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Kageyama, Sanpei
Department of Mathematics, Faculty of School Education, Hiroshima University

https://doi.org/10.5109/13354

出版情報：Bulletin of informatics and cybernetics. 21 (1/2), pp.37-39, 1984-03. Research Association of Statistical Sciences

Bulletin of Informatics and Cybernetics Vol. 21, No. 1~2, 1984

ON A FUNDAMENTAL BOUND OF BALANCED ARRAYS*

By

Sanpei KAGEYAMA**

Abstract

Balanced arrays of strength t in N assemblies with m constraints and s symbols are useful in the construction of fractional factorial designs and to various combinatorial areas of design of experiments. To construct such arrays with the maximum possible number, m, of constraints is a very important problem both in the statistical design of experiments and combinatorial mathematics. In this note, balanced arrays satisfying a bound $m \leqq N$ are completely characterized.

1. Introduction

Let A be an $m \times N$ matrix whose elements are 0, 1, $\cdots$, or $s-1$. Consider the s^t t-vector, $X=(x_1, x_2, \cdots, x_t)'$, which can be formed where $x_i=0, 1, \cdots, s-1$ for $i=1, 2, \cdots, t$, and associate with each vector X a positive integer $\lambda(x_1, x_2, \cdots, x_t)$ which is invariant under any permutations of $(x_1, x_2, \cdots, x_t)$. If, for every t-rowed submatrix of A, the s^t distinct vectors X occur as columns $\lambda(x_1, x_2, \cdots, x_t)$ times, then the matrix A is called a balanced array of strength t in N assemblies with m constraints, s symbols and index parameters $\lambda(x_1, x_2, \cdots, x_t)$. For short, this is denoted by $BA(m, N, s, t)$.

Rafter and Seiden [1] noticed that $m \leqq N$ holds for *all* balanced arrays. It appears that this statement is not correct in general. The inequality $m \leqq N$ is the fundamental bound on the number of constraints, and can also be derived by considering the meaning of an s^m factorial design. In this note, we shall characterize completely balanced arrays of validating the bound $m \leqq N$.

2. Discussions

Let $O_{a\times b}$ and $J_{a\times b}$ be $a \times b$ matrices whose elements are all zero and unity, respectively. Let I_a be the identity matrix of order a. In this case, we can show the following theorem:

THEOREM. *In a $BA(m, N, s, t)$ with $t \geqq 2$ except for any juxtaposition of $O_{m\times l_1}$, $J_{m\times l_2}$, $2J_{m\times l_3}$, $\cdots$, or $(s-1)J_{m\times l_s}$ satisfying $N \geqq l_i \geqq 0$ and $\sum_{i=1}^{s} l_i = N$, an inequality $m \leqq N$ always holds.*

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* Prepared while the author was visiting the Indian Statistical Institute, Calcutta, India, during August 1982 to March 1983.

** Department of Mathematics, Faculty of School Education, Hiroshima University, Shinonome, Hiroshima 734, Japan.

PROOF. Let A be a $BA(m, N, s, t)$ with $\lambda(x_1, x_2, \cdots, x_t)$ for $t \geqq 2$. Then it is well known that A is also a $BA(m, N, s, 2)$ with appropriate index parameters $\lambda^*(x_1, x_2)$. In this case, it can be shown that

$$|AA'| = |(a_1 - a_2)I_m + a_2 J_{m \times m}|$$
$$= (a_1 - a_2)^{m-1}\{a_1 + (m-1)a_2\}$$

with

$$a_1 = \sum_{x_2=0}^{s-1} \sum_{x_1=1}^{s-1} x_1^2 \lambda^*(x_1, x_2),$$

$$a_2 = \sum_{x_2=1}^{s-1} \sum_{x_1=1}^{s-1} x_1 x_2 \lambda^*(x_1, x_2)$$

and $a_1 \geqq a_2 \geqq 0$. If $|AA'| \neq 0$, then it follows that

$$m = rank(AA') = rank(A) \leqq N,$$

i.e., an inequality $m \leqq N$ holds. Thus, we now investigate the possibility of $|AA'| = 0$ by considering two cases. Note that if $a_2 = 0$, then $a_1 \geqq 0$. In this case if $a_1 > 0$, then $|AA'| \neq 0$, and if $a_1 = 0$, then the following case (I) comes out.

Case (I). $a_1 = 0$, which then implies $a_2 = 0$. Then $|AA'| = 0$. It is obvious that $a_1 = 0$ iff there only exist $\lambda^*(0, x_2)$ for some x_2 $(=0, 1, \cdots$, or $s-1)$. Furthermore, since $\lambda^*(0, x_2) = \lambda^*(x_2, 0)$ from the definition of balanced arrays, it holds that $\lambda^*(0, x_2) = 0$ for all $x_2 = 1, 2, \cdots, s-1$. Hence, there is the only possibility of the positive value of $\lambda^*(0, 0)$, that is, the original array is of form $O_{m \times N}$.

Case (II). $a_1 \neq 0$, $a_2 \neq 0$ and $a_1 - a_2 = 0$. In this case, since $\lambda^*(x_1, x_2) = \lambda^*(x_2, x_1)$, it follows that

$$(*) \qquad a_1 - a_2 = \sum_{\substack{x_2=0 \\ x_1 \neq x_2}}^{s-1} \sum_{x_1=1}^{s-1} (x_1^2 - x_1 x_2) \lambda^*(x_1, x_2)$$
$$= \sum_{\substack{x_2=0 \\ x_1 > x_2}}^{s-1} \sum_{x_1=1}^{s-1} b_{x_1 x_2} \lambda^*(x_1, x_2)$$

where $b_{x_1 x_2}$'s are positive constants depending on values of x_1 and x_2. The relation (*) implies that if $a_1 - a_2 = 0$, then there only exist some $\lambda^*(x, x)$ for $x = 0, 1, 2, \cdots, s-1$. Thus, the original array will be only of form

$$[O_{m \times l_1} : J_{m \times l_2} : 2J_{m \times l_3} : \cdots : (s-1) J_{m \times l_s}]$$

for non-negative integers l_i satisfying $\sum_{i=1}^{s} l_i = N$. Other cases about a_i's always yield that $|AA'| \neq 0$. Thus, the proof is completed.

When $s = 2$ (two-symbol), the theorem yields the following.

COROLLARY. *In a $BA(m, N, 2, t)$ with $t \geqq 2$ except for a type $[O_{m \times l} : J_{m \times (N-l)}]$ satisfying $N \geqq l \geqq 0$, an inequality $m \leqq N$ always holds.*

REMARK. When $l = 0$ and N, the two-symbol original balanced array will be $J_{m \times N}$ and $O_{m \times N}$, respectively.

A type of some juxtaposition of $O_{m \times l_1}$, $J_{m \times l_2}$, $2J_{m \times l_3}$, $\cdots$, or $(s-1) J_{m \times l_s}$ is a *trivial* balanced array for integers l_i satisfying $N \geqq l_i \geqq 0$ and $\sum_{i=1}^{s} l_i = N$. In this sense, it follows that, in a non-trivial balanced array, the number of assemblies is always bounded below by the number of constraints.

Acknowledgement

The author wishes to express his thanks to Dr. G. M. Saha, Indian Statistical Institute, Calcutta, for his suggestion of a problem of this note, and to Dr. M. Kuwada, Hiroshima University, for his useful comments.

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Communicated by Ch. Asano
Received April 16, 1983