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ON A FUNDAMENTAL BOUND OF BALANCED ARRAYS*

By

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Abstract

Balanced arrays of strength t in N assemblies with m constraints and s symbols are useful in the construction of fractional factorial designs and to various combinatorial areas of design of experiments. To construct such arrays with the maximum possible number, m , of constraints is a very important problem both in the statistical design of experiments and combinatorial mathematics. In this note, balanced arrays satisfying a bound $m \leq N$ are completely characterized.

1. Introduction

Let A be an $m \times N$ matrix whose elements are $0, 1, \dots$, or $s-1$. Consider the s^t t -vector, $X = (x_1, x_2, \dots, x_t)'$, which can be formed where $x_i = 0, 1, \dots, s-1$ for $i = 1, 2, \dots, t$, and associate with each vector X a positive integer $\lambda(x_1, x_2, \dots, x_t)$ which is invariant under any permutations of $(x_1, x_2, \dots, x_t)$. If, for every t -rowed submatrix of A , the s^t distinct vectors X occur as columns $\lambda(x_1, x_2, \dots, x_t)$ times, then the matrix A is called a balanced array of strength t in N assemblies with m constraints, s symbols and index parameters $\lambda(x_1, x_2, \dots, x_t)$. For short, this is denoted by $BA(m, N, s, t)$.

Rafter and Seiden [1] noticed that $m \leq N$ holds for *all* balanced arrays. It appears that this statement is not correct in general. The inequality $m \leq N$ is the fundamental bound on the number of constraints, and can also be derived by considering the meaning of an s^m factorial design. In this note, we shall characterize completely balanced arrays of validating the bound $m \leq N$.

2. Discussions

Let $O_{a \times b}$ and $J_{a \times b}$ be $a \times b$ matrices whose elements are all zero and unity, respectively. Let I_a be the identity matrix of order a . In this case, we can show the following theorem:

THEOREM. In a $BA(m, N, s, t)$ with $t \geq 2$ except for any juxtaposition of $O_{m \times l_1}$, $J_{m \times l_2}$, $2J_{m \times l_3}$, $\dots$, or $(s-1)J_{m \times l_s}$ satisfying $N \geq l_i \geq 0$ and $\sum_{i=1}^s l_i = N$, an inequality $m \leq N$ always holds.

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PROOF. Let A be a $BA(m, N, s, t)$ with $\lambda(x_1, x_2, \dots, x_t)$ for $t \geq 2$. Then it is well known that A is also a $BA(m, N, s, 2)$ with appropriate index parameters $\lambda^*(x_1, x_2)$. In this case, it can be shown that

$$\begin{aligned} |AA'| &= |(a_1 - a_2)I_m + a_2 J_{m \times m}| \\ &= (a_1 - a_2)^{m-1} \{a_1 + (m-1)a_2\} \end{aligned}$$

with

$$\begin{aligned} a_1 &= \sum_{x_2=0}^{s-1} \sum_{x_1=1}^{s-1} x_1^2 \lambda^*(x_1, x_2), \\ a_2 &= \sum_{x_2=1}^{s-1} \sum_{x_1=1}^{s-1} x_1 x_2 \lambda^*(x_1, x_2) \end{aligned}$$

and $a_1 \geq a_2 \geq 0$. If $|AA'| \neq 0$, then it follows that

$$m = \text{rank}(AA') = \text{rank}(A) \leq N,$$

i.e., an inequality $m \leq N$ holds. Thus, we now investigate the possibility of $|AA'| = 0$ by considering two cases. Note that if $a_2 = 0$, then $a_1 \geq 0$. In this case if $a_1 > 0$, then $|AA'| \neq 0$, and if $a_1 = 0$, then the following case (I) comes out.

Case (I). $a_1 = 0$, which then implies $a_2 = 0$. Then $|AA'| = 0$. It is obvious that $a_1 = 0$ iff there only exist $\lambda^*(0, x_2)$ for some $x_2 (=0, 1, \dots, \text{or } s-1)$. Furthermore, since $\lambda^*(0, x_2) = \lambda^*(x_2, 0)$ from the definition of balanced arrays, it holds that $\lambda^*(0, x_2) = 0$ for all $x_2 = 1, 2, \dots, s-1$. Hence, there is the only possibility of the positive value of $\lambda^*(0, 0)$, that is, the original array is of form $O_{m \times N}$.

Case (II). $a_1 \neq 0$, $a_2 \neq 0$ and $a_1 - a_2 = 0$. In this case, since $\lambda^*(x_1, x_2) = \lambda^*(x_2, x_1)$, it follows that

$$\begin{aligned} (*) \quad a_1 - a_2 &= \sum_{\substack{x_2=0 \\ x_1 \neq x_2}}^{s-1} \sum_{x_1=1}^{s-1} (x_1^2 - x_1 x_2) \lambda^*(x_1, x_2) \\ &= \sum_{x_2=0}^{s-1} \sum_{\substack{x_1=1 \\ x_1 > x_2}}^{s-1} b_{x_1 x_2} \lambda^*(x_1, x_2) \end{aligned}$$

where $b_{x_1 x_2}$'s are positive constants depending on values of x_1 and x_2 . The relation (*) implies that if $a_1 - a_2 = 0$, then there only exist some $\lambda^*(x, x)$ for $x = 0, 1, 2, \dots, s-1$. Thus, the original array will be only of form

$$[O_{m \times l_1} : J_{m \times l_2} : 2J_{m \times l_3} : \dots : (s-1)J_{m \times l_s}]$$

for non-negative integers l_i satisfying $\sum_{i=1}^s l_i = N$. Other cases about a_i 's always yield that $|AA'| \neq 0$. Thus, the proof is completed.

When $s=2$ (two-symbol), the theorem yields the following.

COROLLARY. In a $BA(m, N, 2, t)$ with $t \geq 2$ except for a type $[O_{m \times l} : J_{m \times (N-l)}]$ satisfying $N \geq l \geq 0$, an inequality $m \leq N$ always holds.

REMARK. When $l=0$ and N , the two-symbol original balanced array will be $J_{m \times N}$ and $O_{m \times N}$, respectively.

A type of some juxtaposition of $O_{m \times l_1}, J_{m \times l_2}, 2J_{m \times l_3}, \dots$, or $(s-1)J_{m \times l_s}$ is a *trivial*

balanced array for integers l_i satisfying $N \geq l_i \geq 0$ and $\sum_{i=1}^s l_i = N$. In this sense, it follows that, in a non-trivial balanced array, the number of assemblies is always bounded below by the number of constraints.

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