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Tagawa, Shojiro
College of Arts and Sciences, Chiba University

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AN EXTENSION OF *Q*-ANALYSIS

By

Shojiro TAGAWA*

Abstract

Q-analysis is a new method to research the mathematical structures associated with relations (as opposed to functions or mappings) and their applications to communities, town planning, design, and systems studies. It is therefore concerned with the problem how to make the "soft sciences" into hard sciences. Scientific data, called "hard data", is the result of observing set-membership. This leads to a binary relation, being Yes/No or 1/0 type. But, it is not sufficient in many applications, in fact, integers are already treated in Q-analysis. In this paper, we present another extension in which real numbers in the interval [0, 1] are considered.

1. Introduction

Q-analysis was introduced by Atkin [1]. It is a method to reseach the mathematical structures associated with relations. In this section, we outline the basic concept of Q-analysis.

Let X and Y be two finite sets, where

$$X=\{x_1, \cdots, x_m\} \quad \text{and} \quad Y=\{y_1, \cdots, y_n\}.$$

A relation λ between X and Y is a subset of $X\times Y$ (the cartesian product of X and Y). Each such relation λ may be represented by an incidence matrix $\Lambda=(\lambda_{ij})$ in which

$$\lambda_{ij}=\begin{cases} 1 & (x_i, y_j)\in\lambda \\ 0 & \text{otherwise} \end{cases} \qquad i=1, \cdots, m,\ j=1, \cdots, n.$$

Let $M=(m_{ij})$ be a matrix whose entries are integers. This matrix corresponds to a weighted relation μ. Take a set of parameters θ_i which are to characterise the matrix M. Such $\{\theta_i\}$ is called the set of slicing parameters for M and gives a relation λ by defining its incidence matrix $\Lambda=(\lambda_{ij})$ in terms of elements m_{ij} as follows:

$$\lambda_{ij}=\begin{cases} 1 & \text{if } m_{ij}\geqq\theta_k(m_{ij}) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the integer matrix M is converted to the incidence matrix Λ. But, the original structure may be deformed by this conversion. The purpose of this paper is to realize a natural representation of the structure by considering real numbers without slicing

* College of Arts and Sciences, Chiba University, Yayoi, Chiba, 260 Japan.

parameters.

Now, we continue to state about Q-analysis. A simplicial complex is used to represent a mathematical relation $\lambda \subset X \times Y$ in the following way.

For each $x_i \in X$, let $\{j_1, \cdots, j_{p+1}\}$ be the set of all j's such that $\lambda_{ij}=1$. Then, p-simplex $\sigma_p(x_i)$ is constructed by assigning elements $y_{j_1}, \cdots, y_{j_{p+1}}$ of Y with vertices of $\sigma_p(x_i)$ in a Euclidean space. It is denoted by

$$\sigma_p(x_i)=\langle y_{j_1}, \cdots, y_{j_{p+1}} \rangle .$$

The simplicial complex K which consists of $\sigma_p(x_i)$, $i=1, \cdots, m$ and all their faces is denoted by $K_X(Y; \lambda)$. $K_Y(X; \lambda^{-1})$ is the conjugate simplicial complex obtained by replacing X and Y in $K_X(Y; \lambda)$.

Given two simplices σ_p, σ_r in K, they are joined by a chain of connection if there exists a finite sequence of simplices

$$\sigma_{\alpha_1}, \sigma_{\alpha_2}, \cdots, \sigma_{\alpha_h}$$

such that

(i) σ_{α_1} is a face of σ_p,

(ii) σ_{α_h} is a face of σ_r,

(iii) σ_{α_i} and $\sigma_{\alpha_{i+1}}$ share a common face, say, σ_{β_i}, $i=1, 2, \cdots, (h-1)$.

This sequence is said to be a chain of q-connection (or a q-connectivity) if q is the least of the integers

$$\alpha_1, \beta_1, \beta_2, \cdots, \beta_{h-1}, \alpha_h .$$

The length of the chain is taken as $(h-1)$. As a special case, a p-simplex σ_p must be p-connected to itself, by a chain of length 0, although it cannot be $(p+1)$-connected to any simplex. If σ_p and σ_r are q-connected, then they are also $(q-1)$-. $(q-2)$-, $\cdots$, 1-, 0-connected in K.

An equivalence relation, γ_q, for a fixed q, on the simplices of a simplicial complex $K=K_X(Y; \lambda)$ is defined by

$$(\sigma_p, \sigma_r) \in \gamma_q \quad \text{if and only if } \sigma_p \text{ is } q\text{-connected to } \sigma_r.$$

The equivalence classes under γ_q are the members of the quotient set K/γ_q. The cardinality of K/γ_q is denoted by Q_q. This equals the number of distinct q-connected components in K. When we find all the values of

$$Q_0, Q_1, Q_2, \cdots, Q_N$$

where $N=dim\, K$, it is said that we have performed a Q-analysis on K.

In order to represent the more precise structure, the concept of hierarchy is introduced in [2]. We also extend this hierarchy in this paper.

2. An Extension

We first consider an incidence matrix where entries are real numbers in [0, 1].

Let $X=\{x_1, \cdots, x_m\}$ and $Y=\{y_1, \cdots, y_n\}$ be two finite sets. A relation λ is represented by an incidence matrix $\Omega=(\omega_{ij})$ where $0 \leqq \omega_{ij} \leqq 1$ for $i=1, \cdots, m$, $j=1, \cdots, n$.

$$\begin{array}{c|c} & Y \\ \hline X & \Omega \end{array} \qquad \Omega=(\omega_{ij})$$

The above relation λ is a subset of $X\times Y$ defined by

$$(x_i, y_j)\in\lambda \quad \text{if and only if } \omega_{ij}>0.$$

For each $x_i\in X$, let $\{j_1, \cdots, j_{p+1}\}$ be the set of all j's such that $\omega_{ij}>0$. Then, we construct a p-simplex $\sigma_p(x_i)$ by assigning elements $y_{j_1}, \cdots, y_{j_{p+1}}$ of Y with vertices of $\sigma_p(x_i)$ in a Euclidean space. We construct a simplicial complex $K_X(Y; \lambda)$ in the same way as in Section 1. We now introduce the concept of degree.

DEFINITION 2.1. For each simplex $\sigma_p(x_i)$, we define degree $\tau(\sigma_p(x_i))$ of $\sigma_p(x_i)$ by

$$\tau(\sigma_p(x_i))=\omega_{ij_1}+\omega_{ij_2}+\cdots+\omega_{ij_{p+1}}.$$

Note that the degree of $\sigma_p(x_i)$ is different from the dimension p of $\sigma_p(x_i)$. We next consider a chain of connection.

For any two simplices σ_p, σ_r in $K_X(Y; \lambda)$, a chain of connection

$$\sigma_{\alpha_1}, \sigma_{\alpha_2}, \cdots, \sigma_{\alpha_h}$$

is defined in the same way as in Section 1. We denote by $\sigma_{\alpha_i}\cap\sigma_{\alpha_{i+1}}$ a common face of σ_{α_i} and $\sigma_{\alpha_{i+1}}$. Let each degree of σ_{α_i} be given by

$$\tau(\sigma_{\alpha_i})=\omega_{\delta_i j_1}+\omega_{\delta_i j_2}+\cdots+\omega_{\delta_i j_{p_i}}.$$

DEFINITION 2.2. We consider a chain of connection described above. We define a degree of a common face $\sigma_{\alpha_i}\cap\sigma_{\alpha_{i+1}}$ by

$$\tau(\sigma_{\alpha_i}\cap\sigma_{\alpha_{i+1}})=\omega_{\delta_i j_{l_1}}\omega_{\delta_{i+1} j_{l_1}}+\omega_{\delta_i j_{l_2}}\omega_{\delta_{i+1} j_{l_2}}+\cdots+\omega_{\delta_i j_{l_s}}\omega_{\delta_{i+1} j_{l_s}}$$

where $y_{j_{l_1}}, \cdots, y_{j_{l_s}}$ belong to both σ_{α_i} and $\sigma_{\alpha_{i+1}}$. If the minimum of

$$\tau(\sigma_{\alpha_1}), \tau(\sigma_{\alpha_1}\cap\sigma_{\alpha_2}), \cdots, \tau(\sigma_{\alpha_{h-1}}\cap\sigma_{\alpha_h}), \tau(\sigma_{\alpha_h})$$

is ξ, then σ_p and σ_r are called to be connected with degree ξ.

If σ_p and σ_r are connected with degree ξ, then they are connected with degree η for all $\eta\in(0, \xi)$. It is easy to show that σ_p and σ_r are q-connected if and only if they are connected with degree $(q+1)$. Therefore, the connectivity of degree ξ is a natural extension of the q-connectivity.

LEMMA 2.1. *The connectivity of degree ξ is an equivalence relation on $K_X(Y; \lambda)$.*

We omit the proof of Lemma 2.1. Let Ξ_ξ be the number of equivalence classes on $K_X(Y; \lambda)$ induced by the connectivity with degree ξ. Since the number of ξ is not finite, any vector corresponding to a Q-vector is not finite dimensional. We hence introduce a new vector which plays the same role as a Q-vector.

On the other hand, Q-hierarchy is introduced in [2], which is a very good tool to represent a structure. We first introduce a hierarchy which extends a Q-hierarchy.

DEFINITION 2.3. We consider a simplicial complex $K_X(Y; \lambda)$ and the connectivity with degree $\xi>0$. Then, a multi-level digraph H is a triple (V, E, ψ) where

V=the set of all the mutually different equivalence classes by the connectivity with degree ξ for all $\xi>0$,

$E=\{(u, v) \mid u \subsetneqq v$ for $u, v \in V\}$,

ψ=the function from V to $(0, +\infty)$ defined by $\psi(u)=max\{\xi \mid u$ is an equivalence class under the connectivity with degree $\xi\}$ for all $u \in V$.

We can now state a new vector.

DEFINITION 2.4. We consider a multi-level digraph H in Definition 2.3. We select numbers $\xi_1, \xi_2, \cdots, \xi_l$ which are all mutually different numbers $\psi(u)$ for all $u \in V$ and satisfy

$$\xi_1>\xi_2>\xi_3> \cdots >\xi_l>0 .$$

Then, the structure vector Ξ is defined by

$$\Xi=(\overset{\xi_1}{\Xi}_{\xi_1}, \overset{\xi_2}{\Xi}_{\xi_2}, \cdots, \overset{\xi_l}{\Xi}_{\xi_l}) .$$

Note that the structure vectore Ξ is an extension of the Q-vector. We next show a property of a multi-level digraph introduced in Definition 2.3.

PROPOSITION 2.1. *Consider a multi-level digraph H in Definition* 2.3. *Then, H is a forest.*

PROOF. It suffices to show that there are no vertices $u, v, w \in V$ where $v \neq w$, $(u, v) \in E$ and $(u, w) \in E$. To show the contradiction, we assume that three verticies u, v and w satisfy the above conditions. By the difinition of V, we have

$$u \subsetneqq v \quad \text{and} \quad u \subsetneqq w .$$

Then, v and w meet, i. e.

$$v \cap w \neq \phi .$$

This implies that any two elements of v or w are connected with degree $min\{\psi(v), \psi(w)\}$. Furthermore, we have

$$v=w$$

since v and w are equivalence classes. But, this contradicts the assumptions for u, v and w. *Q.E.D.*

3. An Example

We consider a simple example. In an institute, there are research projects A, B, C and D which needs mathematics, linguistics, computer science, sociology, and informatics as the following table.

	X \ Y	y_1 A	y_2 B	y_3 C	y_4 D
x_1	mathematics	0.1	0.1	0.1	0.1
x_2	linguistics	0.5	0	0.3	0
x_3	comp. sci.	0.5	0.1	0.8	0
x_4	sociology	0	0.8	0	0
x_5	informatics	0	0	0	0.9

The necessity is measured by, for example, time or the amount of documents, and so on. Moreover, it is normalized. In this example, the basic simplices are as follows:

$$\sigma(x_1)=\langle y_1, y_2, y_3, y_4\rangle, \quad \tau(\sigma(x_1))=0.4$$

$$\sigma(x_2)=\langle y_1, y_3\rangle, \quad \tau(\sigma(x_2))=0.8$$

$$\sigma(x_3)=\langle y_1, y_2, y_3\rangle, \quad \tau(\sigma(x_3))=1.4$$

$$\sigma(x_4)=\langle y_2\rangle, \quad \tau(\sigma(x_4))=0.8$$

$$\sigma(x_5)=\langle y_4\rangle, \quad \tau(\sigma(x_5))=0.9.$$

We then obtain the multi-level digraph $H=(V, E, \phi)$ where

$$\begin{aligned} V=\{&\{\sigma(x_3)\}, \{\sigma(x_5)\}, \{\sigma(x_2)\}, \{\sigma(x_4)\}, \{\sigma(x_3), \sigma(x_2)\}, \\ &\{\sigma(x_1)\}, \{\sigma(x_3), \sigma(x_2), \sigma(x_1)\}, \{\sigma(x_3), \sigma(x_2), \sigma(x_1), \sigma(x_5)\}, \\ &\{\sigma(x_3), \sigma(x_2), \sigma(x_1), \sigma(x_5), \sigma(x_4)\}\} \\ =\{&v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}, \end{aligned}$$

$$E=\{(v_i, v_j)\,|\,v_i \subsetneqq v_j \ \mathit{for}\ v_i, v_j \in V\},$$

$$\phi(v_1)=1.4 \quad \phi(v_2)=0.9 \quad \phi(v_3)=0.8$$

$$\phi(v_4)=0.8 \quad \phi(v_5)=0.47 \quad \phi(v_6)=0.4$$

$$\phi(v_7)=0.14 \quad \phi(v_8)=0.09 \quad \phi(v_9)=0.08.$$

If there are three edges (v_i, v_j), (v_j, v_k) and (v_i, v_k), then we omit edge (v_i, v_k). Thus, we obtain the skeleton of H, which is sketched in the following figure. The structure vector Ξ is

$$\Xi=\begin{pmatrix} \overset{1.4}{1} & \overset{0.9}{2} & \overset{0.8}{4} & \overset{0.49}{3} & \overset{0.4}{4} & \overset{0.14}{3} & \overset{0.09}{2} & \overset{0.08}{1} \end{pmatrix}.$$

We now compare the structure vector with Q-vector by slicing parameter θ. If we set $\theta=0.05$, then we have

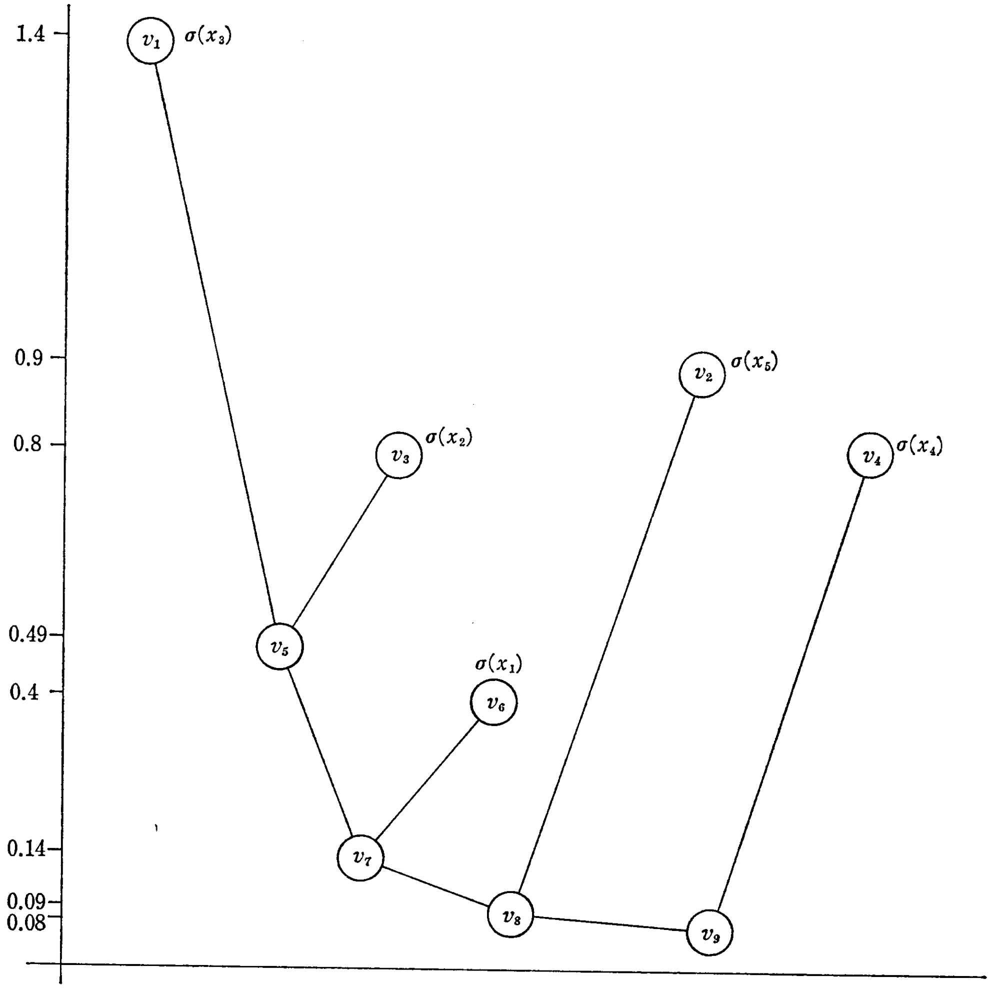
$$Q=\begin{pmatrix} \overset{3}{1} & \overset{2}{1} & \overset{1}{1} & \overset{0}{1} \end{pmatrix}.$$

Consider simplex $\sigma(x_1)$ of mathematics. This Q-vector implies that mathematics is the most widely used among these projects. On the other hand, by setting $\theta=0.2$, we observe that mathematics is no more used in any projects. The necessity of mathematics is all or nothing. But, by our method, mathematics is reasonably evaluated.

4. Concluding Remarks

We have extended Q-analysis to represent more detail structures. Such an extension will be possible for other concepts, say, backcloth, holes and so on.

Algorithms for our methods will be also developed.



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