpha$ $(\alpha\in\Lambda)$ is introduced. The categorical join dependency is a reasonable abstraction of the usual join dependency and it induces a left adjoint functor from $R(X)$ into a full subcategory $D(X)$ of $R(X)$ consisting of all join dependent models. As in Kato [6] it turns out that the join dependency of a relational model is regarded as a notion in the quotient category $R[X]$. In the final section we introduce a notion of adjoint models as a dual of relational models and their morphisms in case that a functor $X:\mathcal{X}\rightarrow\mathcal{C}$ has a right adjoint functor $Y:\mathcal{C}\rightarrow\mathcal{X}$ and $\mathcal{X}$ has an image factorization system $(\mathcal{E}^*, \mathcal{M}^*)$. Then the category $R\{Y\}$ of adjoint models is naturally constructed and we extend Kato's result [6] that $R\{Y\}$ is equivalent to $R[X]$ as categories.

2. Categories of Relational Models

Relational database models due to E. F. Codd are defined as subsets of a cartesian product of sets indexed by a set of atributes. In this section we will generalize the

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notion of relational database models and define a category consisting of them. Throughout the rest of the paper we assume that an image factorization system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ and a functor $X:\mathcal{X}\to\mathcal{C}$ are given.

DEFINITION 2.1. A (relational) model (with respect to $X:\mathcal{X}\to\mathcal{C}$) is defined to be a pair (Q, q) of an object Q in $\mathcal{X}$ and a monomorphism $q:\bar{Q}\to XQ$ in $\mathcal{C}$ such that $q\in\mathcal{M}$. A morphism $f:(Q, q)\to(R, r)$ of a model (Q, q) into another model (R, r) is defined to be a morphism $f:Q\to R$ in $\mathcal{X}$ such that there exists a morphism $\bar{f}:\bar{Q}\to\bar{R}$ in $\mathcal{C}$ rendering the following square commutative:

$$\begin{array}{ccc} \bar{Q} & \overset{q}{\rightarrowtail} & XQ \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle Xf} \\ \bar{R} & \underset{r}{\rightarrowtail} & XR. \end{array}$$

(The uniqueness of $\bar{f}:\bar{Q}\to\bar{R}$ follows from the injectivity of $r:\bar{R}\to XR$. An arrow "$\rightarrowtail$" represents a monomorphism in $\mathcal{M}$.)

Let $R(X)$ be the category of all models and all morphisms between them. Identity morphisms and the composition of morphisms in $R(X)$ are given in a trivial fashion.

LEMMA 2.2. *A morphism $f:(Q, q)\to(R, r)$ is an isomorphism in $R(X)$ if and only if $f:Q\to R$ is an isomorphism in $\mathcal{X}$ and $\bar{f}:\bar{Q}\to\bar{R}$ is an epimorphism in $\mathcal{E}$.*

By analogy with A. Kato [6] we define an equivalence relation on the class of all morphisms in category $R(X)$ as follows:

DEFINITION 2.3. Let $f_1, f_2:(Q, q)\to(R, r)$ be two morphisms in $R(X)$. We say that f_1 is equivalent to f_2, denoted by $f_1\sim f_2:(Q, q)\to(R, r)$, if $\bar{f}_1:\bar{Q}\to\bar{R}$ and $\bar{f}_2:\bar{Q}\to\bar{R}$ (in Definition 2.1) are identical, that is, $\bar{f}_1=\bar{f}_2$.

It is obvious that the relation "$\sim$" is an equivalence relation. The next proposition states that this equivalence relation "$\sim$" is preserved by the composition of morphisms in $R(X)$.

PROPOSITION 2.4. *If $f_1\sim f_2:(Q, q)\to(R, r)$ and $g_1\sim g_2:(R, r)\to(S, s)$ in $R(X)$ then $g_1f_1\sim g_2f_2:(Q, q)\to(S, s)$.*

We are now ready to obtain a quotient category [8; p. 51] of $R(X)$ classifying morphisms by the equivalence relation "$\sim$" defined above. We denote the quotient category by $R[X]$. That is, objects in $R[X]$ are the same ones as in $R(X)$ and a morphism $[f]:(Q, q)\to(R, r)$ in $R[X]$ is an equivalence class of a morphism $f:(Q, q)\to(R, r)$ in $R(X)$.

PROPOSITION 2.5. *Let $f:(Q, q)\to(R, r)$ be a morphism in $R(X)$. If $f:Q\to R$ is an isomorphism in $\mathcal{X}$ and if there exists a morphism $g:(R, r)\to(Q, q)$ with $fg\sim 1_R$, then $f:(Q, q)\to(R, r)$ is an isomorphism in $R(X)$.*

In the above proposition the existence of a morphism $g:(R, r)\to(Q, q)$ with $fg\sim 1_R:(R, r)\to(R, r)$ means that $[f]:(Q, q)\to(R, r)$ is a retraction [5; p. 19] in $R[X]$.

COROLLARY 2.6. *Let $f:(Q, q)\to(R, r)$ be a morphism in $R(X)$. If $f:Q\to R$ is an isomorphism in $\mathcal{X}$ and if $[f]:(Q, q)\to(R, r)$ is an isomorphism in $R[X]$, then $f:(Q, q)\to(R, r)$ is an isomorphism in $R(X)$.*

By using the diagonal fill-in lemma [1, p. 39] we have the following lemma.

LEMMA 2.7. *Let $f:(Q, q)\to(R, r)$ be a morphism in $R(X)$. If there exists a morphism $g:R\to Q$ with $gf=1_Q$ and if $\bar{f}:\bar{Q}\to\bar{R}$ is in $\mathcal{E}$, then $[f]:(Q, q)\to(R, r)$ is an isomorphism in $R[X]$.*

EXAMPLE 2.8. Let C be a category with products and A a set. A is regarded as a discrete category. Denote by C^A the functor category of all functors from A into C, i.e., C^A is a product cotegory of A copies of C. Let $\prod_A : C^A\to C$ be the product functor $(\langle S_\alpha\rangle_{\alpha\in A}\mapsto\prod_\alpha S_\alpha)$. Then the category $R(\prod_A)$ of relational models is obtained and the case $C=$**Set** was considered by Kato [6].

3. Generalized Join Dependencies

In [6] Kato described that observations of join dependencies about relational models as stated in Example 2.8 are sufficient if they are considered in quotient category $R[X]$. This section generalizes the result due to Kato [6] for the category of relational models defined in the previous section.

As in the section 2, we assume that an image factorization system $(\mathcal{E}, \mathcal{M})$ in a category C and a functor $X:\mathcal{X}\to C$ are given. Moreover supposed that category C has products and pullbacks, and supposed that a family of functors $X_\alpha:\mathcal{X}\to C$ $(\alpha\in\Lambda)$ and a family of natural transformations $k_\alpha : X\dot{\to} X_\alpha$ $(\alpha\in\Lambda)$ are given.

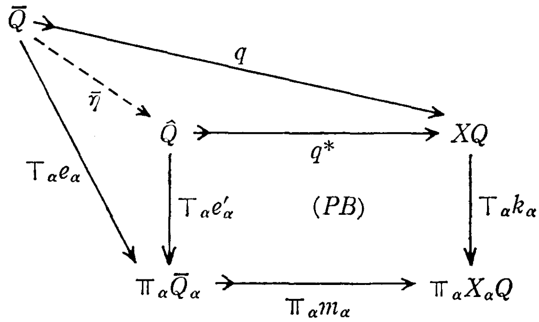
Let (Q, q) be an object of category $R(X)$. Then consider an $(\mathcal{E}, \mathcal{M})$-factorization of the composite $k_\alpha q:\bar{Q}\rightarrowtail XQ\to X_\alpha Q$ $(\alpha\in\Lambda)$, which is rendered by the commutative square

$$\begin{array}{ccc} \bar{Q} & \overset{q}{\rightarrowtail} & XQ \\ {\scriptstyle e_\alpha}\downarrow & & \downarrow{\scriptstyle k_\alpha} \\ \bar{Q}_\alpha & \underset{m_\alpha}{\rightarrowtail} & X_\alpha Q\,. \end{array}$$

(Notice that $k_\alpha=k_{\alpha,Q}:XQ\to X_\alpha Q$. An arrow "$\twoheadrightarrow$" represents an epimorphism in $\mathcal{E}$.) Let $pr_\alpha : \prod_\alpha X_\alpha Q\to X_\alpha Q$ $(\alpha\in\Lambda)$ and $\overline{pr}_\alpha : \prod_\alpha \bar{Q}_\alpha\to\bar{Q}_\alpha$ $(\alpha\in\Lambda)$ be systems of projections of products $\prod_\alpha X_\alpha Q$ and $\prod_\alpha\bar{Q}_\alpha$, respectiqely. We denote by $\top_\alpha k_\alpha : XQ\to\prod_\alpha X_\alpha Q$ a unique morphism with $pr_\alpha(\top_\alpha k_\alpha)=k_\alpha$ for each $\alpha\in\Lambda$ and by $\prod_\alpha m_\alpha : \prod_\alpha\bar{Q}_\alpha\to\prod_\alpha X_\alpha Q$ a unique morphism with $pr_\alpha(\prod_\alpha m_\alpha)=m_\alpha\overline{pr}_\alpha$ for each $\alpha\in\Lambda$. Construct a pullback of two morphisms $\top_\alpha k_\alpha$ and $\prod_\alpha m_\alpha$ with common codomain.

$$\begin{array}{ccc} \hat{Q} & \overset{q^*}{\dashrightarrow} & XQ \\ {\scriptstyle \top_\alpha e'_\alpha}\downarrow & (PB) & \downarrow{\scriptstyle \top_\alpha k_\alpha} \\ \prod_\alpha\bar{Q}_\alpha & \underset{\prod_\alpha m_\alpha}{\longrightarrow} & \prod_\alpha X_\alpha Q \end{array}$$

Trivially q^* is in $\mathcal{M}$ [1, Exercise 2.4.11; 2, Proposition 2.11] since m_α is in $\mathcal{M}$ for each $\alpha\in\Lambda$. Hence we have a new object (Q, q^*) in $R(X)$. By the universal property of pullbacks there exists a unique morphism $\bar{\eta}:\bar{Q}\to\hat{Q}$ in $\mathcal{X}$ making the following daigram commute.



This implies that $1_Q:(Q, q)\to(Q, q^*)$ is a morphism in $R(X)$. Note that $1_Q:Q\to Q$ is an identity morphism in $\mathcal{X}$ but $1_Q:(Q, q)\to(Q, q^*)$ is not always an identity morphism in $R(X)$. It follows at once from the construction of q^* that $q^{**}=q^*$ since e'_α is in $\mathcal{E}$ for each $\alpha\in\Lambda$.

PROPOSITION 3.1. *Let (Q, q) and (R, r) be two objects in $R(X)$. If $f:(Q, q)\to(R, r^*)$ is a morphism in $R(X)$, then so is $f:(Q, q^*)\to(R, r^*)$.*

COROLLARY 3.2. *If $f:(Q, q)\to(R, r)$ is a morphism in $R(X)$, then so is $f:(Q, q^*)\to(R, r^*)$.*

The last corollary shows that * gives a functor called the join functor:

$$*:R(X)\longrightarrow R(X)$$

$$\begin{array}{ccc} (Q, q) & & (Q, q^*) \\ f\downarrow & \longmapsto & \downarrow f \\ (R, r) & & (R, r^*) \end{array}.$$

During the rest of this section we assume that $\top_\alpha k_\alpha: XQ\to\Pi_\alpha X_\alpha Q$ is a monomorphism for each object Q in $\mathcal{X}$. The following results are basic properties of the join functor $*:R(X)\to R(X)$.

LEMMA 3.3. *Let $f, g:(Q, q)\to(R, r)$ be two morphisms in $R(X)$. If $f\sim g:(Q, q)\to(R, r)$, then $f\sim g:(Q, q^*)\to(R, r^*)$.*

COROLLARY 3.4. *An object (Q, q) in $R(X)$ is join dependent (JD), that is, $q=q^*$ as a subobject of XQ if and only if $[1_Q]:(Q, q)\to(Q, q^*)$ is an isomorphism in $R[X]$.*

THEOREM 3.5. *Let (Q, q) and (R, r) be two object in $R(X)$. If (R, r) is isomorphic to (Q, q) in $R[X]$ and if $q=q^*$ (JD), then $r=r^*$ (JD).*

Let $D(X)$ be the full subcategory of $R(X)$ consisting of all objects (Q, q) with $q=q^*$ (*JD*). Then Proposition 3.1 indicates that the inclusion functor $I:D(X)\subset R(X)$ is a right adjoint of the join functor $*:R(X)\to D(X)$:

$$\frac{(Q, q)\xrightarrow{f} I(R, r^*)}{(Q, q^*)\xrightarrow[f]{} (R, r^*)}.$$

EXAMPLE 3.6. Let $\{A_\alpha | \alpha\in\Lambda\}$ be a family of sets with $A=\bigcup_\alpha A_\alpha$. Moreover, let $\Pi_\alpha:\mathbf{Set}^A\to\mathbf{Sst}$ be a functor $(\langle S_a\rangle_{a\in A}\mapsto\Pi_{a\in A_\alpha}S_a)$ and $k_\alpha:\Pi_A\dot{\to}\Pi_\alpha$ a natural transformation of projections $(\alpha\in\Lambda)$. Under this situation the join dependency discussed in the present section is the usual join dependency [6].

4. Categories of Adjoint Models

In this section we will define a category of adjoint models in case that a functor $X:\mathcal{X}\to\mathcal{C}$ has a left adjoint functor $Y:\mathcal{C}\to\mathcal{X}$. Also a sufficient condition for the category of adjoint models to be isomorphic to the quotient category $R[X]$ of relational models will be given. In addition we assume that $Y\dashv X:\mathcal{C}\rightharpoonup\mathcal{X}$ is an adjunction [8, p. 78] and that $(\mathcal{E}^*, \mathcal{M}^*)$ is an image factorization system of $\mathcal{C}$.

DEFINITION 4.1. An adjoint model (with respect to an adjunction $Y\dashv X:\mathcal{C}\rightharpoonup\mathcal{X}$) is defined to be a pair $\langle\bar{Q}, \hat{q}\rangle$ of an object $\bar{Q}$ in $\mathcal{C}$ and an epimorphism $\hat{q}:Y\bar{Q}\to Q$ in $\mathcal{X}$ such that $\hat{q}\in\mathcal{E}^*$ and an adjoint $q:\bar{Q}\to XQ$ of $\hat{q}:Y\bar{Q}\to Q$ is in $\mathcal{M}$. A morphism $\bar{f}:\langle\bar{Q}, \hat{q}\rangle\to\langle\bar{R}, \hat{r}\rangle$ of an adjoint model $\langle\bar{Q}, \hat{q}\rangle$ into another adjoint model $\langle\bar{R}, \hat{r}\rangle$ is defined to be a morphism $\bar{f}:\bar{Q}\to\bar{R}$ in $\mathcal{C}$ such that there exists a morphism $f:Q\to R$ in $\mathcal{X}$ rendering the following square commutative:

$$\begin{array}{ccc} Y\bar{Q} & \overset{\hat{q}}{\twoheadrightarrow} & Q \\ {\scriptstyle Y\bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y\bar{R} & \underset{\hat{r}}{\twoheadrightarrow} & R\,. \end{array}$$

(The uniqueness of $f:Q\to R$ follows from the surjectivity of $\hat{q}:Y\bar{Q}\to Q$. Arrows "$\rightarrowtail$" and "$\twoheadrightarrow$" in $\mathcal{X}$ represent monomorphisms in $\mathcal{M}^*$ and epimorphisms in $\mathcal{E}^*$, respectively.)

Let $R\{Y\}$ be the category of all adjoint models and all morphisms between them. Identity morphisms and the composition of morphisms in $R\{Y\}$ are given as usual.

Taking an adjoint of the last commutative square we have the following commutative diagrams:

$$\langle\bar{Q}, \hat{q}\rangle \xrightarrow{\ \bar{f}\ } \langle\bar{R}, \hat{r}\rangle$$

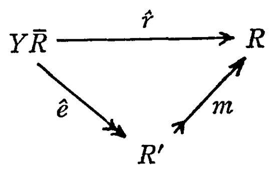
$$\dfrac{\begin{array}{ccc} Y\bar{Q} & \xrightarrow{Y\bar{f}} & Y\bar{R} \\ {\scriptstyle \hat{q}}\downarrow & & \downarrow{\scriptstyle \hat{r}} \\ Q & \xrightarrow[f]{} & R \end{array}}{\begin{array}{ccc} \bar{Q} & \xrightarrow{\bar{f}} & \bar{R} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ XQ & \xrightarrow[Xf]{} & XR \end{array}}\,.$$

Thus we naturally obtain a functor

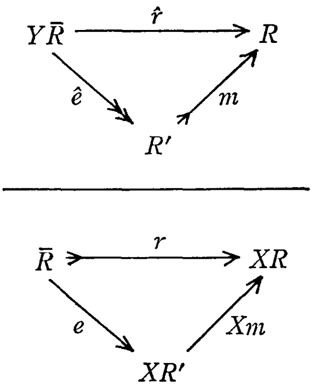
$$\begin{array}{rcl} F: R\{Y\} & \longrightarrow & R(X) \\ \langle\bar{Q}, \hat{q}\rangle & & (Q, q) \\ {\scriptstyle \bar{f}}\downarrow & \longmapsto & \downarrow{\scriptstyle f} \\ \langle\bar{R}, \hat{r}\rangle & & (R, r)\,. \end{array}$$

The functor $F: R\{Y\} \to R(X)$ is called *the Armstrong functor* because it corresponds to a construction [9] of characteristic relations (data models) from data models defined by first order predicates satisfying equivalence laws.

Let (R, r) be an object in $R(X)$ and let

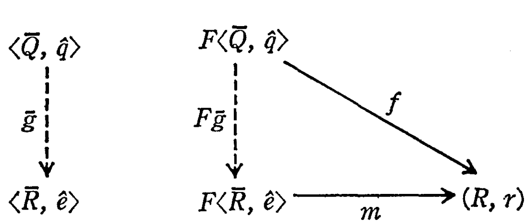


be an $(\mathcal{E}^*, \mathcal{M}^*)$-factorization of an adjoint $\hat{r}$ of r. Consider an adjoint of the last commutative triangle:



Since $Xm \cdot e = r \in \mathcal{M}$ by definition of (R, r) it follows that $e \in \mathcal{M}$ and hence we obtain an object $\langle \bar{R}, \hat{e} \rangle$ in $R\{Y\}$. It is trivial that $F\langle \bar{R}, \hat{e} \rangle = (R', e)$ and $m: (R', e) \to (R, r)$ is a morphism in $R(X)$, where $\bar{m} = 1_{\bar{R}}$. The next lemma shows that morphism $m: F\langle \bar{R}, \hat{e} \rangle \to (R, r)$ in $R(X)$ is a universal arrow [8] from F to (R, r) and the object $\langle \bar{R}, \hat{e} \rangle$ in $R\{Y\}$ is cofree [1] over (R, r).

LEMMA 4.2. *Let $\langle \bar{Q}, \hat{q} \rangle$ be an object in $R\{Y\}$ and (R, r) an object in $R(X)$. For any morphism $f: F\langle \bar{Q}, \hat{q} \rangle \to (R, r)$ in $R(X)$ there exists a unique morphism $\bar{g}: \langle \bar{Q}, q \rangle \to \langle \bar{R}, \hat{e} \rangle$ in $R\{Y\}$ such that the triangle*

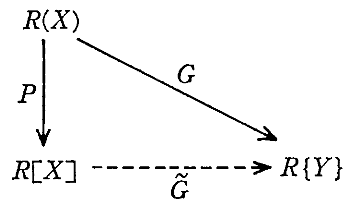


is commutative.

COROLLARY 4.3. *Armstrong functor $F: R\{Y\} \to R(X)$ has a right adjoint functor $G: R(X) \to R\{Y\}$ $((R, r) \mapsto \langle \bar{R}, \hat{e} \rangle)$ and $m: (R', e) \to (R, r)$ is a component of count $FG \dot{\to} 1_{R(X)}$ of adjunction $F \dashv G: R\{Y\} \rightharpoonup R(X)$.*

Denote by $\tilde{F}: R\{Y\} \to R[X]$ the composite of Armstrong functor $F: R\{Y\} \to R(X)$

followed by quotient functor $P: R(X) \to R[X]$ (cf. §2). From definition of functor G there exist a unique functor $\tilde{G}: R[X] \to R\{Y\}$ rendering the triangle of functors



commutative. Since $GF = 1_{R\{X\}}$ (the identity functor of $R\{Y\}$), it is immediate that $\tilde{G}\tilde{F} = 1_{R\{Y\}}$. The totality of all morphisms $[m]: \tilde{F}\tilde{G}(R, r) = (R', e) \to (R, r)$ for all object (R, r) in $R(X)$ constitutes a natural transformation $[m]: \tilde{F}\tilde{G} \dot{\to} 1_{R[X]}$.

THEOREM 4.4. *If every monomorphism in $\mathcal{M}^*$ is a split monomorphisms* [8; *p.* 19], *then $R\{Y\}$ is equivalent to $R[X]$, i.e., $[m]: \tilde{F}\tilde{G} \dot{\to} 1_{R[X]}$ is a natural isomorphism.*

The functor $\pi_A: C^A \to C$ in Example 2.8 has a left adjoint $\Delta_A: C \to C^A$ since C has products. In the case $C = \mathbf{Set}_0$, the category of nonempty sets, we have $\mathbf{Set}_0[\pi_A] \cong \mathbf{Set}_0\{\Delta_A\}$ (an equivalence of categories) because every monomorphism in $\mathcal{M}^* = \langle \mathbf{Mon} \rangle_{a \in A}$ is a split monomorphism. This case was initially observed by Kato [6, Theorem 5.1].

References

[1] ARBIB, M. A. and MANES, E. G.: *Arrows, structures, and functors, The categorical imperative*, Academic Press, New York, 1975.

[2] ———: *Foundations of system theory: The Hankel matrix*, J. Comput. System Sci., **20** (1980), 330-378.

[3] ———: *Generalized Hankel matrices and system realization*, SIAM J. Math. Anal., **11** (1980), 405-424.

[4] EHRIG, H., KREOWSKI, H.-J., MAGGIOLO-SCHETTINI, A., ROSEN, B. K. and WINKOWSKI, J.: *Transformations of structures: an algebraic approach*, Math. Systems Theory, **14** (1981), 305-334.

[5] HUZINO, S.: *Foundations of mathematical programmings* (Japanese), Asakura Shoten, Tokyo, 1979.

[6] KATO, A.: *A categorical approach to the representative independency of relational data models and their application to dependencies* (Japanese), Kyoto Univ. Sûrikaisekikenkyûsho Kôkyûroku No. **423** (1981), 275-293.

[7] ———: *An abstract relational model and natural join functors*, Bull. Infromatics and Cybernetics, **20** (1983), 95-106.

[8] MAC LANE, S.: *Categories for the working mathematicans*, Springer-Verlag, New York, 1971.

[9] TAKESHIMA, T.: *On a generalization of relational data model dependencies by first order predicates satisfying equivalence laws* (Japanese), Kyoto Univ. Sûrikaisekikenkyûsho Kôkyûroku No. **423** (1981), 257-274.

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