

## REMARKS ON SOME SMOOTHED EMPIRICAL DISTRIBUTION FUNCTIONS AND PROCESS

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## REMARKS ON SOME SMOOTHED EMPIRICAL DISTRIBUTION FUNCTIONS AND PROCESSES\*

By

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### Abstract

It is shown that under mild assumptions, a convolution-smoothed empirical process exhibits essentially the same asymptotic properties as the standard empirical process such as: a pointwise law of iterated logarithm, weak convergence to Brownian bridge, and the Chung-Smirnov property. Some remarks of statistical and probabilistic interests are made. A list of open questions is also included.

### I. Introduction

Let  $X_1, X_2, \dots, X_n$  be i.i.d. random variables with common distribution function (d.f.)  $F$  on the real line. Denote by  $F_n$  the standard empirical d.f., i.e.,  $F_n(x) = \text{proportion of observations} \leq x$ . By now there is a rich body of literature on the study of  $F_n$  and the related empirical process  $Z_n = n^{1/2}(F_n - F)$  (see Gaenssler and Stute [8]). When  $F$  is absolutely continuous with density  $f$ , it is natural to look for some smooth estimates  $\hat{F}_n$  of  $F$ . For example, Kronmal and Tarter [12] proposed using a trigonometric series estimate in connection with address calculation sorting. Recently, Efron [7] and Boos and Monahan [5] have suggested using  $\hat{F}_n$  instead of  $F_n$  to generate bootstrap samples. Their initial Monte Carlo experiments appeared to be encouraging.

Typically  $\hat{F}_n$  is constructed by taking an indefinite integral (if it exists) of some density estimates based on some "delta sequence" as studied by Walter and Blum [27]. Among these which are not convolution-based are, for instance, approaches based on orthogonal expansions (e.g., Schwartz [19], Kronmal and Tarter [12], and Walter [26]); approaches based on spline interpolations (e.g., Boneva, Kendall and Stefanov [4], Wahba [25], Lii and Rosenblatt [13]); and approaches based on nonparametric maximum likelihood considerations (e.g., Geman and Huang [9], Blum and Walter [3]). Another "delta sequence" approach is the kernel method, by far the most studied in density estimation, which dates back to Rosenblatt [17] in 1956. The kernel density estimates have the form

$$f_n(x) = \int \frac{1}{b_n} w\left(\frac{x-t}{b_n}\right) dF_n(t)$$

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$$= \frac{1}{nb_n} \sum_{i=1}^n w\left(\frac{x-X_i}{b_n}\right)$$

where  $w$  is a density and  $\{b_n\}$  is a so-called "bandwidth" sequence which tends to 0 as  $n$  tends to  $\infty$ . The smoothed estimate  $\hat{F}_n$  of  $F$  based on the kernel method is thus given by a convolution:

$$\begin{aligned} (1) \quad \hat{F}_n(x) &= \int_{-\infty}^x f_n(u) du \\ &= \iint_{-\infty}^x \frac{1}{b_n} w\left(\frac{u-t}{b_n}\right) du dF_n(t) \\ &= \int W\left(\frac{x-t}{b_n}\right) dF_n(t) \\ &= \frac{1}{n} \sum_{i=1}^n W\left(\frac{x-X_i}{b_n}\right) \end{aligned}$$

where  $W(u) = \int_{-\infty}^u w(t) dt$ . Note that  $\left\{\frac{1}{b_n} w\left(\frac{u}{b_n}\right)\right\}$  is a "delta sequence", while  $\left\{W\left(\frac{u}{b_n}\right)\right\}$  is a sequence of d.f.'s converging weakly to the d.f. of the unit mass at 0. (In the spirit of Walter and Blum [27], we may call such a sequence a "Heaviside sequence".)

While some estimates of the non-convolution type seem to have computational appeal (such as the trigonometric series method), and others may have global optimality at the finite sample level (such as the spline estimates and MLE's), they frequently suffer from the drawback of not being d.f.'s themselves. In addition, many estimates based on global optimality are constructed implicitly, making statistical analysis quite intractable. The convolution-based approach is easier to handle, and the computation issue is made less serious because of current advances in computer science, and by using fast Fourier transform technique as described by Silverman [21].

For the remainder of this discussion, we shall concentrate our efforts on the convolution-based estimates (1) or, more generally,

$$\begin{aligned} (2) \quad \hat{F}_n(x) &= \int W_n(x-t) dF_n(t) \\ &= \frac{1}{n} \sum_{i=1}^n W_n(x-X_i), \end{aligned}$$

where  $\{W_n\}$  is a Heaviside sequence. Note that there are delta sequences not of the kernel type. For example, the Landau sequence described in Walter and Blum [27], or the Fejer sequence encountered in time series.

A detailed listing of research on (1) or (2) is given in the references. Most results seem to indicate that the asymptotic behaviour of the smoothed estimates and that of the standard empirical d.f. are quite similar. For example, Yamato [31] showed that (2) has the same asymptotic covariance as  $F_n$ , while Winter [30] showed that (1) has the Chung-Smirnov property.

In this article, we consider the smoothed empirical processes

$$\hat{Z}_n(x) = n^{1/2} [\hat{F}_n(x) - E\hat{F}_n(x)]$$

or

$$Z_n^*(x) = n^{1/2} [\hat{F}_n(x) - F(x)]$$

with  $\hat{F}_n$  given by (1) or (2). We shall demonstrate that under mild conditions, a pointwise law of iterated logarithm holds for (1) by using a result of Hall [10], and that  $\hat{Z}_n$  and  $Z_n$  are uniformly close by appealing to a result of Stute [24]. Consequently,  $\hat{Z}_n$  inherits many asymptotic properties of  $Z_n$ . We shall also make remarks on these results which have both statistical and probabilistic interests,

## 2. Main Results and Remarks

In what follows,  $\| \cdot \|$  will stand for the supremum norm over  $\mathbf{R}^1$ , and "w.p.1." will be an abbreviation for "with probability one".  $F$  is assumed to have a density.

**THEOREM 1.** *The following pointwise law of iterated logarithm holds for  $\hat{Z}_n$  with  $\hat{F}_n$  given by (2):*

$$(3) \quad \overline{\lim}_{n \rightarrow \infty} \pm [2 \log \log n \cdot F(x)(1-F(x))]^{-1/2} \hat{Z}_n(x) = 1$$

w.p.1 for each  $x$  such that  $F(x) \neq 0, 1$ .

**PROOF OF THEOREM 1.** We verify that the hypothesis of Theorem 1 in Hall [10] is satisfied. Let

$$\sigma_{mn} = \text{cov} [W_m(x - X_1), W_n(x - X_1)]$$

$$\sigma_n^2 = \sigma_{nn}.$$

Since  $\{W_n\}$  is obviously a sequence of functions of bounded variation on  $\mathbf{R}^1$ , it remains to check that

$$(4) \quad \sigma_{mn} / \sigma_n^2 \longrightarrow 1 \quad \text{as } m, n \rightarrow \infty \text{ with } n/m \rightarrow 1$$

and that

$$(5) \quad (\log n)^4 \left[ \int dW_n(t) \right]^2 / n \sigma_n^2 \log \log n \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now as mentioned earlier,  $\sigma_n^2 \rightarrow F(x)(1-F(x))$  as  $n \rightarrow \infty$  (see Yamato [31]), so clearly (5) holds. To see that (4) holds, it is enough to show that  $|\sigma_{mn} - \sigma_n^2| \rightarrow 0$ , or equivalently, that

$$(6) \quad |E[W_m(x - X_1) \cdot W_n(x - X_1)] - E[W_n^2(x - X_1)]| \longrightarrow 0$$

as  $m, n \rightarrow \infty$ . Writing the expression in (6) as

$$|E\{W_n(x - X_1)[W_m(x - X_1) - W_n(x - X_1)]\}|$$

we see that the result follows easily since  $W_n$  is bounded and that  $EW_n(x - X_1)$  and  $EW_m(x - X_1)$  both tend to  $F(x)$  in the limit as  $n, m \rightarrow \infty$ .

**THEOREM 2.** *As in Stute [24], let  $\{a_n\}$  be a sequence of positive real numbers satisfying condition*

$$(i) \quad a_n \downarrow 0 \quad \text{as } n \rightarrow \infty$$

$$(A) \quad (ii) \quad \log \frac{1}{a_n} = o(n a_n)$$

$$(iii) \quad \log \log n = o\left(\log \frac{1}{a_n}\right).$$

Let  $\tau_n = \int_{|t| > a_n} dW_n(t)$  and  $\theta_n = \left(a_n \log \frac{1}{a_n}\right)^{1/2}$ . Then

$$(6) \quad \|\hat{Z}_n - Z_n\| = O[\theta_n + \tau_n (\log \log n)^{1/2}]$$

w.p.1.

COROLLARY 1. Suppose  $\{a_n\}$  satisfies (A) (i)-(iii) and  $\{W_n\}$  satisfies the tail condition

$$(T) \quad \tau_n = o((\log \log n)^{-1/2}).$$

Then  $\hat{Z}_n$  converges weakly to the rescaled Brownian bridge  $B^\circ(F)$ .

COROLLARY 2. Under condition (A) (i)-(iii), on a rich enough probability space (à la Komlós, Major, Tusnády [11]), there exist a version of  $\hat{Z}_n$  and a sequence of Brownian bridges  $B_n^\circ$  such that

$$(7) \quad \|\hat{Z}_n - B_n^\circ(F)\| = O[n^{-1/2} \log n + \|\hat{Z}_n - Z_n\|]$$

w.p.1, where  $\|\hat{Z}_n - Z\|$  is given by (6).

(There is a similar result using a Kiefer process instead of Brownian bridges.)

COROLLARY 3. Suppose  $\{a_n\}$  satisfies (A) (i)-(iii) and suppose  $\tau_n = o(1)$

Then

$$(8) \quad \overline{\lim}_{n \rightarrow \infty} (\log \log n)^{-1/2} \|\hat{Z}_n\| = 2^{-1/2}$$

w.p.1.

(This result is in particular an improvement over Theorems 3.2 and 3.3 of Winter [30] on the Chung-Smirnov property of  $\hat{Z}_n$  with  $\hat{F}_n$  defined by (1).)

PROOF OF THEOREM 2. Write

$$\begin{aligned} |\hat{Z}_n(x) - Z_n(x)| &= \left| \int Z_n(x-t) dW_n(t) - Z_n(x) \int dW_n(t) \right| \\ &\leq \int |Z_n(x-t) - Z_n(x)| dW_n(t) \\ &= \int_{|t| \leq a_n} + \int_{|t| > a_n} . \\ \left\| \int_{|t| \leq a_n} \right\| &\leq \sup_{\substack{|t| \leq a_n \\ \text{all } x}} |Z_n(x-t) - Z_n(x)| \cdot \int_{|t| \leq a_n} dW_n(t) \\ &= O(\theta_n) \end{aligned}$$

w.p.1. by Theorem 0.2 of Stute [24], while

$$\begin{aligned} \left\| \int_{|t| > a_n} \right\| &\leq 2 \|Z_n\| \cdot \int_{|t| > a_n} dW_n(t) \\ &= O[\tau_n (\log \log n)^{1/2}] \end{aligned}$$

w.p.1. by the law of iterated logarithm of  $Z_n$ . Thus (6) is proved. The corollaries follow by standard results on  $Z_n$  (see for example Billingsley [2] or Csörgő and Révész [6].)

#### REMARKS.

- (a)  $a_n = n^{-\lambda} \log n$ ,  $0 < \lambda < 1$ , is an example of  $\{a_n\}$  satisfying (A) (i)-(iii) in Theorem 2.  
 (b) For kernel-type smoothers (1), if  $W$  (as a d.f.) has absolute  $k^{th}$  moment,  $k > 0$ , and is symmetric about 0, then

$$\tau_n = \int_{|v| > a_n/b_n} dW(v) \leq O\left[\left(\frac{b_n}{a_n}\right)^k\right].$$

So we can choose, for example,

$$b_n = o[a_n(\log \log n)^{-1/2k}]$$

in order to satisfy condition (T). Various combinations of choices of  $\{a_n\}$ ,  $\{b_n\}$  and  $k$  are possible for the corollaries to hold. As an illustration, take  $k=1$ ,  $a_n = n^{-1/10}$ ,  $b_n = n^{-1/5}$ , then Corollary (3) holds.

For kernels  $W$  with compact support, we can allow  $a_n = c b_n$  with  $0 < c < \infty$  chosen so that  $w$  vanishes outside  $[-c, c]$ . In this case  $\tau_n = 0$  and  $\{b_n\}$  should satisfy the same condition as  $\{a_n\}$ .

(c) There are refinements on the oscillation behaviour of  $Z_n$  (see Shorack and Wellner [20].) For example, let  $a_n = n^{-1} (\log n)^\delta$ ,  $-\infty < \delta < 1$ . Then the rate  $\theta_n$  in Theorem 2 is replaced by  $a_n^{1/2} (\log n)^{1-\delta/2} (\log \log n)^{-1}$ . However, with  $\delta = -1$ ,  $na_n \not\rightarrow \infty$  as  $n \rightarrow \infty$ . A kernel density estimate constructed with this choice of the bandwidth sequence will not be consistent. Nevertheless, since  $b_n \rightarrow 0$  as  $n \rightarrow \infty$  is both necessary and sufficient for  $\left\{W\left(\frac{t}{b_n}\right)\right\}$  to be a Heaviside sequence, the choice of  $\{a_n\}$  indicated above when  $W$  has compact support will still give rise to consistent estimates  $\hat{F}_n$ .

(d) The difference between  $\hat{Z}_n$  and  $Z_n^*$  lies in the bias term. Under additional assumptions on  $F$  and  $\{W_n\}$ , the bias can be made to decrease at a certain rate. (See Winter [30], Singh [22] and Reiss [16].) Similarly, the mean square error issue can be analysed just as the density estimate case. (See Azzalini [1].) However, Read [15] has shown that the standard empirical d.f.  $F_n$  is inadmissible w.r.t. integrated square loss by exhibiting a (biased) continuous piecewise linear estimate of  $F$  dominating  $F_n$ . It would be of interest to investigate if versions of  $\hat{F}_n$  exist which dominate  $F_n$ .

### 3. Open Questions

There are obviously many interesting open questions regarding the statistical and probabilistic behaviour of  $\hat{F}_n$  or  $\hat{Z}_n$ . We list a few here, hoping that there will be further research work on this topic:

- (i) What kinds of optimality properties can  $\hat{F}_n$  inherit from  $F_n$ ?
- (ii) Is there a Hájek-Beran type representation theorem for  $\hat{Z}_n$  in the context of regular estimators?
- (iii) What is the Prokhorov distance between  $\hat{F}_n$  and  $F$ ? (This has relevance in robustness.)

- (iv) Would similar smoothing procedures be useful in the setting with censoring? (i.e., replace  $F_n$  by the Kaplan-Maier product limit function.)
- (v) How does  $\hat{F}_n$  compare with other possible competitors? For instance, how does (1) or (2) compare with the estimates constructed by orthogonal series expansions as in Kronmal and Tarter [12]?
- (vi) How do bootstrap estimates behave when the bootstrap sample is generated according to  $\hat{F}_n$  rather than  $F_n$ ?

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