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ON SEQUENTIAL ESTIMATION OF THE REGRESSION FUNCTION*

By

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1. Introduction and Summary

Let (X, Y) be a two dimensional random variable having a joint density function f and let g be the marginal density function of X. We assume that $E_f Y$ is finite and define the regression function $m(x)$ (for regression of Y on X) by $m(x)=E[Y|X=x]$. Nadaraya [3] and Watson [7] and Schuster [6] have studied the asymptotic properties of the estimate $\hat{m}_n(x)$ of $m(x)$ defined by

$$(1) \qquad \hat{m}_n(x)=\frac{\sum_{i=1}^{n} Y_i K\left(\frac{x-X_i}{h_n}\right)}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right)}$$

where $K(u)$ is a probability density function on $(-\infty, +\infty)$, $\{h_n\}$ is a monotonically decreasing sequence of positive numbers converging to zero and $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ are n independent observations of (X, Y).

In many practical situations the number of observations N_t which we observe in time $(0, t]$ is a random variable. We call N_t a stopping random variable. We assume that $(X_1, Y_1), (X_2, Y_2), \cdots$ are independent observations of (X, Y) and need not be independent of the random variable N_t. In this paper we propose an estimate $m_{N_t}(x)$ of the regression function $m(x)$ based on $(X_1, Y_1), (X_2, Y_2), \cdots, (X_{N_t}, Y_{N_t})$ and given by

$$(2) \qquad m_{N_t}(x)=\frac{\sum_{i=1}^{N_t} \frac{Y_i}{h_i} K\left(\frac{x-X_i}{h_i}\right)}{\sum_{i=1}^{N_t} \frac{1}{h_i} K\left(\frac{x-X_i}{h_i}\right)}$$

We note that the expression for the estimate $m_{N_t}(x)$ is motivated by the recursive type of estimate of a probability density function first proposed by Yamato [8]. Suppose $x_1, x_2, \cdots, x_l$ are l distinct points. We have shown that under certain regularity conditions $(N_t h_{N_t})^{1/2}\{m_{N_t}(x_1)-m(x_1), \cdots, m_{N_t}(x_l)-m(x_l)\}$ is asymptotically normally distributed with mean vector 0 and diagonal covariance matrix $C=[c_{ij}]$ with $c_{ii}=\frac{\operatorname{Var}[Y|X=x_i]}{g(x_i)}\nu\int_{-\infty}^{\infty} K^2(u)du$ where $\nu(\nu<1)$ is as defined in the next section. For simplicity we have proved the theorem for the special case $l=2$. The method of proof remains valid in the more general case. The theorem can be regarded as the appropriate

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extension of the earlier result due to Schuster [6].

2. Main Result

We assume that the probability density function K and the sequence $\{h_n\}$ are chosen to satisfy the following conditions:

(i) $K(u)$ and $|uK(u)|$ are bounded

(ii) $\int_{-\infty}^{\infty} uK(u)du=0$

(iii) $\int_{-\infty}^{\infty} u^2K(u)du<\infty$

(iv) $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{h_n}{h_j}\right)=\nu<1$

(v) $\frac{1}{n}\sum_{j=1}^{n} h_j^2\leqq C_1h_n^2 \quad (C_1>0),\ n=1, 2, 3, \cdots$

(vi) $\lim_{n\to\infty} nh_n^3=\infty$ and $\lim_{n\to\infty} nh_n^5=0$

REMARK 1. If K is the standard normal probability density function, then conditions (i), (ii) and (iii) are satisfied.

REMARK 2. If $h_n=n^{-\delta}$, $\frac{1}{5}<\delta<\frac{1}{3}$, then conditions (iv), (v) and (vi) are satisfied. (For a proof see [1], p. 26 and p. 46.)

We define the following:

$$
\begin{aligned}
g(x)&=\int_{-\infty}^{\infty} f(x, y)dy \\
w(x)&=\int_{-\infty}^{\infty} yf(x, y)dy \qquad (3)\\
v(x)&=\int_{-\infty}^{\infty} y^2f(x, y)dy .
\end{aligned}
$$

Hence,

$$\mathrm{Var}\,[Y\,|\,X=x]=\frac{v(x)}{g(x)}-\frac{w^2(x)}{g^2(x)}\,.$$

Suppose x_1 and x_2 are two distinct points. We define for $i=1, 2, \cdots, n$ and $s=1, 2$

$$
\begin{aligned}
&U_i^*(x_s)=\frac{1}{h_i}K\left(\frac{x_s-X_i}{h_i}\right)\\
&V_i^*(x_s)=Y_iU_i^*(x_s)\\
&U_i(x_s)=h_n^{1/2}\{U_i^*(x_s)-EU_i^*(x_s)\}\\
&V_i(x_s)=h_n^{1/2}\{V_i^*(x_s)-EV_i^*(x_s)\}\\
&\bar{U}_n(x_s)=\sum_{i=1}^{n}U_i(x_s)
\end{aligned}
$$

$$(4)\qquad \bar{V}_n(x_s)=\sum_{i=1}^{n} V_i(x_s)$$

$$W_i=(U_i(x_1),\ V_i(x_1),\ U_i(x_2),\ V_i(x_2))'$$

$$n^{1/2}Z_n=(\bar{U}_n(x_1),\ \bar{V}_n(x_1),\ \bar{U}_n(x_2),\ \bar{V}_n(x_2))'$$

$$A=\nu\int_{-\infty}^{\infty} K^2(u)du\begin{pmatrix} g(x_1) & w(x_1) & 0 & 0 \\ w(x_1) & v(x_1) & 0 & 0 \\ 0 & 0 & g(x_2) & w(x_2) \\ 0 & 0 & w(x_2) & v(x_2)\end{pmatrix}$$

Let Z be a four variate normal random variable with mean vector 0 and covariance matrix A.

We now prove the following lemmas.

LEMMA 1. *Suppose K satisfies conditions* (i) *and* (iii) *and the sequence $\{h_n\}$ satisfies condition* (iv). *Let g', w' and v' exist and be bounded. Then the following results hold for $s=1, 2$ and $r=1, 2$.*

(a) $\lim_{n\to\infty} \mathrm{Var}\{n^{-1/2}\bar{U}_n(x_s)\}=\nu g(x_s)\int_{-\infty}^{\infty} K^2(u)du$

(b) $\lim_{n\to\infty} \mathrm{Var}\{n^{-1/2}\bar{V}_n(x_s)\}=\nu v(x_s)\int_{-\infty}^{\infty} K^2(u)du$

(c) $\lim_{n\to\infty} \mathrm{Cov}\{n^{-1/2}\bar{U}_n(x_s),\ n^{-1/2}\bar{V}_n(x_s)\}=\nu w(x_s)\int_{-\infty}^{\infty} K^2(u)du$

(d) $\lim_{n\to\infty} \mathrm{Cov}\{n^{-1/2}\bar{U}_n(x_1),\ n^{-1/2}\bar{U}_n(x_2)\}=0$

(e) $\lim_{n\to\infty} \mathrm{Cov}\{n^{-1/2}\bar{V}_n(x_1),\ n^{-1/2}\bar{V}_n(x_2)\}=0$

(f) $\lim_{n\to\infty} \mathrm{Cov}\{n^{-1/2}\bar{U}_n(x_s),\ n^{-1/2}\bar{V}_n(x_r)\}=0, \quad r\neq s.$

PROOF. We sketch the proof of part (a) and part (d) of the Lemma. The proof of the other parts are similar and will be omitted. To obtain part (a) we have

$$\begin{aligned}\mathrm{Var}\{n^{-1/2}\bar{U}_n(x_s)\}&=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{h_n}{h_j}\right)\int_{-\infty}^{\infty}\{K(u)\}^2 g(x_s-h_j u)du\\&\quad-\frac{h_n}{n}\sum_{j=1}^{n}\left\{\int_{-\infty}^{\infty}K(u)g(x_s-h_j u)du\right\}^2\\&=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{h_n}{h_j}\right)\left[g(x_s)\int_{-\infty}^{\infty}\{K(u)\}^2du+O(h_j)\right]\\&\quad-\frac{h_n}{n}\sum_{j=1}^{n}[g(x_s)+O(h)]_j .\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\mathrm{Var}\{n^{-1/2}\bar{U}_n(x_s)\}=\nu g(x_s)\int_{-\infty}^{\infty}K^2(u)du\,,\qquad s=1,\ 2.$$

To prove part (d) we have used the method similar to that of Schuster [6]:

$$E\left[\frac{1}{h_i}K\left(\frac{x_1-X_i}{h_i}\right)K\left(\frac{x_2-X_i}{h_i}\right)\right]=O(h_i).$$

Now,

$$\begin{aligned}\operatorname{Cov}\{n^{-1/2}\bar{U}_n(x_1),\ n^{-1/2}\bar{U}_n(x_2)\} &= \frac{1}{n}\sum_{i=1}^{n} E\{U_i(x_1)U_i(x_2)\} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{h_n}{h_i}E\left\{\frac{1}{h_i}K\left(\frac{x_1-X_i}{h_i}\right)K\left(\frac{x_2-X_i}{h_i}\right)\right\}\right. \\ &\qquad \left. -h_nE\left\{\frac{1}{h_i}K\left(\frac{x_1-X_i}{h_i}\right)\right\}E\left\{\frac{1}{h_i}K\left(\frac{x_2-X_i}{h_i}\right)\right\}\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{h_n}{h_i}\cdot O(h_i)-h_n\{g(x_1)+O(h_i)\}\{g(x_2)+O(h_i)\}\right] \\ &= O(h_n).\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\operatorname{Cov}\{n^{-1/2}\bar{U}_n(x_1),\ n^{-1/2}\bar{U}_n(x_2)\}=0.$$

Let $C=(c_1, d_1, c_2, d_2)'$ be any real vector in R^4.

LEMMA 2. *Suppose K satisfies conditions* (i) *and* (iii) *and the sequence $\{h_n\}$ satisfies condition* (iv) *and $nh_n^3\to\infty$ as $n\to\infty$. Let $E_f|Y|^3$ be finite and let g', w' and v' exist and be bounded. If $g(x_i)>0$ for $i=1, 2$, then $C'Z_n$ converges in distribution to a normal random variable with mean* 0 *and variance $C'AC$.*

PROOF. We shall establish the asymptotic normality of $C'Z_n$ by showing that

$$\lim_{n\to\infty}\left\{\frac{\sum_{i=1}^{n}E|C'W_i|^3}{n^{3/2}(\operatorname{Var}(C'Z_n))^{3/2}}\right\}=0. \qquad \text{(See [2], p. 275.)} \tag{5}$$

Using Lemma 1, we have

$$\begin{aligned}\lim_{n\to\infty}\operatorname{Var}(C'Z_n) &= \nu\int_{-\infty}^{\infty}K^2(u)du\left[\sum_{s=1}^{2}\{c_s^2g(x_s)+d_s^2v(x_s)+2c_sd_sw(x_s)\}\right] \\ &= C'AC>0.\end{aligned} \tag{6}$$

The positive definiteness of the matrix A follows from the fact that $g(x)v(x)-w^2(x)=g^2(x)\operatorname{Var}[Y|X=x]$ and $g(x_i)>0$ for $i=1, 2$. It now suffices to prove that

$$\lim_{n\to\infty}n^{-3/2}\left\{\sum_{i=1}^{n}E|C'W_i|^3\right\}=0$$

Using the hypothesis and the arguments similar to those in Lemma 1 it can be shown that

$$E|U_i(x_s)|^3=O(h_n^{3/2}h_i^{-2})=O(h_n^{-1/2})$$

and

$$E|V_i(x_s)|^3=O(h_n^{3/2}h_i^{-3})=O(h_n^{-3/2}).$$

Hence

$$n^{-3/2}\left\{\sum_{i=1}^{n} E|C'W_i|^3\right\} \leqq n^{-3/2}|C|^3 \sum_{i=1}^{n} E|W_i|^3$$

$$\leqq 8n^{-3/2}|C|^3 \sum_{i=1}^{n} \max_{s=1,2} \{E|U_i(x_s)|^3,\ E|V_i(x_s)|^3\}$$

$$= O\{(nh_n^3)^{-1/2}\} = o(1).$$

This completes the proof.

We define

$$Z_n^* = h_n^{1/2} \cdot n^{-1/2}\left\{\sum_{i=1}^{n}(U_i^*(x_1)-g(x_1)),\ \sum_{i=1}^{n}(V_i^*(x_1)-w(x_1)),\right. \tag{7}$$

$$\left.\sum_{i=1}^{n}(U_i^*(x_2)-g(x_2)),\ \sum_{i=1}^{n}(V_i^*(x_2)-w(x_2))\right\}.$$

LEMMA 3. *Suppose conditions* (i) *through* (vi) *are satisfied. Let* $E_f|Y|^3$ *be finite and let* g', g'', w', w'' *and* v' *exist and be bounded. If* $g(x_i)>0$ *for* $i=1, 2$, *then* $C'Z_n^*$ *converges in distribution to a normal random variable with mean* 0 *and variance* $C'AC$.

PROOF. We have

$$C'Z_n - C'Z_n^* = h_n^{1/2} \cdot n^{-1/2} \sum_{i=1}^{n} \sum_{s=1}^{2} [c_s\{g(x_s)-EU_i^*(x_s)\} + d_s\{w(x_s)-EV_i^*(x_s)\}]. \tag{8}$$

Using the hypothesis it can be shown that for $i=1, 2, \cdots, n$ and $s=1, 2$

$$EU_i^*(x_s) = g(x_s) + O(h_i^2)$$

and

$$EV_i^*(x_s) = w(x_s) + O(h_i^2).$$

Hence,

$$\left|\sum_{i=1}^{n}(g(x_s)-EU_i^*(x_s))\right| = O\left(\sum_{i=1}^{n} h_i^2\right), \tag{9}$$

and

$$\left|\sum_{i=1}^{n}(w(x_s)-EV_i^*(x_s))\right| = O\left(\sum_{i=1}^{n} h_i^2\right). \tag{10}$$

From (8), (9) and (10) we get

$$C'Z_n - C'Z_n^* = O\left(h_n^{1/2} n^{-1/2} \sum_{i=1}^{n} h_i^2\right)$$

$$= O\left((nh_n)^{1/2} \frac{1}{n} \sum_{i=1}^{n} h_i^2\right)$$

$$= O((nh_n)^{1/2} h_n^2)$$

$$= O((nh_n^5)^{1/2})$$

$$= o(1).$$

The proof now follows from Lemma 2.

Suppose N_t $(t>0)$ is a stopping random variable such that $\frac{N_t}{t} \xrightarrow{p} \pi$ $(\pi>0)$ as $t \to \infty$. This implies that for any $\varepsilon>0$, there exists $t_0 = t_0(\varepsilon)$ such that for $t \geqq t_0$ we have

$$P[|N_t-\pi t|\geqq \pi t\varepsilon]<\varepsilon .$$

We define

$$N_1=N_1(t,\ \varepsilon)=[\pi t(1-\varepsilon)] \tag{11}$$

and

$$N_2=N_2(t,\ \varepsilon)=[\pi t(1+\varepsilon)] \tag{12}$$

where $[x]$ is the integral part of x.

We note that for any $0<\varepsilon<\frac{1}{2}$ and $t>\frac{1}{\pi\varepsilon}$ the numbers N_1 and N_2 defined above satisfy the following inequalities:

$$\frac{N_2}{N_1}<\frac{1+\varepsilon}{1-2\varepsilon}$$

and

$$\frac{N_2-N_1}{N_1}<\frac{3\varepsilon}{1-2\varepsilon} .$$

We define

$$R_i=\sum_{s=1}^{2}\{c_sU_i^*(x_s)+d_sV_i^*(x_s)\}, \qquad i=1,\ 2,\ \cdots$$

$$\mu=\sum_{s=1}^{2}\{c_sg(x_s)+d_sw(x_s)\}$$

$$S_n=\sum_{i=1}^{n}\{R_i-\mu\}, \qquad n=1,\ 2,\ \cdots$$

$$Q=\max_{N_1<n\leqq N_2}\left|\sum_{i=N_1+1}^{n}[R_i-E\{R_i\}]\right| .$$

It can be seen that $h_n^{1/2}n^{-1/2}S_n=C'Z_n^*$. Replacing n by N_t in the expressions for $\bar{U}_n(x_s)$, $\bar{V}_n(x_s)$, Z_n and Z_n^*, we define $\bar{U}_{N_t}(x_s)$, $\bar{V}_{N_t}(x_s)$, Z_{N_t} and $Z_{N_t}^*$ respectively.

In order to study the asymptotic distribution of $C'Z_{N_t}^*$ we find it convenient to choose a specific sequence $\{h_n=n^{-\delta},\ n=1,\ 2,\ \cdots\}$, where δ is some positive number. With this choice of $\{h_n\}$ we have the following lemma. Let C_1 be a generic constant.

LEMMA 4. *Suppose K satisfies conditions* (i) *and* (iii) *and* $\{h_n=n^{-\delta}\}$, $\delta>0$. *If g' and v' exist and are bounded, then for any* $0<\varepsilon<\frac{1}{2}$, $t>\frac{1}{\pi\varepsilon}$ *we have*

$$P\left\{Q\geqq\varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\right\}<\frac{C_1\varepsilon^{1/3}}{(1-2\varepsilon)}\left\{\frac{(1+\varepsilon)}{(1-2\varepsilon)}\right\}^{\delta}$$

PROOF. By Kolmogorov's inequality

$$P\left[Q\geqq\varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\right]\leqq\frac{\sum_{i=N_1+1}^{N_2}E\{R_i^2\}}{\varepsilon^{2/3}\left\{\frac{N_1}{h_{N_1}}\right\}} . \tag{13}$$

Using the hypothesis it can be shown that

$$E\{R_i^2\}=O\left(\frac{1}{h_i}\right)=O\left(\frac{1}{h_{N_2}}\right) \qquad \text{if} \quad N_1<i\leqq N_2 . \tag{14}$$

From (13) and (14), we get

$$P\left[Q \geqq \varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\right] \leqq \frac{C_1(N_2-N_1)}{3\varepsilon^{2/3}N_1}\left(\frac{h_{N_1}}{h_{N_2}}\right)$$

$$\leqq \frac{C_1\varepsilon^{1/3}}{(1-2\varepsilon)}\left(\frac{1+\varepsilon}{(1-2\varepsilon)}\right)^{\delta}.$$

LEMMA 5. *Suppose K satisfies conditions* (ii), *and* (iii) *and* $\{h_n=n^{-\delta}\}$, $\delta>0$. *If g', g'', w' and w'' exist and are bounded and if* $0<\varepsilon<\frac{1}{2}$, $t>\frac{1}{\pi\varepsilon}$ *and* $\frac{1}{5}<\delta$, *then for all* $N_1<n \leqq N_2$,

$$\left|\sqrt{\frac{h_{N_1}}{N_1}}\sum_{i=N_1+1}^{n}[E\{R_i\}-\mu]\right| < \frac{C_1\varepsilon}{(1-2\varepsilon)}.$$

PROOF. Using computations similar to those in Lemma 3, we obtain for all n such that $N_1<n\leqq N_2$

$$\left|\sum_{i=N_1+1}^{n}[E\{R_i\}-\mu]\right| \leqq \frac{C_1}{3}(N_2-N_1)\sum_{i=N_1+1}^{n}h_i^2$$

$$\leqq \frac{C_1}{3}(N_2-N_1)h_{N_1}^2.$$

Hence,

$$\left|\sqrt{\frac{h_{N_1}}{N_1}}\sum_{i=N_1+1}^{n}[E\{R_i\}-\mu]\right| \leqq \frac{C_1(N_2-N_1)}{3N_1}(N_1h_{N_1}^5)^{1/2}$$

$$=\frac{C_1(N_2-N_1)}{3N_1}(N_1^{1-5\delta})^{1/2}$$

$$\leqq \frac{C_1\varepsilon}{(1-2\varepsilon)}.$$

LEMMA 6. *Suppose K satisfies conditions* (i), (ii) *and* (iii) *and* $\{h_n=n^{-\delta}\}$, $\frac{1}{5}<\delta<\frac{1}{3}$ *and* $\frac{N_t}{t}\xrightarrow{p}\pi$ $(\pi>0)$ *as* $t\to\infty$. *Let* $E_f|Y|^3$ *be finite and let* g', g'', w', w'' *and* v' *exist and be bounded. If* $x_1\neq x_2$, $g(x_i)>0$ *for* $i=1, 2$, *then* $C'Z^*_{N_t}$ *converges in distribution to a normal random variable with mean* 0 *and variance* $C'AC$ *as* t *tends to infinity.*

PROOF. The proof resembles that of Theorem 1 in Renyi [5]. Let ε $\left(\varepsilon<\frac{1}{2}\right)$ be an arbitrarily small positive number. Let $t\geqq t_0$ where $t_0=t_0(\varepsilon)>\frac{1}{\pi\varepsilon}$ and let N_1 and N_2 be chosen as before.

We have for $y>0$

$$P[C'Z^*_{N_t}<y]=\sum_{n=1}^{\infty}P[C'Z^*_n<y\,;N_t=n]$$

$$=\sum_{|n-\pi t|<\pi t\varepsilon}P[C'Z^*_n<y\,;N_t=n]$$

$$+\sum_{|n-\pi t|\geqq\pi t\varepsilon}P[C'Z^*_n<y\,;N_t=n]$$

$$\leqq\sum_{|n-\pi t|<\pi t\varepsilon}P[C'Z^*_n<y\,;N_t=n]+\varepsilon.$$

Hence,

$$|P[C'Z^*_{N_t}<y]-\sum_{|n-\pi t|<\pi t\varepsilon} P[C'Z^*_n<y\,;N_t=n]|<\varepsilon\,. \tag{15}$$

Introducing the random variables R_i, S_n and Q as defined in (11), we have for any n such that $N_1<n\leqq N_2$

$$\begin{aligned}
P[C'Z^*_n<y\,;N_t=n]&=P\Big[\sqrt{\frac{h_n}{n}}\Big\{\sum_{i=1}^{N_1}(R_i-\mu)+\sum_{i=N_1+1}^{n}(R_i-E(R_i))\\
&\qquad+\sum_{i=N_1+1}^{n}(E(R_i)-\mu)\Big\}<y\,;N_t=n\Big]\\
&\leqq P\Big[S_{N_1}<y\sqrt{\frac{N_2}{h_{N_2}}}-\sum_{i=N_1+1}^{n}(E(R_i)-\mu)+Q\,;N_t=n\Big]\\
&=P\Big[\sqrt{\frac{h_{N_1}}{N_1}}S_{N_1}<y\sqrt{\frac{N_2}{N_1}\Big(\frac{h_{N_1}}{h_{N_2}}\Big)}-\sqrt{\frac{h_{N_1}}{N_1}}\sum_{i=N_1+1}^{n}\{E(R_i)-\mu\}\\
&\qquad+\sqrt{\frac{h_{N_1}}{N_1}}Q\,;N_t=n\Big]\\
&\leqq P\Big[C'Z^*_{N_1}<y\sqrt{\Big(\frac{N_2}{N_1}\Big)^{1+\delta}}+\frac{C_1\varepsilon}{(1-2\varepsilon)}+\sqrt{\frac{h_{N_1}}{N_1}}Q\,;N_t=n\Big],
\end{aligned}\tag{16}$$

by Lemma 5. Hence,

$$\begin{aligned}
&\sum_{|n-\pi t|<\pi t\varepsilon}P[C'Z^*_n<y\,;N_t=n]\\
&\leqq P\Big[C'Z^*_{N_1}<y\sqrt{\Big(\frac{N_2}{N_1}\Big)^{1+\delta}}+\frac{C_1\varepsilon}{(1-2\varepsilon)}+\sqrt{\frac{h_{N_1}}{N_1}}Q\,;|N_t-\pi t|<\pi t\varepsilon\Big]\\
&\leqq P\Big[C'Z^*_{N_1}<y\Big(\frac{1+\varepsilon}{(1-2\varepsilon)}\Big)^{(1+\delta)/2}+\frac{C_1\varepsilon}{(1-2\varepsilon)}+\varepsilon^{1/3}\,;\\
&\qquad Q<\varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\,;|N_t-\pi t|<\pi t\varepsilon\Big]+P\Big[Q\geqq\varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\Big]\\
&\leqq P\Big[C'Z^*_{N_1}<y\Big(\frac{1+\varepsilon}{1-2\varepsilon}\Big)^{(1+\delta)/2}+\frac{C_1\varepsilon}{(1-2\varepsilon)}+\varepsilon^{1/3}\Big]+\frac{C_1\varepsilon^{1/3}}{(1-2\varepsilon)}\Big(\frac{1+\varepsilon}{1-2\varepsilon}\Big)^{\delta},
\end{aligned}\tag{17}$$

by Lemma 4.

From (16) we get in a similar manner

$$\begin{aligned}
&\sum_{|n-\pi t|<\pi t\varepsilon}P[C'Z^*_n<y\,;N_t=n]\\
&\geqq P\Big[C'Z^*_{N_1}<y-\frac{C_1\varepsilon}{(1-2\varepsilon)}-\sqrt{\frac{h_{N_1}}{N_1}}Q\,;|N_t-\pi t|<\pi t\varepsilon\Big]\\
&\geqq P\Big[C'Z^*_{N_1}<y-\frac{C_1\varepsilon}{(1-2\varepsilon)}-\varepsilon^{1/3}\,;Q<\varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\,;|N_t-\pi t|<\pi t\varepsilon\Big]\\
&\geqq P\Big[C'Z^*_{N_1}<y-\frac{C_1\varepsilon}{(1-2\varepsilon)}-\varepsilon^{1/3}\Big]-P\Big[Q\geqq\varepsilon^{1/3}\sqrt{\frac{N_1}{h_{N_1}}}\Big]-P[|N_t-\pi t|\geqq\pi t\varepsilon]\\
&\geqq P\Big[C'Z^*_{N_1}<y-\frac{C_1\varepsilon}{(1-2\varepsilon)}-\varepsilon^{1/3}\Big]-\frac{C_1\varepsilon^{1/3}}{(1-2\varepsilon)}\Big(\frac{1+\varepsilon}{1-2\varepsilon}\Big)^{\delta}-\varepsilon\,.
\end{aligned}\tag{18}$$

From (15), (17) and (18) we conclude that for $t \geqq t_0$

$$P[C'Z^*_{N_t}<y] \leqq P\left[C'Z^*_{N_1}<y\left(\frac{1+\varepsilon}{1-2\varepsilon}\right)^{(1+\delta)/2}+\frac{C_1\varepsilon}{(1-2\varepsilon)}+\varepsilon^{1/3}\right]+\frac{C_1\varepsilon^{1/3}}{(1-2\varepsilon)}\left(\frac{1+\varepsilon}{1-2\varepsilon}\right)^{\delta}+\varepsilon$$

and

$$P[C'Z^*_{N_t}<y] \geqq P\left[C'Z^*_{N_1}<y-\frac{C_1\varepsilon}{(1-2\varepsilon)}-\varepsilon^{1/3}\right]-\frac{C_1\varepsilon^{1/3}}{(1-2\varepsilon)}\left(\frac{1+\varepsilon}{1-2\varepsilon}\right)^{\delta}-2\varepsilon .$$

Similar statements hold for $y<0$. We now invoke Lemma 3 and the continuity of the distribution function of a normal random variable to complete the proof.

We are now in a position to prove the main theorem of this paper.

THEOREM. *Suppose K satisfies conditions* (i), (ii) *and* (iii), *and* $\{h_n=n^{-\delta}\}$, $\frac{1}{5}<\delta<\frac{1}{3}$ *and* $\frac{N_t}{t}\xrightarrow{p}\pi$ $(\pi>0)$ *as* $t\to\infty$. *Let* $E_f|Y|^3$ *be finite and let* g', g'', w', w'' *and* v' *exist and be bounded. If* $x_1\neq x_2$, $g(x_i)>0$ *for* $i=1, 2$, *then* $(N_t h_{N_t})^{1/2}(m_{N_t}(x_1)-m(x_1), m_{N_t}(x_2)-m(x_2))'$ *converges in distribution to* Z^* *as* t *tends to infinity where* Z^* *is a bivariate normal random variable with mean vector* 0 *and diagonal covariance matrix* $C=[c_{ij}]$ *where*

$$c_{ii}=\frac{\mathrm{Var}\,[Y|X=x_i]}{g(x_i)}\nu\int_{-\infty}^{\infty}K^2(u)du \qquad i=1, 2 .$$

PROOF. Using the Cramér-Wold theorem (Theorem (xi) on page 123 of [4]) we conclude from Lemma 6 that $Z^*_{N_t}$ converges in distribution to Z as t tends to infinity. The proof of the theorem now immediately follows from this result in conjunction with Theorem (iii) on page 388 of [4].

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