INVERSE FUNCTIONAL EQUATIONS FOR BELLMAN'S ALLOCATION PROCESS

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By

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Abstract

This paper studies an inversion of functional equation for Bellman's infinite-stage allocation process. Obtaining a sufficient condition under which Bellman's (main) functional equation has a unique invertible solution, we propose additive- and recursive-inverse functional equations which have a common solution. It is proved that the additive-inverse functional equation has a unique solution in a moderate class of functions. The solution is obtained from two successive approximation methods.

1. Introduction

One of the most interesting features of dynamic programming is found in the functional equation

$$f(x) = \operatorname*{Max}_{0 \le y \le x} [g(y) + h(x-y) + f(ay + b(x-y))] \qquad x \ge 0$$

(1)

$$f(0) = 0$$

which was introduced by Bellman [1, Chap. 1]. The equation (1) seems to involve all the essences of dynamic programming. Bellman proved among many others that under some moderate assumptions the equation (1) has a unique solution which is continuous at 0.

This paper continues Bellman's work [1, Chap. 1] with three purposes of developing his functional equation treatment to a kind of dual theory. The first purpose is to prove under the additional assumption " invertibility of g, h " the existence of a unique " invertible " solution f of (1). The existence in turn enables us to propose two inverse functional equations

$$u(z) = \operatorname*{Min}_{0 \le y \le x = f^{-1}(z)} [(1-b)x + (b-a)y + u(z - g(y) - h(x-y))] \qquad z \ge 0$$

(5)

$$u(0) = 0$$

and

$$v(z) = \operatorname*{Min}_{0 \le y \le x = f^{-1}(z)} \frac{1}{b} [(b-a)y + v(z - g(y) - h(x-y))] \qquad z \ge 0$$

(6)

$$v(0) = 0$$

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where $0 \leqq b < 1$ for (5) and for $0 < b < 1$ for (6), respectively. The equation (5) does not admit the trivial solution. But, (6) admits. The former will turn out to be more attractive than the latter. If we formally weaken the condition $x = f^{-1}(z)$ to $x < \infty$ in (5), then the resulting equation becomes an infinite-version of the recursive formula

$$u_1(z) = \underset{\substack{g(y_1) + h(y_2) = z \\ y_1, y_2 \geqq 0}}{\text{Min}} [(1-a)y_1 + (1-b)y_2]$$

$$u_{n+1}(z) = \underset{y_1, y_2 \geqq 0}{\text{Min}} [(1-a)y_1 + (1-b)y_2 + u_n(z - g(y_1) - h(y_2))]$$

for Bellman's finite-stage process [1, p. 45, No. 20]. However, both in problem setting and in its analysis, imposing the constraint $x = f^{-1}(z)$ will play a crucial role in this paper.

The second purpose is to show that both equations (5) and (6) have a common solution f^{-1}. The proof justifies the introduction of (5) and (6). The third purpose is to prove the uniqueness of the solution of (5) in the class of functions which are non-negative and continuous at 0.

The inverse theory of finite-stage dynamic programming which escapes the functional equation treatment has been extensively studied by the author [2-9].

2. Bellman's Basic Functional Equation

In this section we shall prepare Bellman's basic functional equation for an infinite-stage allocation process [1, Chap. 1].

Under the following assumptions

a. $g, h : [0, \infty) \rightarrow [0, \infty)$ are continuous and $g(0) = h(0) = 0$.

a′. g, h are strictly increasing and $g(\infty) = h(\infty) = \infty$.

b. If $m(x) = \underset{0 \leqq y \leqq x}{\text{Max}} \text{Max}(g(y), h(y))$ and $c = \text{Max}(a, b)$, then $\sum_{n=0}^{\infty} m(c^n x) < \infty$ for all $x \geqq 0$.

c. $0 \leqq a < 1$, $0 \leqq b < 1$.

we consider the functional equation

(1)
$$f(x) = \underset{0 \leqq y \leqq x}{\text{Max}} [g(y) + h(x-y) + f(ay + b(x-y))] \qquad x \geqq 0$$

$$f(0) = 0.$$

This is called *main functional equation*. Let $f(x)$, if it exists, be the maximum value of $\sum_{n=0}^{\infty} [g(y_n) + h(x_n - y_n)]$ over the infinite-stage resource-allocation process $x_{n+1} = a y_n + b(x_n - y_n)$, $0 \leqq y_n \leqq x_n$, $n \geqq 0$ with an initial resource $x_0 = x$, then the maximum total reward function f satisfies the main functional equation (1). However, every solution of (1) does not represent the maximum value of this allocation process. This fact leads us to the mathematical problem of existence and uniqueness. The following basic result is due to Bellman.

THEOREM 1 (Bellman [1, p. 12]) *Under Assumptions a, b and c, there exists a unique solution of (1) which is continuous at 0. Moreover, this function is continuous on $[0, \infty)$.*

The proof is based upon the successive approximation method. Let us define the sequence $\{f_n\}$ as follows.

(2) $$f_1(x) = \underset{0 \le y \le x}{\text{Max}} [g(y) + h(x-y)] \qquad x \ge 0$$

(3) $$f_{n+1}(x) = \underset{0 \le y \le x}{\text{Max}} [g(y) + h(x-y) + f_n(ay + b(x-y))] \qquad x \ge 0, \ n \ge 1.$$

Then it converges to the unique solution of (1).

3. Invertible Functional Equation

Our first aim is to obtain an invertible solution of (1). Under the additional assumption a', this will be the case.

LEMMA 1. *Under Assumptions a, a', b and c, the sequence $\{f_n\}$ enjoys the following properties.*

(i) $f_n : [0, \infty) \to [0, \infty)$ *is continuous strictly increasing and $f_n(0)=0$, $f_n(\infty)=\infty$ for $n \ge 1$.*

(ii) $f_n(x) \le f_{n+1}(x)$ *on $[0, \infty)$ for $n \ge 1$. Moreover, if $a>0$ and $b>0$, then $f_n(x) < f_{n+1}(x)$ on $(0, \infty)$.*

(iii) *$\{f_n\}$ converges uniformly on any finite interval of $[0, \infty)$. The convergence is monotone increasing.*

(iv) *The limit function $f : [0, \infty) \to [0, \infty)$ is continuous strictly increasing and $f(0)=0$, $f(\infty)=\infty$. Moreover, it is a solution of (1).*

PROOF. Bellman's proof is fundamental [1, p. 12-16]. It suffices to prove only the following additional results not claimed in his proof.

A. Strict increasingness of f_n.

Let $x_1 < x_2$. Let $y_1(x)$ be a maximizer of (2). Then we have

$$f_1(x_1) = g(y_1(x_1)) + h(x_1 - y_1(x_1))$$
$$< g(y_1(x_1)) + h(x_2 - y_1(x_1)) \quad \text{(strict increasingness of } h\text{)}$$
$$\le \underset{0 \le y \le x_2}{\text{Max}} [g(y) + h(x-y)] \qquad (0 \le y_1(x_1) \le x_1 < x_2)$$
$$= f_1(x_2).$$

Further let f_n be strictly increasing and $y_n(x)$ be a maximizer of (3). Then we have

$$f_{n+1}(x_1) = g(y_n(x_1)) + h(x_1 - y_n(x_1)) + f_n(ay_n(x_1) + b(x_1 - y_n(x_1)))$$
$$< g(y_n(x_1)) + h(x_2 - y_n(x_1)) + f_n(ay_n(x_1) + b(x_2 - y_n(x_1)))$$
$$\text{(strict increasingness of } h \text{ and } f_n\text{)}$$
$$\le \underset{0 \le y \le x_2}{\text{Max}} [g(y) + h(x_2 - y) + f_n(ay + b(x_2 - y))]$$
$$(0 \le y_n(x_1) \le x_1 < x_2)$$
$$= f_{n+1}(x_2).$$

B. $f_1(\infty) = \infty$.

For any large $M>0$, there exists an $x>0$ such that $g(x/2) \ge M$ and $h(x/2) \ge M$. Hence, for any y in $[0, x]$, it holds that either $g(y) \ge M$ or $h(x-y) \ge M$. Thus we obtain

$$f_1(x) = \underset{0 \le y \le x}{\text{Max}} [g(y) + h(x-y)]$$
$$\ge M.$$

(Similarly, $f_n(\infty)=\infty$ is inductively proved).

C. $f_n(x)<f_{n+1}(x)$.

Let $0<a,b<1$. Then for $x>0$ we have

$$f_1(x)=g(y_1(x))+h(x-y_1(x))$$

$$<g(y_1(x))+h(x-y_1(x))+f_1(ay_1(x)+b(x-y_1(x)))$$

$$(f_1(x)>0 \text{ for } x>0)$$

$$\leqq \underset{0\leqq y\leqq x}{\text{Max}}[g(y)+h(x-y)+f_1(ay+b(x-y))]$$

$$=f_2(x).$$

Assuming that $f_{n-1}(x)<f_n(x)$ on $(0,\infty)$, we further obtain

$$f_n(x)=g(y_{n-1}(x))+h(x-y_{n-1}(x))+f_{n-1}(ay_{n-1}(x)+b(x-y_{n-1}(x)))$$

$$<g(y_{n-1}(x))+h(x-y_{n-1}(x))+f_n(ay_{n-1}(x)+b(x-y_{n-1}(x)))$$

$$(f_n(x)>0 \text{ for } x>0)$$

$$\leqq \underset{0\leqq y\leqq x}{\text{Max}}[g(y)+h(x-y)+f_n(ay+b(x-y))]$$

$$=f_{n+1}(x).$$

D. $f_n(\infty)=\infty$ for $n\geqq 2$ and $f(\infty)=\infty$.

This (strict) monotonicity together with $f_1(\infty)=\infty$ implies $f_n(\infty)=\infty$ for $n\geqq 2$ and $f(\infty)=\infty$.

E. $f(0)=0$.

This is a direct consequence of $f_n(0)=0$ for $n\geqq 1$.

F. Strict increasingness of f.

Let $0\leqq x_1<x_2$ and $y(x)$ be a maximizer of (1). Then we have

$$f(x_1)=g(y(x_1))+h(x_1-y(x_1))+f(ay(x_1)+b(x_1-y(x_1)))$$

$$<g(y(x_1))+h(x_2-y(x_1))+f(ay(x_1)+b(x_2-y(x_1)))$$

$$\leqq \underset{0\leqq y\leqq x_2}{\text{Max}}[g(y)+h(x_2-y)+f(ay+b(x_2-y))]$$

$$(0\leqq y(x_1)\leqq x_1<x_2)$$

$$=f(x_2).$$

This completes the proof of Lemma 1.

Thus, Lemma 1 leads us to show

THEOREM 2. *Under Assumptions a, a', b and c, there exists a unique solution of* (1) *which is continuous at 0. Moreover, this function is strictly increasing continuous and goes to infinity as so does x.*

4. Two Inverse Functional Equations

Throughout the remainder of the paper we always assume a, a', b and c. Then Theorem 2 shows us that the unique solution f of (1) which is continuous at 0 is invertible. The inverse function f^{-1} satisfies $f^{-1}(0)=0$, $f^{-1}(\infty)=\infty$ and strict increasingness like as g, h and f.

The next lemma will take an important part in the remainder of the paper.

LEMMA 2. *The condition* $0 \leq y \leq x = f^{-1}(z)$ *implies*

(4)
$$z - g(y) - h(x - y) \geq f(ay + b(x - y))$$
$$\geq f(d f^{-1}(z))$$
$$\geq 0$$

where $d = \mathrm{Min}(a, b)$.

PROOF. Let $0 \leq y \leq x = f^{-1}(z)$. Then we have

$$z = f(x)$$
$$= \underset{0 \leq y' \leq x}{\mathrm{Max}} [g(y') + h(x - y') + f(ay' + b(x - y'))]$$
$$\geq g(y) + h(x - y) + f(ay + b(x - y)).$$

Since $f(x) \geq 0$ for $x \geq 0$, this leads to

$$z - g(y) - h(x - y) \geq f(ay + b(x - y))$$
$$\geq f(dx)$$
$$= f(d f^{-1}(z))$$
$$\geq 0,$$

which completes the proof.

Lemma 2 enables us to consider the functional equation

(5)
$$u(z) = \underset{0 \leq y \leq x = f^{-1}(z)}{\mathrm{Min}} [(1 - b)x + (b - a)y + u(z - g(y) - h(x - y))] \qquad z \geq 0$$
$$u(0) = 0$$

rather than the equation

(5′)
$$u(z) = \underset{\substack{0 \leq y \leq x = f^{-1}(z) \\ 0 \leq g(y) + h(x - y) \leq z}}{\mathrm{Min}} [(1 - b)x + (b - a)y + u(z - g(y) - h(x - y))] \qquad z \geq 0$$
$$u(0) = 0.$$

That is, the constraint $0 \leq g(y) + h(x - y) \leq z$ is redundant for treating functional equations involving optimization under the condition $0 \leq y \leq x = f^{-1}(z)$. We call (5) *additive-inverse functional equation* of (1). We always keep our mind on the equality

$$(1 - b)x + (b - a)y = (1 - a)y + (1 - b)(x - y)$$
$$= x - (ax + b(x - y)).$$

It is considered that $u(z)$, if it exists, represents the minimum value of

$$\sum_{n=0}^{\infty} [(1 - b)x_n + (b - a)y_n]$$

over the infinite-stage total-reward-decrease process $z_{n+1} = z_n - g(y_n) - h(x_n - y_n)$, $0 \leq y_n \leq x_n = f^{-1}(z_n)$, $n \geq 0$ with a preassigned total reward $z_0 = z$. We see that any total cost is roughly evaluated as follows.

$$\sum_{n=0}^{\infty} [(1 - b)x_n + (b - a)y_n] \geq (1 - c) \sum_{n=0}^{\infty} x_n$$
$$\geq (1 - c)f^{-1}(z).$$

Thus, the minimum value is larger than or equal to $(1-c)f^{-1}(z)$. This lower bound makes the additive-inverse functional equation more attractive. This equation does not admit the trivial solution.

On the other hand, another inverse functional equation is introduced as follows. For $0 < b < 1$ we consider

(6)
$$v(z) = \underset{0 \leq y \leq x = f^{-1}(z)}{\text{Min}} \frac{1}{b} [(b-a)y + v(z - g(y) - h(x-y))] \qquad z \geq 0$$
$$v(0) = 0.$$

This is called *recursive-inverse functional equation* of (1). It is considered that $v(z)$, if it exists, represents the minimum value of $\sum_{n=0}^{\infty} \frac{(b-a)}{b^{n+1}} y_n$ over the total-reward-decrease process stated above. Any path with $y_n \equiv 0$ yields zero value. Thus, the minimum value attains zero for $b \geq a$. The recursive-inverse functional equation admits the trivial solution for $b \geq a$. This triviality escapes a further discussion of the equation (6).

For N-stage main dynamic program (DP), the recursive-inverse DP is introduced in [2-5, 7] and the additive-inverse DP in [8]. For infinite-stage DP, the recursive inversion is applied in [6]. The author [9] has investigated the additive inversion of continuous-time DP.

The derivation of equations (5) and (6) will be justified in the proof of Theorem 3.

5. Common Solution

THEOREM 3. *Both equations (5) and (6) have a common solution f^{-1}.*

PROOF. Let f be the unique solution of (1) which is continuous at 0. Then for any $z \geq 0$ there exists a unique $x \geq 0$ with

(7)
$$z = f(x) = \underset{0 \leq y \leq x}{\text{Max}} [g(y) + h(x-y) + f(ay + b(x-y))].$$

From (7) we have

(8)
$$z \geq g(y) + h(x-y) + f(ay + b(x-y)) \qquad \text{for} \quad 0 \leq y \leq x = f^{-1}(z)$$

which is equivalent to

$$f^{-1}(z - g(y) - h(x-y)) \geq ay + b(x-y) \qquad \text{for} \quad 0 \leq y \leq x = f^{-1}(z).$$

That is, (7) implies

(10)
$$(b-a)y + f^{-1}(z - g(y) - h(x-y)) \geq bx \qquad \text{for} \quad 0 \leq y \leq x = f^{-1}(z).$$

First, adding $(1-b)x$ to both-hand sides of (10), we obtain

$$(1-b)x + (b-a)y + f^{-1}(z - g(y) - h(x-y)) \geq x \qquad \text{for} \quad 0 \leq y \leq x = f^{-1}(z).$$

This leads to

(11)
$$\underset{0 \leq y \leq x = f^{-1}(z)}{\text{Inf}} [(1-b)x + (b-a)y + f^{-1}(z - g(y) - h(x-y))] \geq f^{-1}(z).$$

Second, dividing both-hand sides of (10) by b $(0 < b < 1)$, we have

$$\frac{1}{b} [(b-a)y + f^{-1}(z - g(y) - h(x-y))] \geq x \qquad \text{for} \quad 0 \leq y \leq x = f^{-1}(z).$$

This in turn yields

(12) $$\text{Inf}_{0 \le y \le x = f^{-1}(z)} \frac{1}{b}[(b-a)y + f^{-1}(z - g(y) - h(x-y))] \ge f^{-1}(z).$$

Furthermore, (7) together with its maximizer $y(x)$ implies

(13) $$z = g(y(x)) + h(x - y(x)) + f(a y(x) + b(x - y(x))) \qquad \text{for} \quad x = f^{-1}(z).$$

Accordingly we see that (13) yields both

(11′) $$(1-b)x + (b-a)y(x) + f^{-1}(z - g(y(x)) - h(x - y(x))) = f^{-1}(z)$$

and

(12′) $$\frac{1}{b}[(b-a)y(x) + f^{-1}(z - g(y(x)) - h(x - y(x)))] = f^{-1}(z)$$

where $x = f^{-1}(z)$. From (11), (11′), (12) and (12′), we have a common solution f^{-1} of (5) and (6). This completes the proof.

6. Maximality and Minimality

Our second aim is to obtain a uniqueness theorem. This will be done after discussion on maximal and minimal solutions. Each solution will be obtained through the usual approach for uniqueness of solution to dynamic programming functional equation [1, p. 15, 16].

THEOREM 4. *The f^{-1} is the largest solution of* (5) *which is continuous at* 0.

PROOF. Let v be any solution of (5) which is continuous at 0. Then we have for a minimizer $y(x)$ of (5)

(14)
$$v(z) = \underset{0 \le y \le x = f^{-1}(z)}{\text{Min}} [(1-b)x + (b-a)y + v(z - g(y) - h(x-y))]$$
$$\le (1-b)x + (b-a)y(x) + v(z - g(y(x)) - h(x - y(x)))$$

and

(15) $$f^{-1}(z) = (1-b)x + (b-a)y(x) + f^{-1}(z - g(y(x)) - h(x - y(x)))$$

where $x = f^{-1}(z)$. From (14), (15) we have

(16) $$v(z) - f^{-1}(z) \le v(z - g(y(x)) - h(x - y(x))) - f^{-1}(z - g(y(x)) - h(x - y(x)))$$

and

(17)
$$z - g(y(x)) - h(x - y(x)) = f(x - (1-b)x - (b-a)y(x))$$
$$= f(a y(x) + b(x - y(x)))$$
$$\le f(cx)$$
$$= f(c f^{-1}(z)).$$

Let us now define

(17) $$U(w) = \underset{0 \le z \le w}{\text{Sup}} (v(z) - f^{-1}(z)).$$

Then U is continuous at 0 and has the value 0 there. From (16), (17) and (18) we obtain

(19) $$U(w) \le U(f(c f^{-1}(w))),$$

whence, by iteration,

(20) $$U(w) \leqq U(f(c^n f^{-1}(w))) \qquad \text{for} \quad n \geqq 1.$$

Here we remark that $f(c^n f^{-1}(w))$ decreases monotonically to 0 as n goes to ∞. Since U is continuous at 0 and $U(0) = 0$, upon letting $n \to \infty$, we obtain $U(w) \leqq 0$, and thus that $v(z) \leqq f^{-1}(z)$. This completes the proof of maximality.

This maximality theorem in turn yields a minimality of f^{-1} for $0 \leqq a, b < 1/2$.

THEOREM 5. *If* $0 \leqq a, b < 1/2$, *then* f^{-1} *is the smallest solution of (5) which is continuous strictly increasing and goes to infinity as so does z.*

PROOF. Let v be any solution of (5) which is continuous strictly increasing and $v(\infty) = \infty$, and $\hat{y}(x)$ be a corresponding minimizer. Then we have

(21) $$v(z) = (1-b)x + (b-a)\hat{y}(x) + v(z - g(\hat{y}(x)) - h(x - \hat{y}(x)))$$

and

(22) $$f^{-1}(z) = \operatorname*{Min}_{0 \leqq y \leqq x = f^{-1}(z)} [(1-b)x + (b-a)y + f^{-1}(z - g(y) - h(x-y))]$$
$$\leqq (1-b)x + (b-a)\hat{y}(x) + f^{-1}(z - g(\hat{y}(x)) - h(x - \hat{y}(x)))$$

where $x = f^{-1}(z)$. From (21), (22) we in turn get

(23) $$f^{-1}(z) - v(z) \leqq f^{-1}(z - g(\hat{y}(x)) - h(x - \hat{y}(x))) - v(z - g(\hat{y}(x)) - h(x - y(x))),$$

(24) $$v(z) \geqq (1-b)x + (b-a)\hat{y}(x)$$
$$= x - a\hat{y}(x) - b(x - \hat{y}(x))$$
$$\geqq x - cx$$
$$= (1-c)f^{-1}(z)$$

and

(25) $$z - g(\hat{y}(x)) - h(x - \hat{y}(x)) = v^{-1}(v(z) - (1-b)x - (b-a)\hat{y}(x))$$
$$= v^{-1}(v(z) - x + a\hat{y}(x) + b(x - \hat{y}(x))).$$

Here we remark that Theorem 4 shows

(26) $$v(z) - x = v(z) - f^{-1}(z) \leqq 0.$$

From (25), (26) we have

(27) $$z - g(\hat{y}(x)) - h(x - \hat{y}(x)) \leqq v^{-1}(a\hat{y}(x) + b(x - \hat{y}(x)))$$
$$\leqq v^{-1}(cx)$$
$$= v^{-1}(cf^{-1}(z)).$$

On the other hand, (24) leads to

(28) $$v^{-1}(u) \leqq f\left(\frac{u}{1-c}\right).$$

Combining (27) and (28), we have

(29) $$z - g(\hat{y}(x)) - h(x - \hat{y}(x)) \leqq f\left(\frac{c}{1-c} f^{-1}(z)\right).$$

Let us now define

(30) $$L(w) = \operatorname*{Max}_{0 \leqq z \leqq w} (f^{-1}(z) - v(z)).$$

Then L is continuous and has the value 0 at $w=0$.

From (23), (29), (30) we obtain

(31)
$$L(w) \leq L\left(f\left(\frac{c}{1-c} f^{-1}(w)\right)\right),$$

whence, by iteration,

$$L(w) \leq L\left(f\left(\left(\frac{c}{1-c}\right)^n f^{-1}(w)\right)\right) \qquad n \geq 1.$$

Here we remark that $0 \leq c = \text{Max}(a, b) < 1/2$. Since L is continuous and $L(0)=0$, upon letting $n \to \infty$, we obtain $L(w) \leq 0$, and that $f^{-1}(z) \leq v(z)$. This completes the proof of minimality.

COROLLARY 1. *If* $0 \leq a, b < 1/2$, *then* f^{-1} *is the unique solution of* (5) *which is continuous strictly increasing and goes to infinity as so does z.*

COROLLARY 2. *Any solution* v *of* (5) *which is nonnegative and continuous at* 0 *satisfies*

$$(1-c)f^{-1}(z) \leq v(z) \leq f^{-1}(z).$$

7. Uniqueness

Corollary 2 gives us both upper and lower bounds. Theorem 4 shows us that the upper bound is most desirable. However, the lower bound is more improvable. This improvement is based upon the following lemma, which leads directly to uniqueness of the solution f^{-1}.

LEMMA 3. *Any solution* v *of* (5) *which is nonnegative and continuous at* 0 *satisfies*

(i) $\dfrac{1-c}{1-d} f^{-1}(z) \leq v(z) \leq f^{-1}(z)$. *If* $a=b$, *then* $v=f^{-1}$.

(ii) $(1-c^n)f^{-1}(z) \leq v(z) \leq f^{-1}(z)$ *for all* $n \geq 1$. *Hence* $v=f^{-1}$.

PROOF. Theorem 4 shows that $v \leq f^{-1}$. It suffices to prove the lower boundedness.

(i) First, from Corollary 2 and Lemma 2, we have

$$v(z) = \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [(1-a)y + (1-b)(x-y) + v(z-g(y)-h(x-y))]$$

$$\geq \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [(1-c)y + (1-c)(x-y) + (1-c)f^{-1}(z-g(y)-h(x-y))]$$

$$\geq (1-c) \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x + f^{-1}(f(ay+b(x-y)))]$$

$$= (1-c) \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x + ay + b(x-y)]$$

$$\geq (1-c) \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x + dx]$$

$$= (1-c)(1+d)f^{-1}(z).$$

Second, this inequality in turn yields

$$v(z) = \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [(1-a)y + (1-b)(x-y) + v(z-g(y)-h(x-y))]$$

$$\geq \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [(1-c)y + (1-c)(x-y) + (1-c)(1+d)f^{-1}(z-g(y)-h(x-y))]$$

$$\geq (1-c)(1+d+d^2)f^{-1}(z).$$

We have successively

$$v(z) \geqq (1-c)\Big(\sum_{i=0}^{n} d^i\Big)f^{-1}(z) \qquad n \geqq 1.$$

Hence, by letting $n \to \infty$, we obtain

$$v(z) \geqq \frac{1-c}{1-d} f^{-1}(z).$$

(ii) First, from Corollary 2 and Lemma 2, we have

$$
\begin{aligned}
v(z) &= \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [(1-a)y + (1-b)(x-y) + v(z - g(y) - h(x-y))] \\
&\geqq \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x - (ay + b(x-y)) + (1-c)f^{-1}(f(ay + b(x-y)))] \\
&= \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x - c(ay + b(x-y))] \\
&\geqq \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x - c(cx)] \\
&= (1 - c^2)f^{-1}(z).
\end{aligned}
$$

Second, this inequality in turn yields

$$
\begin{aligned}
v(z) &\geqq \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x - (ay + b(x-y)) + (1 - c^2)f^{-1}(z - g(y) - h(x-y))] \\
&\geqq \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [x - (ay + b(x-y)) + (1 - c^2)(ay + b(x-y))] \\
&\geqq (1 - c^3)f^{-1}(z).
\end{aligned}
$$

Further, by iteration, we get

$$v(z) \geqq (1 - c^n)f^{-1}(z) \qquad n \geqq 1.$$

Hence, upon letting $n \to \infty$, we obtain

$$v(z) \geqq f^{-1}(z).$$

This completes the proof.

The following theorem covers both Corollary 1 in this paper and Theorem 10 in [8].

THEOREM 6. *The functional equation (5) has the unique solution f^{-1} which is non-negative and continuous at 0.*

PROOF. This is a direct consequence of Lemma 3(ii).

8. Two Successive Approximations

The third aim is to prove that two successive approximations yield the solution f^{-1} of (5). One approximation whose n-th feasible action space is time-invariant may be monotone increasing. The other approximation which is restricted to time-variant region is always monotone decreasing.

LEMMA 4. *Any sequence $\{v_n\}$ generated by*

(32) $$v_{n+1}(z) = \operatorname*{Min}_{0 \leq y \leq x = f^{-1}(z)} [(1-b)x + (b-a)y + v_n(z - g(y) - h(x-y))] \quad z \geqq 0,\ n \geqq 0$$

where

(33) $$v_0 \ \text{is continuous and}\ 0 \leqq v_0(z) \leqq f^{-1}(z)$$

satisfies

(i) v_n *is continuous.*

(ii) $(1-c^n)f^{-1}(z)\leqq v_n(z)\leqq f^{-1}(z)$ *i.e.,* $-c^n f^{-1}(z)\leqq v_n(z)-f^{-1}(z)\leqq 0$. *Thus,* $\{v_n\}$ *converges to* f^{-1}.

(iii) *If* $v_0=0$, *then the convergence is monotone increasing.*

PROOF. (i) The continuity is trivial. (ii) Let $y_n(x)$ be a minimizer of (32). Then we have

$$(34) \qquad v_{n+1}(z)=(1-a)y_n(x)+(1-b)(x-y_n(x))+v_n(z-g(y_n(x))-h(x-y_n(x)))$$

where $x=f^{-1}(z)$. Since $v_0\geqq 0$, this leads to

$$(35) \qquad v_1(z)\geqq (1-a)y_1(x)+(1-b)(x-y_1(x))$$
$$\geqq (1-c)y_1(x)+(1-c)(x-y_1(x))$$
$$=(1-c)f^{-1}(z).$$

On the other hand, $v_0\leqq f^{-1}$ implies

$$v_1(z)=\underset{0\leqq y\leqq x=f^{-1}(z)}{\text{Min}}[(1-b)x+(b-a)y+v_0(z-g(y)-h(x-y))]$$
$$\leqq \underset{0\leqq y\leqq x=f^{-1}(z)}{\text{Min}}[(1-b)x+(b-a)y+f^{-1}(z-g(y)-h(x-y))]$$
$$=f^{-1}(z).$$

Further assuming that $(1-c^n)f^{-1}\leqq v_n\leqq f^{-1}$, we have

$$v_{n+1}(z)\geqq x-(ay_n(x)+b(x-y_n(x)))+(1-c^n)f^{-1}(z-g(y_n(x))-h(x-y_n(x)))$$
$$\geqq x-(ay_n(x)+b(x-y_n(x)))+(1-c^n)(ay_n(x)+b(x-y_n(x)))$$
$$\geqq (1-c^{n+1})f^{-1}(z)$$

and

$$v_{n+1}(z)\leqq \underset{0\leqq y\leqq x=f^{-1}(z)}{\text{Min}}[(1-b)x+(b-a)y+f^{-1}(z-g(y)-h(x-y))]$$
$$=f^{-1}(z).$$

(iii) Let $v_0=0$. Then (35) leads to $v_1(z)\geqq v_0(z)=0$. Assuming that $v_n\geqq v_{n-1}$, we have

$$v_{n+1}(z)=\underset{0\leqq y\leqq x=f^{-1}(z)}{\text{Min}}[(1-b)x+(b-a)y+v_n(z-g(y)-h(x-y))]$$
$$\geqq \underset{0\leqq y\leqq x=f^{-1}(z)}{\text{Min}}[(1-b)x+(b-a)y+v_{n-1}(z-g(y)-h(x-y))]$$
$$=v_n(z).$$

This completes the proof.

Lemma 4 gives us the proof of Theorem 7.

THEOREM 7. *Any sequence* $\{v_n\}$ *generated by* (32), (33) *converges to the solution* f^{-1} *of* (5). *If* $v_0=0$, *then the convergence is monotone increasing.*

Another approximation is finite-stage approximation for the infinite-stage inverse process as follows.

THEOREM 8. *Let* $\{f_n\}$ *be defined in* (2), (3). *Let* $\{u_n\}$ *be defined as follows.*

$$(36) \qquad u_1(z)=f_1^{-1}(z)$$

(37) $\qquad u_{n+1}(z) = \underset{0 \le y \le x = f_{n+1}^{-1}(z)}{\text{Min}} [(1-a)y + (1-b)(x-y) + u_n(z-g(y)-h(x-y))]$

$$z \ge 0, \quad n \ge 1.$$

Then we have

(i) $\quad u_n = f_n^{-1} \quad n \ge 1.$

(ii) $\{u_n\}$ *converges monotone-decreasingly to the solution* f^{-1} *of* (5).

PROOF. (i) Assume that $u_1 = f_1^{-1}$. Let $0 \le y \le x = f_2^{-1}(z)$. Then we have

$$z = f_2(x) = \underset{0 \le y' \le x}{\text{Max}} [g(y') + h(x-y') + f_1(ay' + b(x-y'))]$$

$$\ge g(y) + h(x-y) + f_1(ay + b(x-y)).$$

From $u_1 = f_1^{-1}$, we obtain

(38) $\qquad (1-a)y + (1-b)(x-y) + u_1(z-g(y)-h(x-y)) \ge x$

for all y with $0 \le y \le x = f_2^{-1}(z)$. Further it follows that for a maximizer $y_2(x)$ of (3) for $n = 1$

(39) $\qquad (1-a)y_2(x) + (1-b)(x-y_2(x)) + u_1(z-g(y_2(x))-h(x-y_2(x))) = x.$

From (38), (39) we obtain

$$f_2^{-1}(z) = \underset{0 \le y \le x = f_2^{-1}(z)}{\text{Min}} [(1-a)y + (1-b)(x-y) + u_1(z-g(y)-h(x-y))],$$

which implies $u_2 = f_2^{-1}$. Moreover, $u_n = f_n^{-1}$ is inductively proved.

(ii) Since $\{f_n\}$ converges monotone-increasingly to f (Lemma 1), the inverse function $f_n^{-1} = u_n$ converges monotone-decreasingly to f^{-1}. On the other hand, f^{-1} is a solution of (5) (Theorem 3). This completes the proof.

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