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# CATEGORICAL RELATIONAL DATABASE MODELS

By

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#### Abstract

A category of relational models and its quotient category are defined and their basic properties are discussed. Moreover a categorical join dependency and a category of adjoint models, which dualize the notion of relational models, are studied.

#### 1. Introduction

Theory of categories [1, 8] founded by S. Mac Lane and S. Eilenberg can be applied to various fields of mathematics and has the advantage that it serves a global point of view for logical structures of many mathematical objects. For instance category theory has been a useful tool in studying theories of automata, mathematical languages, systems, graphs and programmings [1, 2, 3, 4, 5] as mathematical foundations of computer science.

The purpose of this paper is to investigate a categorical aspect of relational detabase This categorical viewpoint for theory of detabase models was initiated by the models. work of A. Kato [6, 7]. Given an image factorization system ( $\mathcal{E}$ ,  $\mathcal{M}$ ) of a category  $\mathcal{C}$ and a functor  $X: \mathfrak{X} \rightarrow \mathcal{C}$ , we define relational models and their morphisms in section 2. Then we can naturally obtain a category R(X) of relational models and morphisms between them. Also we show some basic properties of the category R(X) of relational models and consider an equivalence relation on the class of all morphisms in R(X) to obtain a quotient category R[X] of R(X). In section 3 a join dependency associated with a family of natural transformations  $k_{\alpha}: X \to X_{\alpha} \ (\alpha \in A)$  is introduced. The categorical join dependency is a reasonable abstraction of the usual join dependency and it induces a left adjoint functor from R(X) into a full subcategory D(X) of R(X) consisting of all join dependent models. As in Kato [6] it turns out that the join dependency of a relational model is regarded as a notion in the quotient category R [X]. In the final section we introduce a notion of adjoint models as a dual of relational models and their morphisms in case that a functor  $X: \mathcal{X} \to \mathcal{C}$  has a right adjoint functor  $Y: \mathcal{C} \to \mathcal{X}$ and  $\mathcal{X}$  has an image factorization system ( $\mathcal{E}^*, \mathcal{M}^*$ ). Then the category  $R\{Y\}$  of adjoint models is naturally constructed and we extend Kato's result [6] that  $R\{Y\}$  is equivalent to R[X] as categories.

#### 2. Categories of Relational Models

Relational database models due to E.F. Codd are defined as subsets of a cartesian product of sets indexed by a set of atributes. In this section we will generalize the

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notion of relational database models and define a category consisting of them. Throughout the rest of the paper we assume that an image factorization system  $(\mathcal{C}, \mathcal{M})$  in a category  $\mathcal{C}$  and a functor  $X: \mathfrak{X} \rightarrow \mathcal{C}$  are given.

DEFINITION 2.1. A (relational) model (with respect to  $X: \mathfrak{X} \to \mathcal{C}$ ) is defined to be a pair (Q, q) of an object Q in  $\mathfrak{X}$  and a monomorphism  $q: \overline{Q} \to XQ$  in  $\mathcal{C}$  such that  $q \in \mathcal{M}$ . A morphism  $f: (Q, q) \to (R, r)$  of a model (Q, q) into another model (R, r) is defined to be a morphism  $f: Q \to R$  in  $\mathfrak{X}$  such that there exists a morphism  $\overline{f}: \overline{Q} \to \overline{R}$  in  $\mathcal{C}$  rendering the following square commutative:



(The uniqueness of  $\overline{f}: \overline{Q} \to \overline{R}$  follows from the injectivity of  $r: \overline{R} \to XR$ . An arrow " $\to$ " represents a monomorphism in  $\mathcal{M}$ .)

Let R(X) be the category of all models and all morphisms between them. Identity morphisms and the composition of morphisms in R(X) are given in a trivial fashion.

LEMMA 2.2. A morphism  $f:(Q, q) \rightarrow (R, r)$  is an isomorphism in R(X) if and only if  $f:Q \rightarrow R$  is an isomorphism in  $\mathfrak{X}$  and  $\overline{f}:\overline{Q} \rightarrow \overline{R}$  is an epimorphism in  $\mathcal{E}$ .

By analogy with A. Kato [6] we define an equivalence relation on the class of all morphisms in category R(X) as follows:

DEFINITION 2.3. Let  $f_1, f_2: (Q, q) \to (R, r)$  be two morphisms in R(X). We say that  $f_1$  is equivalent to  $f_2$ , denoted by  $f_1 \sim f_2: (Q, q) \to (R, r)$ , if  $\overline{f_1}: \overline{Q} \to \overline{R}$  and  $\overline{f_2}: \overline{Q} \to \overline{R}$  (in Definition 2.1) are identical, that is,  $\overline{f_1}=\overline{f_2}$ .

It is obvious that the relation " $\sim$ " is an equivalence relation. The next proposition states that this equivalence relation " $\sim$ " is preserved by the composition of morphisms in R(X).

PROPOSITION 2.4. If  $f_1 \sim f_2: (Q, q) \rightarrow (R, r)$  and  $g_1 \sim g_2: (R, r) \rightarrow (S, s)$  in R(X) then  $g_1 f_1 \sim g_2 f_2: (Q, q) \rightarrow (S, s)$ .

We are now ready to obtain a quotient category [8; p. 51] of R(X) classifying morphisms by the equivalence relation " $\sim$ " defined above. We denote the quotient category by R[X]. That is, objects in R[X] are the same ones as in R(X) and a morphism  $[f]:(Q, q) \rightarrow (R, r)$  in R[X] is an equivalence class of a morphism  $f:(Q, q) \rightarrow (R, r)$  in R(X).

**PROPOSITION 2.5.** Let  $f:(Q, q) \rightarrow (R, r)$  be a morphism in R(X). If  $f:Q \rightarrow R$  is an isomorphism in  $\mathcal{X}$  and if there exists a morphism  $g:(R, r) \rightarrow (Q, q)$  with  $fg \sim 1_R$ , then  $f:(Q, q) \rightarrow (R, r)$  is an isomorphism in R(X).

In the above proposition the existence of a morphism  $g:(R, r)\to(Q, q)$  with  $fg\sim 1_R:(R, r)\to(R, r)$  means that  $[f]:(Q, q)\to(R, r)$  is a retraction [5; p. 19] in R[X].

COROLLARY 2.6. Let  $f:(Q, q) \rightarrow (R, r)$  be a morphism in R(X). If  $f:Q \rightarrow R$  is an isomorphism in  $\mathcal{X}$  and if  $[f]:(Q, q) \rightarrow (R, r)$  is an isomorphism in R[X], then  $f:(Q, q) \rightarrow (R, r)$  is an isomorphism in R(X).

By using the diagonal fill-in lemma [1, p. 39] we have the following lemma.

LEMMA 2.7. Let  $f:(Q, q) \rightarrow (R, r)$  be a morphism in R(X). If there exists a morphism  $g: R \rightarrow Q$  with  $gf = 1_Q$  and if  $\overline{f}: \overline{Q} \rightarrow \overline{R}$  is in  $\mathcal{E}$ , then  $[f]:(Q, q) \rightarrow (R, r)$  is an isomorphism in R[X].

EXAMPLE 2.8. Let C be a category with products and A a set. A is regarded as a discrete category. Denote by  $C^A$  the functor category of all functors from A into C, i.e.,  $C^A$  is a product cotegory of A copies of C. Let  $\pi_A: C^A \to C$  be the product functor  $(\langle S_a \rangle_{a \in A} \to \pi_a S_a \rangle$ . Then the category  $R(\pi_A)$  of relational models is obtained and the case C =**Set** was considered by Kato [6].

# 3. Generalized Join Dependencies

In [6] Kato described that observations of join dependencies about relational models as stated in Example 2.8 are sufficient if they are considered in quotient category R[X]. This section generalizes the result due to Kato [6] for the category of relational models defined in the previous section.

As in the section 2, we assume that an image factorization system  $(\mathcal{E}, \mathcal{M})$  in a category  $\mathcal{C}$  and a functor  $X: \mathcal{X} \to \mathcal{C}$  are given. Moreover supposed that category  $\mathcal{C}$  has products and pullbacks, and supposed that a family of functors  $X_{\alpha}: \mathcal{X} \to \mathcal{C}$   $(\alpha \in \Lambda)$  and a family of natural transformations  $k_{\alpha}: X \to X_{\alpha}$   $(\alpha \in \Lambda)$  are given.

Let (Q, q) be an object of category R(X). Then consider an  $(\mathcal{E}, \mathcal{M})$ -factorization of the composite  $k_{\alpha}q: \overline{Q} \to XQ \to X_{\alpha}Q$  ( $\alpha \in \Lambda$ ), which is rendered by the commutative square



(Notice that  $k_{\alpha} = k_{\alpha,Q} : XQ \to X_{\alpha}Q$ . An arrow " $\to$ " represents an epimorphism in  $\mathcal{E}$ .) Let  $pr_{\alpha} : \pi_{\alpha}X_{\alpha}Q \to X_{\alpha}Q$  ( $\alpha \in \Lambda$ ) and  $\overline{pr}_{\alpha} : \pi_{\alpha}\overline{Q}_{\alpha} \to \overline{Q}_{\alpha}$  ( $\alpha \in \Lambda$ ) be systems of projections of products  $\pi_{\alpha}X_{\alpha}Q$  and  $\pi_{\alpha}\overline{Q}_{\alpha}$ , respectiqely. We denote by  $\tau_{\alpha}k_{\alpha} : XQ \to \pi_{\alpha}X_{\alpha}Q$  a unique morphism with  $pr_{\alpha}(\tau_{\alpha}k_{\alpha}) = k_{\alpha}$  for each  $\alpha \in \Lambda$  and by  $\pi_{\alpha}m_{\alpha} : \pi_{\alpha}\overline{Q}_{\alpha} \to \pi_{\alpha}X_{\alpha}Q$  a unique morphism with  $pr_{\alpha}(\pi_{\alpha}m_{\alpha}) = m_{\alpha}\overline{pr}_{\alpha}$  for each  $\alpha \in \Lambda$ . Construct a pullback of two morphisms  $\tau_{\alpha}k_{\alpha}$  and  $\pi_{\alpha}m_{\alpha}$  with common codomain.

Trivially  $q^*$  is in  $\mathcal{M}$  [1, Exercise 2.4.11; 2, Proposition 2.11] since  $m_{\alpha}$  is in  $\mathcal{M}$  for each  $\alpha \in \Lambda$ . Hence we have a new object  $(Q, q^*)$  in R(X). By the universal property of pullbacks there exists a unique morphism  $\overline{\eta}: \overline{Q} \to \widehat{Q}$  in  $\mathcal{X}$  making the following daigram commute.



This implies that  $1_Q:(Q, q) \to (Q, q^*)$  is a morphism in R(X). Note that  $1_Q:Q \to Q$  is an identity morphism in  $\mathcal{X}$  but  $1_Q:(Q, q) \to (Q, q^*)$  is not always an identity morphism in R(X). It follows at once from the construction of  $q^*$  that  $q^{**}=q^*$  since  $e'_{\alpha}$  is in  $\mathcal{E}$  for each  $\alpha \in \Lambda$ .

PROPOSITION 3.1. Let (Q, q) and (R, r) be two objects in R(X). If  $f:(Q, q) \rightarrow (R, r^*)$  is a morphism in R(X), then so is  $f:(Q, q^*) \rightarrow (R, r^*)$ .

COROLLARY 3.2. If  $f:(Q, q) \rightarrow (R, r)$  is a morphism in R(X), then so is  $f:(Q, q^*) \rightarrow (R, r^*)$ .

The last corollary shows that \* gives a functor called the join functor:

$$\begin{array}{c} *: R(X) \longrightarrow R(X) \\ (Q, q) & (Q, q^*) \\ f & \downarrow & \longmapsto & \downarrow & f \\ (R, r) & (R, r^*) \end{array}$$

During the rest of this section we assume that  $\top_{\alpha}k_{\alpha}: XQ \to \prod_{\alpha}X_{\alpha}Q$  is a monomorphism for each object Q in  $\mathcal{X}$ . The following results are basic properties of the join functor  $*: R(X) \to R(X)$ .

LEMMA 3.3. Let  $f, g: (Q, q) \rightarrow (R, r)$  be two morphisms in R(X). If  $f \sim g: (Q, q) \rightarrow (R, r)$ , then  $f \sim g: (Q, q^*) \rightarrow (R, r^*)$ .

COROLLARY 3.4. An object (Q, q) in R(X) is join dependent (JD), that is,  $q=q^*$  as a subobject of XQ if and only if  $[1_Q]: (Q, q) \rightarrow (Q, q^*)$  is an isomorphism in R[X].

THEOREM 3.5. Let (Q, q) and (R, r) be two object in R(X). If (R, r) is isomorphic to (Q, q) in R[X] and if  $q=q^*$  (JD), then  $r=r^*$  (JD).

Let D(X) be the full subcategory of R(X) consisting of all objects (Q, q) with  $q=q^*$ (*JD*). Then Proposition 3.1 indicates that the inclusion functor  $I: D(X) \subset R(X)$  is a right adjoint of the join functor  $*: R(X) \rightarrow D(X)$ :

$$\frac{(Q, q) \xrightarrow{f} I(R, r^*)}{(Q, q^*) \xrightarrow{f} (R, r^*)}$$

EXAMPLE 3.6. Let  $\{A_{\alpha} | \alpha \in A\}$  be a family of sets with  $A = \bigcup_{\alpha} A_{\alpha}$ . Moreover, let  $\pi_{\alpha} : \mathbf{Set}^{A} \to \mathbf{Sst}$  be a functor  $(\langle S_{\alpha} \rangle_{\alpha \in A} \to \pi_{\alpha \in A_{\alpha}} S_{\alpha})$  and  $k_{\alpha} : \pi_{A} \to \pi_{\alpha}$  a natural transformation of projections  $(\alpha \in A)$ . Under this situation the join dependency discussed in the present section is the usual join dependency [6].

## 4. Categories of Adjoint Models

In this section we will define a category of adjoint models in case that a functor  $X: \mathcal{X} \to \mathcal{C}$  has a left adjoint functor  $Y: \mathcal{C} \to \mathcal{X}$ . Also a sufficient condition for the category of adjoint models to be isomorphic to the quotient category R[X] of relational models will be given. In addition we assume that  $Y \to X: \mathcal{C} \to \mathcal{X}$  is an adjunction [8, p. 78] and that  $(\mathcal{C}^*, \mathcal{M}^*)$  is an image factorization system of  $\mathcal{C}$ .

DEFINITION 4.1. An adjoint model (with respect to an adjunction  $Y \rightarrow X: \mathcal{C} \rightarrow \mathfrak{X}$ ) is defined to be a pair  $\langle \overline{Q}, \hat{q} \rangle$  of an object  $\overline{Q}$  in  $\mathcal{C}$  and an epimorphism  $\hat{q}: Y\overline{Q} \rightarrow Q$  in  $\mathfrak{X}$ such that  $\hat{q} \in \mathcal{C}^*$  and an adjoint  $q: \overline{Q} \rightarrow XQ$  of  $\hat{q}: Y\overline{Q} \rightarrow Q$  is in  $\mathcal{M}$ . A morphism  $\overline{f}: \langle \overline{Q}, \hat{q} \rangle$  $\rightarrow \langle \overline{R}, \hat{r} \rangle$  of an adjoint model  $\langle \overline{Q}, \hat{q} \rangle$  into another adjoint model  $\langle \overline{R}, \hat{r} \rangle$  is defined to be a morphism  $\overline{f}: \overline{Q} \rightarrow \overline{R}$  in  $\mathcal{C}$  such that there exists a morphism  $f: Q \rightarrow R$  in  $\mathfrak{X}$  rendering the following square commutative:



(The uniqueness of  $f: Q \to R$  follows from the surjectivity of  $\hat{q}: Y\bar{Q} \to Q$ . Arrows " $\to$ " and " $\to$ " in  $\mathscr{X}$  represent monomorphisms in  $\mathscr{M}^*$  and epimorphisms in  $\mathscr{E}^*$ , respectively.)

Let  $R \{Y\}$  be the category of all adjoint models and all morphisms between them. Identity morphisms and the composition of morphisms in  $R \{Y\}$  are given as usual.

Taking an adjoint of the last commutative square we have the following commutative diagrams:

Thus we naturally obtain a functor

$$F: R \{Y\} \longrightarrow R(X)$$

$$\langle \overline{Q}, \, \hat{q} \rangle \qquad (Q, q)$$

$$\overline{f} \qquad \longmapsto \qquad \downarrow \qquad f$$

$$\langle \overline{R}, \, \hat{r} \rangle \qquad (R, r)$$

The functor  $F: R\{Y\} \rightarrow R(X)$  is called the Armstrong functor because it corresponds to a construction [9] of characteristic relations (data models) from data models defined by first order predicates satisfying equivalence laws.

Let (R, r) be an object in R(X) and let



be an  $(\mathcal{E}^*, \mathcal{M}^*)$ -factorization of an adjoint  $\hat{r}$  of r. Consider an adjoint of the last commutative triangle:



Since  $Xm \cdot e = r \in \mathcal{M}$  by definition of (R, r) it follows that  $e \in \mathcal{M}$  and hence we obtain an object  $\langle \overline{R}, \hat{e} \rangle$  in  $R\{Y\}$ . It is trivial that  $F \langle \overline{R}, \hat{e} \rangle = (R', e)$  and  $m : (R', e) \to (R, r)$  is a morphism in R(X), where  $\overline{m} = 1_{\overline{R}}$ . The next lemma shows that morphism  $m : F \langle \overline{R}, \hat{e} \rangle \to (R, r)$  in R(X) is a universal arrow [8] from F to (R, r) and the object  $\langle \overline{R}, \hat{e} \rangle$  in  $R\{Y\}$  is cofree [1] over (R, r).

LEMMA 4.2. Let  $\langle \bar{Q}, \hat{q} \rangle$  be an object in  $R\{Y\}$  and (R, r) an object in R(X). For any morphism  $f: F \langle \bar{Q}, \hat{q} \rangle \rightarrow (R, r)$  in R(X) there exists a unique morphism  $\bar{g}: \langle \bar{Q}, q \rangle \rightarrow \langle \bar{R}, \hat{e} \rangle$  in  $R\{Y\}$  such that the triangle



is commutative.

COROLLARY 4.3. Armstrong functor  $F: R\{Y\} \rightarrow R(X)$  has a right adjoint functor  $G: R(X) \rightarrow R\{Y\}$  ((R, r)  $\rightarrow \langle \overline{R}, \hat{e} \rangle$ ) and  $m: (R', e) \rightarrow (R, r)$  is a component of count  $FG \rightarrow 1_{R(X)}$  of adjunction  $F \rightarrow G: R\{Y\} \rightarrow R(X)$ .

Denote by  $\widetilde{F}: R\{Y\} \to R[X]$  the composite of Armstrong functor  $F: R\{Y\} \to R(X)$ 

followed by quotient functor  $P: R(X) \to R[X]$  (cf. §2). From definition of functor G there exist a unique functor  $\tilde{G}: R[X] \to R\{Y\}$  rendering the triangle of functors



commutative. Since  $GF=1_{R(X)}$  (the identity functor of  $R\{Y\}$ ), it is immediate that  $\tilde{G}\tilde{F}=1_{R(Y)}$ . The totality of all morphisms  $[m]:\tilde{F}\tilde{G}(R, r)=(R', e)\rightarrow(R, r)$  for all object (R, r) in R(X) constitutes a natural transformation  $[m]:\tilde{F}\tilde{G}\rightarrow 1_{R(X)}$ .

THEOREM 4.4. If every monomorphism in  $\mathcal{M}^*$  is a split monomorphisms [8; p. 19], then  $R\{Y\}$  is equivalent to R[X], i.e.,  $[m]: \tilde{F}\tilde{G} \rightarrow 1_{R[X]}$  is a natural isomorphism.

The functor  $\pi_A: \mathcal{C}^A \to \mathcal{C}$  in Example 2.8 has a left adjoint  $\mathcal{\Delta}_A: \mathcal{C} \to \mathcal{C}^A$  since  $\mathcal{C}$  has products. In the case  $\mathcal{C} = \mathbf{Set}_0$ , the category of nonempty sets, we have  $\mathbf{Set}_0[\pi_A] \cong \mathbf{Set}_0\{\mathcal{\Delta}_A\}$  (an equivalence of categories) because every monomorphism in  $\mathcal{M}^* = \langle \mathbf{Mon} \rangle_{a \in A}$  is a split monomorphism. This case was initially observed by Kato [6, Theorem 5.1].

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