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**Abstract**

Asymptotic behavior of solutions to the compressible Navier-Stokes equation around the plane Couette flow is investigated. It is shown that the plane Couette flow is asymptotically stable for initial disturbances sufficiently small in some $L^2$ Sobolev space if the Reynolds and Mach numbers are sufficiently small. Furthermore, the disturbances behave in large time in $L^2$ norm as solutions of an $n-1$ dimensional linear heat equation with a convective term.

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This paper is concerned with the asymptotic behavior of solutions to the compressible Navier-Stokes equation around the plane Couette flow.

We consider the system of equations

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) + \nabla P(\rho) = \mu \Delta v + (\mu + \mu')\nabla \mathrm{div}\, v \tag{1.2}$$

in an $n$ dimensional infinite layer $\Omega = \mathbf{R}^{n-1} \times (0, \ell)$:

$$\Omega = \{x = (x', x_n)\,;\, x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\, 0 < x_n < \ell\} \quad (n \geq 2).$$

Here $\rho = \rho(x,t)$ and $v = (v^1(x,t), \cdots, v^n(x,t))$ denote the unknown density and velocity at time $t \geq 0$ and position $x \in \Omega$, respectively ; $P = P(\rho)$ is the pressure ; $\mu$ and $\mu'$ are the viscosity coefficients that are assumed to be constants and satisfy $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$.

The system (1.1)–(1.2) is considered under the boundary condition

$$v|_{x_n=0} = 0, \quad v^1|_{x_n=\ell} = V, \; v^2|_{x_n=\ell} = \cdots = v^n|_{x_n=\ell} = 0. \tag{1.3}$$

Here $V \neq 0$ is a given constant.

It is easy to see that the problem (1.1)–(1.3) has the stationary solution $(\rho_s, v_s)$:

$$\rho_s = \rho_*, \quad v_s = \left( \frac{V}{\ell} x_n, 0, \cdots, 0 \right),$$

which is the so called plane Couette flow. Here $\rho_* > 0$ is a given constant.

We are interested in the large time behavior of solutions to problem (1.1)–(1.3) when the initial value $(\rho_0, v_0)$ is sufficiently close to the plane Couette flow $(\rho_s, v_s)$.

The plane Couette flow $(\rho_s, v_s)$ is also the solution of the incompressible Navier-Stokes equation

$$\begin{cases} \operatorname{div} v = 0, \\ \rho_*(\partial_t v + v \cdot \nabla v) + \nabla p = \mu \Delta v, \\ v^1|_{x_n=\ell} = V, \;\; v^2|_{x_n=\ell} = \cdots = v^n|_{x_n=\ell} = 0, \\ v|_{x_n=0} = 0, \end{cases} \tag{1.4}$$

and its stability as a solution of (1.4) has been widely studied. Since the Poincaré inequality holds for the disturbances of the velocity, one can easily see that there exists a number $Re_0 > 0$ such that if the Reynolds number $Re = \rho_* \ell V/\mu$ is less than $Re_0$ then the plane Couette flow is asymptotically stable for any initial disturbance in $L^2(\Omega)$ and it holds that $\|v(t) - v_s\|_{L^2(\Omega)} \leq Ce^{-\delta_0 t}$ for some $\delta_0 > 0$. Furthermore, Romanov [13] proved that the plane Couette flow is stable for any Reynolds number $Re > 0$ under sufficiently small disturbances, i.e., for any Reynolds number $Re > 0$ there exist positive numbers $\varepsilon_0$ and $\delta_1 > 0$ such that if $\|v_0 - v_s\|_{H^1(\Omega)} \leq \varepsilon_0$ then $\|v(t) - v_s\|_{H^1(\Omega)} \leq Ce^{-\delta_1 t}$. We also mention the work by Abe and Shibata [1] where they considered the stability of general laminar flows under $L^n$-disturbances and proved that there exists a number $Re_1 > 0$ such that if the Reynolds number $Re \leq Re_1$ then the plane Couette flow is asymptotically stable for small initial disturbances in $L^n(\Omega)$ and it holds that if $\|v_0 - v_s\|_{L^n(\Omega)} \ll 1$ then $\|v(t) - v_s\|_{L^n(\Omega)} \leq Ce^{-\delta_2 t}$ for some $\delta_2 > 0$.

The purpose of this paper is to study the stability properties of the plane Couette flow under the compressible disturbances. Concerning the compressible flow in an infinite layer, the stability of the motionless state $V = 0$ was investigated in [2, 3, 4] and it was shown that the motionless state is stable for sufficiently small initial disturbances and the disturbances behave in large time as solutions of an $n-1$ dimensional linear heat equation. As for the plane Couette flow ($V \neq 0$) we will prove that the plane Couette flow is asymptotically stable for sufficiently small initial disturbances if the Reynolds number $Re$ and the Mach number $Ma = V/\sqrt{P'(\rho_*)}$ are sufficiently small; and, furthermore, the disturbances behave in large time as solutions of an $n-1$ dimensional linear heat equation with a convective term. Although the result looks similar to that for the case of the motionless state $V=0$, the problem is not classified in the same category as that of the case $V=0$ from a mathematical point of view which will be explained below.

More precise statement of our result is as follows. There exist positive numbers $c_1 > 0$, $c_2 > 0$ and $\varepsilon_1 > 0$ such that if $Re \leq c_1$ and $Ma \leq c_2$, then there exists a unique global solution $(\rho(t), v(t))$ of (1.1)–(1.3) such that $u(t) = (\rho(t) - \rho_s, v(t) - v_s) \in C([0,\infty); H^s(\Omega))$ and $u(t)$ satisfies

$$\|\partial_x^l u(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{l}{2}}) \quad (t \to \infty) \tag{1.5}$$

for $l = 0, 1$ and

$$\|u(t) - u^{(0)}(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}} L(t)) \quad (t \to \infty), \tag{1.6}$$

provided that the initial disturbance $u_0 = (\rho_0 - \rho_s, v_0 - v_s)$ satisfies $\|u_0\|_{(H^s \cap L^1)(\Omega)} \leq \varepsilon_1$. Here $H^s(\Omega)$ is the $L^2$-Sobolev space of order $s \geq [2/n] + 2$; $L(t)$ is defined as $L(t) = 1$ when $n \geq 3$ and $L(t) = \log(1+t)$ when $n = 2$ ; and $u^{(0)}$ is a function of the form $u^{(0)} = (\phi^{(0)}(x', t), 0)$ with $\phi^{(0)}(x', t)$ satisfying

$$\partial_t \phi^{(0)} - \widetilde{\kappa}_1 \partial_{x_1}^2 \phi^{(0)} - \widetilde{\kappa}_2 \Delta'' \phi^{(0)} + \frac{1}{2} V \partial_{x_1} \phi^{(0)} = 0,$$

$$\phi^{(0)}|_{t=0} = \frac{1}{\ell} \int_0^{\ell} (\rho_0(x', x_n) - \rho_s)\, dx_n,$$

where $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ are positive constants depending on $\rho_*$, $\ell$, $V$, $\mu$, $\mu'$ and $P'(\rho_*)$; and $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$.

The proof of our stability result is based on the energy method and the spectral analysis of the linearized operator. The global existence is proved by the energy method by Matsumura and Nishida [12], which also yields the $H^s$-energy bound. The asymptotic behavior in (1.5) and (1.6) is proved by combining the $H^s$-energy bound and the decay estimates of the solutions of the linearized problem

$$\partial_t w + \widetilde{L} w = 0, \quad w|_{t=0} = w_0. \tag{1.7}$$

Here $w = {}^T(\phi, \psi)$ and

$$\widetilde{L} = \begin{pmatrix} \frac{V}{\ell} x_n \partial_{x_1} & \rho_* \mathrm{div} \\ \frac{P'(\rho_*)}{\rho_*} \nabla & -\frac{\mu}{\rho_*} \Delta I_n - \frac{(\mu+\mu')}{\rho_*} \nabla \mathrm{div} \end{pmatrix} + \frac{V}{\ell} \begin{pmatrix} 0 & 0 \\ 0 & x_n \partial_{x_1} I_n + {}^T\boldsymbol{e}_1 \boldsymbol{e}_n \end{pmatrix},$$

where $I_n$ denotes the $n \times n$ identity matrix and ${}^T\boldsymbol{e}_1\boldsymbol{e}_n$ is the matrix with $(i, j)$ components given by $\delta_{1i}\delta_{nj}$. Here and in what follows the superscript ${}^T\cdot$ means transposition.

Concerning the linearized problem, we note that in the case of the incompressible problem (1.4), the linearized operator can be regarded as a simple perturbation from the one around the motionless state $V = 0$. Therefore, the stability problem can be handled by a standard perturbation argument for analytic semigroups based on the analysis in the case $V = 0$ as in Abe and Shibata [1] if the Reynolds number is sufficiently small. On the other hand, in the case of the compressible problem (1.7), due to the first order term $\frac{V}{\ell} x_n \partial_{x_1} \phi$ appearing in the linearization of (1.1), the operator $\widetilde{L}$ cannot be regarded as a simple perturbation from the linearized one around the motionless state $V = 0$. In fact, the domain of the operator $\widetilde{L}$ is different from that in the case $V = 0$. Thus, the results in [2] cannot be directly applied even in case of small Reynolds numbers.

To investigate the decay properties of solutions to (1.7) we consider the Fourier transform of (1.7) in $x' \in \mathbf{R}^{n-1}$:

$$\partial_t \widehat{w} + \widetilde{L}_{\xi'} \widehat{w} = 0, \quad \widehat{w}|_{t=0} = \widehat{w}_0, \tag{1.8}$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ denotes the dual variable.

The operator $\widetilde{L}_{\xi'}$ has different characters between the cases $|\xi'| \ll 1$ and $|\xi'| \gg 1$. We thus decompose the semigroup $e^{-t\widetilde{L}}$ associated with (1.7) into two parts: $e^{-t\widetilde{L}} = \mathcal{F}^{-1}[e^{-t\widetilde{L}_{\xi'}}\,|_{|\xi'| \leq r}] + \mathcal{F}^{-1}[e^{-t\widetilde{L}_{\xi'}}\,|_{|\xi'| \geq r}]$ for some small $r > 0$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. In case $|\xi'| \ll 1$, $\widetilde{L}_{\xi'}$ can be treated as in the case $V = 0$ ([3]). If $|\xi'| \ll 1$, then $\widetilde{L}_{\xi'}$ can be regarded as a perturbation from $\widetilde{L}_0$ and we will find that the spectrum of $-\widetilde{L}$ near the imaginary axis is given by that of $-\widetilde{L}_{\xi'}$ with $|\xi'| \ll 1$, which is parametrized as $-\frac{i}{2} V \xi_1 - \widetilde{\kappa}_1 \xi_1^2 - \widetilde{\kappa}_2 |\xi''|^2 + O(|\xi'|^3)$, where $\xi' = (\xi_1, \xi'')$, $\xi'' = (\xi_2, \cdots, \xi_{n-1})$. On the other hand, if $|\xi'| \gg 1$, the hyperbolic aspect of (1.7) is getting much stronger than that in the case $V = 0$, due to the $(1, 1)$-component $\frac{V}{\ell} x_n \partial_{x_1}$ of $\widetilde{L}$. Therefore, to analyze the part $|\xi'| \gg 1$, we employ the Fourier transformed version of Matsumura-Nishida's energy method and derive the exponential decay property of the corresponding part of the semigroup $e^{-t\widetilde{L}}$.

This paper is organized as follows. In section 2 we rewrite the problem into the system of equations for the perturbation in a non-dimensional form.

In section 3 we introduce some function spaces and state our main results. In sections 4–6 we investigate the linearized problem. As mentioned above, we consider the Fourier transformed problem (1.8). In section 4 we restate our main results on the linearized problem. In section 5 we analyze the case $|\xi'| \ll 1$. The case $|\xi'| \gg 1$ is treated in section 6. Section 7 is devoted to the nonlinear problem and we here give an outline of the proof of (1.5) and (1.6) only, since the proof for the nonlinear problem is similar to those in [4, 6, 11, 12]. In the appendix we give an estimate for the Fourier transform of solutions to the stationary Stokes problem on the half space, which is used in the analysis in section 6.

## 2. Reformulation of the Problem

In this section we rewrite the problem into the one for the perturbation in a non-dimensional form.

Throughout the paper we assume that

$$P'(\rho_*) > 0. \tag{2.1}$$

Without loss of generality we may assume that $V > 0$.

We introduce the following dimensionless variables:

$$x = \widetilde{x}, \quad t = \frac{\ell}{V}\widetilde{t}, \quad v = V\widetilde{v}, \quad \rho = \rho_*\widetilde{\rho}, \quad P = \rho_* V^2 \widetilde{P}.$$

Then the problem (1.1)–(1.3) is transformed into the following dimensionless problem on the layer $\Omega = \mathbf{R}^{n-1} \times (0,1)$:

$$\partial_t \widetilde{\rho} + \mathrm{div}\,(\widetilde{\rho}\widetilde{v}) = 0, \tag{2.2}$$

$$\widetilde{\rho}(\partial_t \widetilde{v} + \widetilde{v} \cdot \nabla \widetilde{v}) + \widetilde{P}'(\widetilde{\rho})\nabla\widetilde{\rho} = \nu\Delta\widetilde{v} + (\nu + \nu')\nabla \mathrm{div}\,\widetilde{v}, \tag{2.3}$$

$$\widetilde{v}|_{x_n=0} = 0, \quad \widetilde{v}|_{x_n=1} = \boldsymbol{e}_1. \tag{2.4}$$

Here $\boldsymbol{e}_1$ is the unit vector $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$, and $\nu$ and $\nu'$ are the nondimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}.$$

The plane Couette flow is transformed into $(\widetilde{\rho}_s, \widetilde{v}_s) = (1, x_n \boldsymbol{e}_1)$. Note that the Reynolds number $Re$ is given by $Re = \nu^{-1}$. We also introduce the Mach number

$$Ma = \frac{1}{\sqrt{\widetilde{P}'(1)}} = \frac{V}{\sqrt{P'(\rho_*)}}.$$

Setting $\widetilde{\rho} = \widetilde{\rho}_s + \gamma^{-2}\phi$ with $\gamma = Ma^{-1}$ and $\widetilde{v} = \widetilde{v}_s + \psi$ in (2.2)–(2.4), we arrive at the initial boundary value problem for the perturbation $u = (\phi, \psi)$:

$$\frac{1}{\gamma^2}\partial_t\phi + \frac{1}{\gamma^2}x_n\partial_{x_1}\phi + \operatorname{div}\psi = f^0, \tag{2.5}$$

$$\partial_t\psi - \nu\Delta\psi - (\nu + \nu')\nabla\operatorname{div}\psi + \nabla\phi + x_n\partial_{x_1}\psi + \psi^n\boldsymbol{e}_1 = g, \tag{2.6}$$

$$\psi|_{x_n=0,1} = 0, \tag{2.7}$$

$$(\phi, \psi)|_{t=0} = (\phi_0, \psi_0). \tag{2.8}$$

Here

$$f^0 = -\frac{1}{\gamma^2}\operatorname{div}(\phi\psi),$$

$$g = -\psi\cdot\nabla\psi - \frac{\phi}{\gamma^2+\phi}\{\nu\Delta\psi + (\nu+\nu')\nabla\operatorname{div}\psi + (P_2(\gamma,\phi) - 1)\nabla\phi\}$$

with

$$P_2(\gamma, \phi) = \frac{1}{\gamma^2}\int_0^1 \widetilde{P}''(1 + \gamma^{-2}\theta\phi)\,d\theta.$$

## 3. Main Results

We first introduce some notation which will be used throughout the paper. For a domain $D$ and $1 \le p \le \infty$ we denote by $L^p(D)$ the usual Lebesgue space on $D$ and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let $l$ be a nonnegative integer. The symbol $W^{l,p}(D)$ denotes the $l$ th order $L^p$ Sobolev space on $D$ with norm $\|\cdot\|_{W^{l,p}(D)}$. When $p = 2$, the space $W^{l,2}(D)$ is denoted by $H^l(D)$ and its norm is denoted by $\|\cdot\|_{H^l(D)}$. $C_0^l(D)$ stands for the set of all $C^l$ functions which have compact support in $D$. We denote by $W_0^{1,p}(D)$ the completion of $C_0^1(D)$ in $W^{1,p}(D)$. In particular, $W_0^{1,2}(D)$ is denoted by $H_0^1(D)$.

We simply denote by $L^p(D)$ (resp., $W^{l,p}(D)$, $H^l(D)$) the set of all vector fields $\psi = {}^T(\psi^1, \cdots, \psi^n)$ on $D$ with $\psi^j \in L^p(D)$ (resp., $W^{l,p}(D)$, $H^l(D)$), $j = 1, \cdots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{W^{l,p}(D)}$, $\|\cdot\|_{H^l(D)}$). For $u = {}^T(\phi, \psi)$ with $\phi \in W^{k,p}(D)$ and $\psi = {}^T(\psi^1, \cdots, \psi^n) \in W^{l,q}(D)$, we define $\|u\|_{W^{k,p}(D)\times W^{l,q}(D)}$ by $\|u\|_{W^{k,p}(D)\times W^{l,q}(D)} = \|\phi\|_{W^{k,p}(D)} + \|\psi\|_{W^{l,q}(D)}$. When $k = l$ and $p = q$, we simply write $\|u\|_{W^{k,p}(D)\times W^{k,p}(D)} = \|u\|_{W^{k,p}(D)}$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $W^{l,p}(\Omega)$, $H^l(\Omega)$) as $L^p$ (resp., $W^{l,p}$, $H^l$). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case $D = (0,1)$ we denote the norm of $L^p(0,1)$ by $|\cdot|_p$. The inner product of $L^2(0,1)$ is denoted by

$$(f,g) = \int_0^1 f(x_n)\overline{g(x_n)}\,dx_n, \quad f,g \in L^2(0,1).$$

Here $\overline{g}$ denotes the complex conjugate of $g$. Furthermore, for $f \in L^1(0,1)$ we denote the mean value of $f$ in $(0,1)$ by $\langle f \rangle$:

$$\langle f \rangle = (f,1) = \int_0^1 f(x_n)\,dx_n.$$

The norms of $W^{l,p}(0,1)$ and $H^l(0,1)$ are denoted by $|\cdot|_{W^{l,p}}$ and $|\cdot|_{H^l}$, respectively.

We often write $x \in \Omega$ as $x = {}^T(x', x_n)$, $x' = {}^T(x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$. Partial derivatives of a function $u$ in $x$, $x'$, $x_n$ and $t$ are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of $u$ in $x$ as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$.

We denote the $k \times k$ identity matrix by $I_k$. In particular, when $k = n+1$, we simply write $I$ for $I_{n+1}$. We also define $(n+1)\times(n+1)$ diagonal matrices $Q_0$ and $\widetilde{Q}$ by

$$Q_0 = \mathrm{diag}\,(1,0,\cdots,0), \quad \widetilde{Q} = \mathrm{diag}\,(0,1,\cdots,1).$$

We then have, for $u = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathbf{R}^{n+1}$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ \psi \end{pmatrix}.$$

We next introduce some notation about integral operators. For a function $f = f(x')$ ($x' \in \mathbf{R}^{n-1}$), we denote its Fourier transform by $\widehat{f}$ or $\mathcal{F} f$:

$$\widehat{f}(\xi') = (\mathcal{F} f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-i\xi'\cdot x'}\,dx'.$$

The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$:

$$(\mathcal{F}^{-1} f)(x) = (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} f(\xi') e^{i\xi'\cdot x'}\,d\xi'.$$

For a function $K(x_n, y_n)$ on $(0,1)\times(0,1)$ we will denote by $Kf$ the integral operator $\int_0^1 K(x_n,y_n) f(y_n)\,dy_n$.

We will denote the resolvent set of a closed operator $A$ by $\rho(A)$ and the spectrum of $A$ by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ we denote the set $\{\lambda \in \mathbf{C};\ |\arg(\lambda - \Lambda)| \le \theta\}$ by $\Sigma(\Lambda, \theta)$:

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C};\ |\arg(\lambda - \Lambda)| \le \theta\}.$$

We first state our main results on the linearized problem.

Let us consider the linearized problem

$$\partial_t u + Lu = 0, u|_{t=0} = u_0. \tag{3.1}$$

Here

$$L = \begin{pmatrix} x_n \partial_{x_1} & \gamma^2 \mathrm{div} \\ \nabla & -\nu \Delta I_n - (\nu + \nu')\nabla \mathrm{div} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & x_n \partial_{x_1} I_n + {}^T\boldsymbol{e}_1 \boldsymbol{e}_n \end{pmatrix}.$$

We denote the solution operator for (3.1) by $\mathcal{U}(t)$.

**Theorem 3.1.** *Suppose that $u_0 = {}^T(\phi_0, \psi_0) \in H^1 \times L^2$ and that $\partial_{x'}\psi_0 \in L^2$. Then (3.1) has a unique solution $u(t) = \mathcal{U}(t)u_0$ and satisfies the estimates*

$$\|\partial_x^l \mathcal{U}(t)u_0\|_2 \le C\{t^{-\frac{l}{2}}\|u_0\|_{H^l \times L^2} + \|\partial_{x'}\psi_0\|_2\} \quad (l = 0, 1)$$

*for $0 < t \le 1$.*

**Theorem 3.2.** *There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $(2\nu + \nu')/\gamma^2 \le 1/\gamma_0^2$, then the following estimates hold uniformly for $t \ge 1$, $u_0 = {}^T(\phi_0, \psi_0) \in H^1 \times L^2$ with $\partial_{x'}\psi_0 \in L^2$ and $u_0 \in L^1$:*

$$\|\partial_x^l \mathcal{U}(t)u_0\|_2 \le C\{t^{-\frac{n-1}{4} - \frac{l}{2}}\|u_0\|_1 + e^{-at}\|u_0\|_{H^1}\} \quad (l = 0, 1),$$

$$\|\mathcal{U}(t)u_0 - G_t *_{x'} \Pi^{(0)}u_0\|_2 \le C\{t^{-\frac{n-1}{4} - \frac{1}{2}}\|u_0\|_1 + e^{-at}\|u_0\|_{H^1}\}.$$

*with some constant $a > 0$. Here*

$$G_t *_{x'} \Pi^{(0)}u_0 = \mathcal{F}^{-1}\left[e^{-(\frac{i}{2}\xi_1 + \kappa_1\xi_1^2 + \kappa_2|\xi''|^2)t}\widehat{\Pi}^{(0)}\widehat{u}_0\right]$$

*with $\widehat{\Pi}^{(0)}\widehat{u}_0 = \langle Q_0 \widehat{u}_0 \rangle$, where $\kappa_1$ and $\kappa_2$ are some positive constants.*

**Remark.** We easily see that $\|G_t *_{x'} \Pi^{(0)}u_0\|_2 = O(t^{-\frac{n-1}{4}})$ if $u_0 \in L^1$. We also see that the function $G_t *_{x'} \Pi^{(0)}u_0$ is written in the form $G_t *_{x'} \Pi^{(0)}u_0 = (\phi^{(0)}(x', t), 0)$ with $\phi^{(0)}(x', t)$ satisfying

$$\partial_t \phi^{(0)} - \kappa_1 \partial_{x_1}^2 \phi^{(0)} - \kappa_2 \Delta'' \phi^{(0)} + \frac{1}{2}\partial_{x_1}\phi^{(0)} = 0,$$

$$\phi^{(0)}|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n,$$

where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$.

We next state our main result on the nonlinear problem. Before stating the result we mention the compatibility condition for $u_0 = (\phi_0, \psi_0)$. We will look for a solution $u = (\phi, \psi)$ of (2.5)–(2.8) in $\cap_{j=0}^{[\frac{s}{2}]} C^j([0,\infty); H^{s-2j})$ satisfying $\int_0^t \|\partial_x \phi\|_{H^{s-1}}^2 + \|\partial_x \psi\|_{H^s}^2\, d\tau < \infty$ for all $t \geq 0$ with $s \geq [n/2]+1$. Therefore, we need to require the compatibility condition for the initial value $u_0 = (\phi_0, \psi_0)$, which is formulated as follows.

Let $u = (\phi, \psi)$ be a smooth solution of (2.5)–(2.8). Then $\partial_t^j u = (\partial_t^j \phi, \partial_t^j \psi)$ $(j \geq 1)$ is inductively determined by

$$\partial_t^j \phi = -x_n \partial_{x_1} \partial_t^{j-1} \phi - \gamma^2 \mathrm{div}\, \partial_t^{j-1} \psi + \gamma^2 \partial_t^{j-1} f^0$$

and

$$\partial_t^j \psi = -T \partial_t^{j-1} \psi - \nabla \partial_t^{j-1} \phi + \partial_t^{j-1} g.$$

Here $T\psi = -\mu \Delta \psi - (\mu + \mu') \nabla \mathrm{div}\, \psi + x_n \partial_{x_1} \psi + \psi^n \boldsymbol{e}_1$.

From these relations we see that $(\partial_t^j \phi, \partial_t^j \psi)\big|_{t=0}$ is inductively given by $(\phi_0, \psi_0)$ in the following way:

$$(\partial_t^j \phi, \partial_t^j \psi)\big|_{t=0} = (\phi_j, \psi_j),$$

where

$$\begin{aligned} \phi_j &= -x_n \partial_{x_1} \phi_{j-1} - \gamma^2 \mathrm{div}\, \psi_{j-1} \\ &\quad + \gamma^2 f_{j-1}^0(\phi_0, \cdots, \phi_{j-1}, \psi_0, \cdots, \psi_{j-1}, \partial_x \psi_0, \cdots, \partial_x \psi_{j-1}), \\ \psi_j &= -T \psi_{j-1} - \nabla \phi_{j-1} \\ &\quad + g_{j-1}(\phi_0, \cdots, \phi_{j-1}, \psi_0, \cdots, \psi_{j-1}, \cdots, \partial_x \phi_{j-1}, \cdots, \partial_x^2 \psi_{j-1}). \end{aligned}$$

Here $f_l^0(\phi_0, \cdots, \phi_l, \cdots)$ is a certain polynomial in $\phi_0, \cdots, \phi_l, \cdots\ ;\ \cdots\cdots$, and so on.

By the boundary condition $\psi|_{x_n=0,1} = 0$ in (2.7), we necessarily have $\partial_t^j \psi\, |_{x_n=0,1} = 0$, and hence,

$$\psi_j\, |_{x_n=0,1} = 0.$$

Assume that $(\phi, \psi)$ is a solution of (2.5)–(2.8) in $\cap_{j=0}^{[\frac{s}{2}]} C^j([0, T_0]; H^{s-2j})$ for some $T_0 > 0$. Then, from the above observation, we need the regularity $(\phi_j, \psi_j) \in H^{s-2j}$ for $j = 0, \cdots, [s/2]$, which, indeed, follows from the fact that $(\phi_0, \psi_0) \in H^s$ with $s \geq [n/2]+1$. Furthermore, it is necessary to require that $(\phi_0, \psi_0)$ satisfies the $\widehat{s}$*-th order compatibility condition*:

$$\psi_j \in H_0^1 \quad \text{for} \ \ j = 0, 1, \cdots, \widehat{s} = \left[\frac{s-1}{2}\right].$$

We are ready to state our main result on the nonlinear problem.

**Theorem 3.3.** (i) *Let $s$ be an integer satisfying $s \geq [n/2]+1$. Then there exist constants $\widetilde{\nu}_0 > 0$, $\widetilde{\gamma}_0 > 0$ and $\varepsilon_0 > 0$ such that if $\nu \geq \widetilde{\nu}_0$ and $(2\nu + \nu')/\gamma^2 \leq 1/\widetilde{\gamma}_0^2$, then for any $u_0 = (\phi_0, \psi_0) \in H^s$ satisfying the $\widehat{s}$-th compatibility condition with $\|u_0\|_{H^s} \leq \varepsilon_0$ there exists a unique global solution $u(t) = (\phi(t), \psi(t)) \in \cap_{j=0}^{[\frac{s}{2}]} C^j([0,\infty); H^{s-2j})$ of* (2.5)–(2.8)*, which satisfies*

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x \phi\|_{H^{s-1}}^2 + \|\partial_x \psi\|_{H^s}^2 \, d\tau \leq C\|u_0\|_{H^s}^2$$

*uniformly for $t \geq 0$ and*

$$\|u(t)\|_\infty \to 0 \quad (t \to \infty).$$

(ii) *Furthermore, let $s \geq [n/2]+2$ and assume that $\nu \geq \widetilde{\nu}_0$ and $(2\nu + \nu')/\gamma^2 \leq 1/\widetilde{\gamma}_0^2$. Assume also that $u_0 = (\phi_0, \psi_0)$ belongs to $H^s \cap L^1$ and satisfies the $\widehat{s}$-th compatibility condition. Then there exists a constant $\varepsilon_1 > 0$ such that if $\|u_0\|_{H^s \cap L^1} \leq \varepsilon_1$, then the solution $u(t) = (\phi(t), \psi(t))$ satisfies*

$$\|\partial_x^l u(t)\|_2 = O(t^{-\frac{n-1}{4} - \frac{l}{2}}) \quad (t \to \infty)$$

*for $l = 0, 1$ and*

$$\|u(t) - G_t *_x \Pi^{(0)} u_0\|_2 = O(t^{-\frac{n-1}{4} - \frac{1}{2}} L(t)) \quad (t \to \infty).$$

*Here $L(t) = 1$ when $n \geq 3$ and $L(t) = \log(1+t)$ when $n = 2$.*

Theorem 3.3 (i) is proved by the energy method by Matsumura and Nishida [12]. The proof of Theorem 3.3 (ii) is based on the $H^s$-energy bound in (i) and Theorems 3.1 and 3.2. We will give an outline of the proof of Theorem 3.3 in section 7.

## 4. The Linearized Problem

In sections 4–6 we will consider the linearized problem and will prove Theorems 3.1 and 3.2.

Theorem 3.1 is proved as follows. It is not difficult to show the unique existence of solutions $u \in C([0,\infty); H^1) \times (C([0,\infty); H_0^1) \cap L^2_{loc}([0,\infty); H^2))$ for all $u_0 \in H^1 \times H_0^1$. Theorem 3.1 then follows by an approximation argument if one shows the estimate presented in Theorem 3.1. In section 6 we will give a proof of the estimate given in Theorem 3.1. (See Proposition 6.2.)

Theorem 3.2 immediately follows from the following theorem.

**Theorem 4.1.** *There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \geq \nu_0$ and $(2\nu + \nu')/\gamma^2 \leq 1/\gamma_0^2$, then for any $u_0 = {}^T(\phi_0, \psi_0) \in (H^1 \times L^2) \cap L^1$ with $\partial_{x'}\psi_0 \in L^2$ the solution $u(t) = \mathcal{U}(t)u_0$ of problem (3.1) is decomposed as*

$$\mathcal{U}(t)u_0 = \mathcal{U}^{(0)}(t)u_0 + \mathcal{U}^{(\infty)}(t)u_0,$$

*where each term on the right-hand side has the following properties.*

(i) *The function $\mathcal{U}^{(0)}(t)u_0$ satisfies the following estimates (4.1) – (4.3) uniformly for $t \geq 1$:*

$$\|\partial_x^l \mathcal{U}^{(0)}(t)u_0\|_2 \leq Ct^{-\frac{n-1}{4}-\frac{l}{2}}\|u_0\|_1 \quad (l = 0, 1), \tag{4.1}$$

$$\|\mathcal{U}^{(0)}(t)u_0 - G_t *_{x'} \Pi^{(0)}u_0\|_2 \leq Ct^{-\frac{n-1}{4}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1, \tag{4.2}$$

*and*

$$\|\partial_x^l \mathcal{U}^{(0)}(t)[\widetilde{Q}u_0]\|_2 \leq Ct^{-\frac{n-1}{4}-\frac{l+1}{2}}\|\widetilde{Q}u_0\|_1 \quad (l = 0, 1). \tag{4.3}$$

*Furthermore, if $u_0 = {}^T(\operatorname{div}\Psi_0, \partial_x\psi_0)$ with $\Psi_0^n|_{x_n=0,1} = 0$ then it holds that*

$$\|\partial_x^l \mathcal{U}^{(0)}(t)u_0\|_2 \leq Ct^{-\frac{n-1}{4}-\frac{l+1}{2}}(\|\Psi_0\|_{L^1} + \|\psi_0\|_{L^1}) \quad (l = 0, 1) \tag{4.4}$$

*for all $t \geq 1$.*

(ii) *There exists a constant $a > 0$ such that $\mathcal{U}^{(\infty)}(t)u_0$ satisfies*

$$\|\partial_x^l \mathcal{U}^{(\infty)}(t)u_0\|_2 \leq Ce^{-at}\|u_0\|_{H^1} \quad (l = 0, 1) \tag{4.5}$$

*for all $t \geq 1$.*

To prove Theorem 4.1 we decompose $\mathcal{U}(t)u_0$ in the following way. Let $r > 0$. Define $\chi^{(0)}(\xi')$ and $\chi^{(\infty)}(\xi')$ by

$$\chi^{(0)}(\xi') = 1 \text{ if } |\xi'| \leq r, \quad \chi^{(0)}(\xi') = 0 \text{ if } |\xi'| \leq r, \text{ and } \chi^{(\infty)} = 1 - \chi^{(0)}.$$

We decompose $\mathcal{U}(t)u_0$ as

$$\mathcal{U}(t)u_0 = U_0(t)u_0 + U_\infty(t)u_0,$$

where
$$U_j(t)u_0 = \mathcal{F}^{-1}\left[\chi^{(j)}e^{-t\widehat{L}_{\xi'}}\widehat{u}_0\right], \quad j = 0, \infty.$$

Here $\widehat{L}_{\xi'}$ is the operator of the form

$$\widehat{L}_{\xi'} = \begin{pmatrix} i\xi_1 x_n & i\gamma^{2\,T}\xi' & \gamma^2\partial_{x_n} \\ i\xi' & \{\nu(|\xi'|^2 - \partial_{x_n}^2) + i\xi_1 x_n\}I_{n-1} + \widetilde{\nu}\xi'^T\xi' & -i\widetilde{\nu}\xi'\partial_{x_n} + \boldsymbol{e}_1' \\ \partial_{x_n} & -i\widetilde{\nu}^T\xi'\partial_{x_n} & \nu(|\xi'|^2 - \partial_{x_n}^2) - \widetilde{\nu}\partial_{x_n}^2 + i\xi_1 x_n \end{pmatrix},$$

which is a closed operator on $H^1(0,1) \times L^2(0,1)$ with domain of definition $D(\widehat{L}_{\xi'}) = H^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$.

**Proposition 4.2.** *There exists $r_0 > 0$ such that if $r \le r_0$, then $U_0(t)u_0$ is written as*
$$U_0(t)u_0 = \mathcal{U}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0,$$
*where $\mathcal{U}^{(0)}(t)u_0$ has the properties in Theorem* 4.1 (i) *and $\mathcal{R}^{(0)}(t)u_0$ satisfies the estimate* (4.5) *in Theorem* 4.1 (ii) *with $\mathcal{U}^{(\infty)}(t)u_0$ replaced by $\mathcal{R}^{(0)}(t)u_0$.*

**Proposition 4.3.** *There exist constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $(2\nu + \nu')/\gamma^2 \le 1/\gamma_0^2$, then the following estimate holds for any fixed $r > 0$ uniformly in $t \ge 1$:*

$$\|U_\infty(t)u_0\|_{H^1} \le Ce^{-at}\|u_0\|_{H^1},$$

*where $a = a(r) > 0$.*

Theorem 4.2 follows from Propositions 4.2 and 4.3 by setting $r = r_0$ and $\mathcal{U}^{(\infty)}(t)u_0 = \mathcal{R}^{(0)}(t)u_0 + U_\infty(t)u_0$.

Proposition 4.2 can be proved in a similar manner in [3]. In section 5 we will give an outline of its proof. The proof of Proposition 4.3 will be given in section 6 by the Fourier transformed version of Matsumura-Nishida's energy method. The proof of the estimate in Theorem 3.1 will be also given in section 6 (Proposition 6.2).

## 5. Proof of Proposition 4.2

Proposition 4.2 can be proved in a similar manner in [3]. So we here give an outline of the proof following the argument in [3].

To investigate (3.1) we take the Fourier transform in $x' \in \mathbf{R}^{n-1}$. We then have the following initial boundary value problem for functions $\phi(x_n, t)$ and

$\psi(x_n,t)$ $(x_n\in(0,1),\ t\ge 0)$:

$$\frac{du}{dt}+\widehat{L}_{\xi'}u=0,\quad u|_{t=0}=u_0, \tag{5.1}$$

where $u={}^T(\phi(x_n,t),\psi'(x_n,t),\psi^n(x_n,t))$, $u_0={}^T(\phi_0(x_n),\psi_0'(x_n),\psi_0^n(x_n))$.

It is not difficult to see that for each fixed $\xi'\in\mathbf{R}^{n-1}$ the operator $-\widehat{L}_{\xi'}$ generates an analytic semigroup $e^{-t\widehat{L}_{\xi'}}$ on $H^1(0,1)\times L^2(0,1)$. (Cf., [2, 3].) Proposition 4.2 is proved by investigating the spectrum of $-\widehat{L}_{\xi'}$ for $|\xi'|\ll 1$. We analyze it regarding the problem as a perturbation from the one with $\xi'=0$.

We consider the resolvent problem

$$\lambda u+\widehat{L}_{\xi'}u=f, \tag{5.2}$$

where $\lambda\in\mathbf{C}$ is the resolvent parameter, $u={}^T(\phi(x_n),\psi'(x_n),\psi^n(x_n))$ and $f={}^T(f^0(x_n),g'(x_n),g^n(x_n))$. To investigate problem (5.2) we write $\widehat{L}_{\xi'}$ in the following form:

$$\widehat{L}_{\xi'}=\widehat{L}_0+\sum_{j=1}^{n-1}\xi_j\widehat{L}_j^{(1)}+\sum_{j,k=1}^{n-1}\xi_j\xi_k\widehat{L}_{jk}^{(2)},$$

where $\xi'={}^T(\xi_1,\cdots,\xi_{n-1})$, and

$$\widehat{L}_0=\begin{pmatrix}0 & 0 & \gamma^2\partial_{x_n}\\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & \boldsymbol{e}_1'\\ \partial_{x_n} & 0 & -\nu_1\partial_{x_n}^2\end{pmatrix},\quad \nu_1=\nu+\widetilde{\nu},$$

$$\widehat{L}_j^{(1)}=\begin{pmatrix}ix_n\delta_{1j} & i\gamma^2\,{}^T\boldsymbol{e}_j' & 0\\ i\boldsymbol{e}_j' & ix_n\delta_{1j}I_{n-1} & -i\widetilde{\nu}\boldsymbol{e}_j'\partial_{x_n}\\ 0 & -i\widetilde{\nu}\,{}^T\boldsymbol{e}_j'\partial_{x_n} & ix_n\delta_{1j}\end{pmatrix},$$

$$\widehat{L}_{jk}^{(2)}=\begin{pmatrix}0 & 0 & 0\\ 0 & \nu\delta_{jk}I_{n-1}+\widetilde{\nu}\boldsymbol{e}_j'\,{}^T\boldsymbol{e}_k' & 0\\ 0 & 0 & \nu\delta_{jk}\end{pmatrix}.$$

Here $\boldsymbol{e}_j'$ denotes the unit vector of $\mathbf{R}^{n-1}$ in $\xi_j$-direction.

We will treat $\widehat{L}_{\xi'}$ as a perturbation from $\widehat{L}_0$. We begin with the analysis of (5.2) with $\xi' = 0$:

$$(\lambda + \widehat{L}_0)u = f.$$

We introduce some quantities. For $k = 1, 2, \cdots$, we define $\lambda_{1,k}$ and $\lambda_{\pm,k}$ by

$$\lambda_{1,k} = -\nu(k\pi)^2$$

and

$$\lambda_{\pm,k} = -\frac{\nu_1}{2}(k\pi)^2 \pm \frac{1}{2}\sqrt{\nu_1^2(k\pi)^4 - 4\gamma^2(k\pi)^2}$$

for $k = 1, 2, \cdots$. An elementary observation shows that $\lambda_{\pm,k}$ are the two roots of $\lambda^2 + \nu_1(k\pi)^2\lambda + \gamma^2(k\pi)^2 = 0$; $\lambda_{-,k} = \overline{\lambda_{+,k}}$ with $\mathrm{Im}\,\lambda_{+,k} = \gamma k\pi\sqrt{1 - \frac{\nu_1^2}{4\gamma^2}(k\pi)^2}$ when $k\pi < 2\gamma/\nu_1$ and $\lambda_{\pm,k} \in \mathbf{R}$ when $k\pi \geq 2\gamma/\nu_1$; and it holds that

$$\lambda_{+,k} = -\frac{\gamma^2}{\nu_1} + O(k^{-2}), \quad \lambda_{-,k} = -\nu_1(k\pi)^2 + O(1)$$

as $k \to \infty$. (See [2, Remarks 3.2 and 3.5].)

**Lemma 5.1.** (i) *The spectrum $\sigma(-\widehat{L}_0)$ is given by*

$$\sigma(-\widehat{L}_0) \subset \{0\} \cup \{\lambda_{1,k}\}_{k=1}^{\infty} \cup \{\lambda_{+,k}, \lambda_{-,k}\}_{k=1}^{\infty} \cup \{-\tfrac{\gamma^2}{\nu_1}\}.$$

*Here* 0 *is an eigenvalue.*

(ii) *The eigenvalue* 0 *of $-\widehat{L}_0$ is simple; the eigenspace is spanned by $u^{(0)} = {}^T(1, 0, \cdots, 0)$; and the associated eigenprojection is given by*

$$\widehat{\Pi}^{(0)}u = \begin{pmatrix} \langle\phi\rangle \\ 0 \end{pmatrix} \quad \textit{for} \quad u = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

(iii) *There exist positive numbers $\eta_0$ and $\theta_0$ with $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that the following estimates hold uniformly for $\lambda \in \rho(-\widehat{L}_0) \cap \Sigma(-\eta_0, \theta_0)$:*

$$\left|(\lambda + \widehat{L}_0)^{-1}f\right|_{H^l \times L^2} \leq \frac{C}{|\lambda|}|f|_{H^l \times L^2}, \quad l = 0, 1,$$

$$\left|\partial_{x_n}^l \widetilde{Q}(\lambda + \widehat{L}_0)^{-1}f\right|_2 \leq \frac{C}{(|\lambda| + 1)^{1-\frac{l}{2}}}\left(1 + \frac{1}{|\lambda|}\right)|f|_{H^{l-1} \times L^2}, \quad l = 1, 2,$$

$$\left|\partial_{x_n}^2 Q_0(\lambda + \widehat{L}_0)^{-1}f\right|_2 \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}}|f|_{H^2 \times H^1}.$$

**Remark.** As for the adjoint operator

$$\widehat{L}_0^* = \begin{pmatrix} 0 & 0 & -\partial_{x_n} \\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & 0 \\ -\gamma^2\partial_{x_n} & {}^T\boldsymbol{e}_1' & -\nu_1\partial_{x_n}^2 \end{pmatrix}$$

with domain of definition $D(\widehat{L}_0^*) = D(\widehat{L}_0)$, one can see that $\sigma(-\widehat{L}_0^*) = \sigma(-\widehat{L}_0)$, and, in particular, 0 is a simple eigenvalue and $\widehat{L}_0^* u^{(0)} = 0$.

Lemma 5.1 can be proved in a similar manner to the proof of [3, Lemmas 3.1 and 3.2]. So we here omit the proof.

We next give some estimates for $(\lambda+\widehat{L}_{\xi'})^{-1}$ for small $\xi'$. Based on Lemma 5.1 we obtain the following estimates.

**Theorem 5.2.** *Let $\eta_0$ and $\theta_0$ be the numbers given in Lemma 5.1. Then there exists a positive number $\widetilde{r}_0 = \widetilde{r}_0(\eta_0,\theta_0)$ such that the set $\Sigma(-\eta_0,\theta_0) \cap \left\{\lambda;\ |\lambda| \geq \frac{\eta_0}{2}\right\}$ is in $\rho(-\widehat{L}_{\xi'})$ for $|\xi'| \leq \widetilde{r}_0$. Furthermore, the following estimates hold for any multi-index $\alpha'$ uniformly in $\lambda \in \Sigma(-\eta_0,\theta_0) \cap \left\{\lambda;\ |\lambda| \geq \frac{\eta_0}{2}\right\}$ and $\xi'$ with $|\xi'| \leq \widetilde{r}_0$:*

$$\left|\partial_{\xi'}^{\alpha'}(\lambda+\widehat{L}_{\xi'})^{-1} f\right|_{H^l\times L^2} \leq \frac{C_{\alpha'}}{|\lambda|}|f|_{H^l\times L^2}, \quad l=0,1,$$

$$\left|\partial_{\xi'}^{\alpha'}\partial_{x_n}^l \widetilde{Q}(\lambda+\widehat{L}_{\xi'})^{-1} f\right|_2 \leq \frac{C_{\alpha'}}{(|\lambda|+1)^{1-\frac{l}{2}}}|f|_{H^{l-1}\times L^2}, \quad l=1,2,$$

$$\left|\partial_{\xi'}^{\alpha'}\partial_{x_n}^2 Q_0(\lambda+\widehat{L}_{\xi'})^{-1} f\right|_2 \leq \frac{C_{\alpha'}}{(|\lambda|+1)^{\frac{1}{2}}}|f|_{H^2\times H^1}.$$

**Proof.** Theorem 5.2 can be proved in the same way as in the proof of [3, Theorem 3.2]. We here give only the necessary estimates.

In the following we will write

$$\widehat{L}^{(1)}(\xi') = \sum_{j=1}^{n-1}\xi_j \widehat{L}_j^{(1)} \quad\text{and}\quad \widehat{L}^{(2)}(\xi') = \sum_{j,k=1}^{n-1}\xi_j\xi_k \widehat{L}_{jk}^{(2)}.$$

We first observe that

$$\left|\widehat{L}_j^{(1)} u\right|_{H^l\times H^{(l-1)_+}} \leq C\left\{|Q_0 u|_{H^l} + |\widetilde{Q}u|_{H^{(l-1)_++1}}\right\} \tag{5.3}$$

and

$$\left|\widehat{L}^{(2)}_{jk}u\right|_{H^l\times H^{(l-1)_+}} \le C|\widetilde{Q}u|_{H^{(l-1)_+}}. \tag{5.4}$$

Let $l = 0, 1, 2$ and $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{\lambda; |\lambda| \ge \frac{\eta_0}{2}\right\}$. It then follows from Lemma 5.1, (5.3) and (5.4) that there exists a positive number $\widetilde{r}_0$ such that if $|\xi'| \le \widetilde{r}_0$, then

$$\left|\left(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi')\right)(\lambda + \widehat{L}_0)^{-1}f\right|_{H^l\times H^{(l-1)_+}} \le \frac{1}{2}|f|_{H^l\times H^{(l-1)_+}}.$$

Therefore, by the Neumann series expansion, we see that

$$I + \left(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi')\right)(\lambda + \widehat{L}_0)^{-1}$$

is invertible on $H^l(0,1) \times H^{(l-1)_+}(0,1)$, $l = 0, 1, 2$, for $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{\lambda; |\lambda| \ge \frac{\eta_0}{2}\right\}$ and $\xi'$ with $|\xi'| \le \widetilde{r}_0$. In particular, we conclude that $\Sigma(-\eta_0, \theta_0) \cap \left\{\lambda; |\lambda| \ge \frac{\eta_0}{2}\right\} \subset \rho(-\widehat{L}_{\xi'})$ and

$$(\lambda + \widehat{L}_{\xi'})^{-1} = (\lambda + \widehat{L}_0)^{-1}\sum_{N=0}^{\infty}(-1)^N\left[\left(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi')\right)(\lambda + \widehat{L}_0)^{-1}\right]^N \tag{5.5}$$

for $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{\lambda; |\lambda| \ge \frac{\eta_0}{2}\right\}$ and $\xi'$ with $|\xi'| \le \widetilde{r}_0$. Using Lemma 5.1 and (5.3)–(5.5), we obtain the desired estimates in Theorem 5.2 as in the proof of [3, Theorem 3.1]. □

As for the spectrum of $-\widehat{L}_{\xi'}$ near $\lambda = 0$, we have the following result.

**Theorem 5.3.** *Let $\eta_0$ and $\widetilde{r}_0$ be the numbers given in Theorem 5.2. Then there exists a positive number $r_0$ with $r_0 \le \widetilde{r}_0$ such that for each $\xi'$ with $|\xi'| \le r_0$ it holds that*

$$\sigma(-\widehat{L}_{\xi'}) \cap \{\lambda; |\lambda| \le \eta_0\} = \{\lambda_0(\xi')\},$$

*where $\lambda_0(\xi')$ is a simple eigenvalue of $-\widehat{L}_{\xi'}$ that has the form*

$$\lambda_0(\xi') = -\frac{i}{2}\xi_1 - \kappa_1\xi_1^2 - \kappa_2|\xi''|^2 + O(|\xi'|^3)$$

*as $|\xi'| \to 0$. Here $\xi'' = (\xi_2, \cdots, \xi_{n-1})$; and $\kappa_1$ and $\kappa_2$ are some positive numbers.*

**Proof.** By Lemma 5.1, (5.3) and (5.4), we see that if $|\lambda| = \eta_0$, then $\lambda \in \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \le \widetilde{r}_0$. In particular,

$$\widehat{\Pi}(\xi') = \frac{1}{2\pi i}\int_{|\lambda|=\eta_0} (\lambda + \widehat{L}_{\xi'})^{-1}\, d\lambda$$

is the eigenprojection for the eigenvalues of $-\widehat{L}_{\xi'}$ lying inside the circle $|\lambda| = \eta_0$. The continuity of $(\lambda+\widehat{L}_{\xi'})^{-1}$ in $(\lambda, \xi')$ then implies that $\dim \mathrm{Range}\, \widehat{\Pi}(\xi') = \dim \mathrm{Range}\, \widehat{\Pi}^{(0)} = 1$. (See [10, Chap. 1, Lemma 4.10 and Chap. 4, Theorem 3.16].) Therefore, we see from Lemma 5.1 that $\sigma(-\widehat{L}_{\xi'}) \cap \{\lambda;\, |\lambda| \le \eta_0\}$ consists of only one simple eigenvalue, say $\lambda_0(\xi')$.

In view of (5.3) and (5.4) we can apply the analytic perturbation theory (cf., [10, Chap. 2, Sect.2.2 and Chap. 7, Remark 2.10]). Since $\lambda^{(0)}(\xi')$ is simple, we can see that $\lambda^{(0)}(\xi')$ and $\Pi(\xi')$ are expanded as

$$\lambda_0(\xi') = \lambda^{(0)} + \sum_{j=1}^{n-1} \xi_j \lambda_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \lambda_{jk}^{(2)} + O(|\xi'|^3),$$

$$\widehat{\Pi}(\xi') = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)} + O(|\xi'|^2)$$

with $\lambda^{(0)} = 0$ and

$$\lambda_j^{(1)} = -(\widehat{L}_j^{(1)} u^{(0)}, u^{(0)}),$$
$$\lambda_{jk}^{(2)} = -(\widehat{L}_{jk}^{(2)} u^{(0)}, u^{(0)}) + (\widehat{L}_j^{(1)} \widehat{S} \widehat{L}_k^{(1)} u^{(0)}, u^{(0)}),$$
$$\widehat{\Pi}_j^{(1)} = -\widehat{\Pi}^{(0)} \widehat{L}_j^{(1)} \widehat{S} - \widehat{S} \widehat{L}_j^{(1)} \widehat{\Pi}^{(0)},$$

where $\widehat{S} = \Big[(I - \widehat{\Pi}^{(0)}) \widehat{L}_0 (I - \widehat{\Pi}^{(0)})\Big]^{-1}$.

Since $\widehat{L}_j^{(1)} u^{(0)} = i \begin{pmatrix} x_n \delta_{1j} \\ \boldsymbol{e}_1' \\ 0 \end{pmatrix}$, we have

$$\lambda_j^{(1)} = -\frac{i}{2}\delta_{1j}.$$

Let us compute $\lambda_{jk}^{(2)}$. It is easy to see that $\widehat{L}_{jk}^{(2)} u^{(0)} = 0$. By a direct computation we have

$$\widehat{S}\widehat{L}_1^{(1)} u^{(0)} = \begin{pmatrix} \frac{i\nu_1}{\gamma^2}(x_n - \frac{1}{2}) \\ \frac{i}{\nu}\{\frac{1}{\gamma^2}(\frac{1}{24}x_n - \frac{1}{12}x_n^3 + \frac{1}{24}x_n^4) + \frac{1}{2}(x_n - x_n^2)\}\boldsymbol{e}_1' \\ -\frac{i}{2\gamma^2}(x_n - x_n^2) \end{pmatrix}$$

and

$$\widehat{S}\widehat{L}_j^{(1)}u^{(0)} = \frac{i}{2\nu}\begin{pmatrix} 0 \\ (x_n - x_n^2)\boldsymbol{e}_j' \\ 0 \end{pmatrix} \quad (j \neq 1).$$

It then follows that

$$\lambda_{11}^{(2)} = -\frac{1}{12}\left\{\left(\frac{\nu_1}{\gamma^2} + \frac{1}{10\nu}\right) + \frac{\gamma^2}{\nu}\right\}, \quad \lambda_{jj}^{(2)} = -\frac{\gamma^2}{12\nu}\ (j \neq 1), \quad \lambda_{jk}^{(2)} = 0 \ \ (j \neq k).$$

Consequently, we obtain

$$\lambda_0(\xi') = -\frac{i}{2}\xi_1 - \kappa_1\xi_1^2 - \kappa_2|\xi''|^2 + O(|\xi'|^3)$$

with

$$\kappa_1 = \frac{1}{12}\left\{\left(\frac{\nu_1}{\gamma^2} + \frac{1}{10\nu}\right) + \frac{\gamma^2}{\nu}\right\}, \quad \kappa_2 = \frac{\gamma^2}{12\nu}.$$

This completes the proof. □

As for the eigenprojection $\widehat{\Pi}(\xi')$ associated with $\lambda_0(\xi')$, we have the following result.

**Theorem 5.4.** (i) *Let $\widehat{\Pi}(\xi')$ be the eigenprojection associated with $\lambda_0(\xi')$. Then there exists a positive number $r_0$ such that for any $\xi'$ with $|\xi'| \leq r_0$ the projection $\widehat{\Pi}(\xi')$ is written in the form*

$$\widehat{\Pi}(\xi')u = \int_0^1 \widehat{\Pi}(\xi', x_n, y_n)u(y_n)\,dy_n$$

*with*

$$\widehat{\Pi}(\xi', x_n, y_n) = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1}\xi_j\widehat{\Pi}_j^{(1)}(x_n, y_n) + \widehat{\Pi}^{(2)}(\xi', x_n, y_n).$$

*Here $\widehat{\Pi}^{(0)} = Q_0$; and $\widehat{\Pi}_j^{(1)}(x_n, y_n)$ $(j = 1, \cdots, n-1)$ and $\widehat{\Pi}^{(2)}(\xi', x_n, y_n)$ satisfy*

$$\left|\partial_{x_n}^k\partial_{y_n}^l\widehat{\Pi}_j^{(1)}(\cdot,\cdot)\right|_{L^\infty((0,1)\times(0,1))} \leq C,$$

$$\left|\partial_{x_n}^k\partial_{y_n}^l\partial_{\xi'}^{\alpha'}\widehat{\Pi}^{(2)}(\xi',\cdot,\cdot)\right|_{L^\infty((0,1)\times(0,1))} \leq C_{\alpha'}|\xi'|^{2-|\alpha'|}$$

*for $0 \leq k, l \leq 1$ and any multi-index $\alpha'$ uniformly in $\xi'$ with $|\xi'| \leq r_0$.*

(ii) *If $Q_0u|_{x_n=0,1} = 0$, then $\widehat{\Pi}(\xi')$ satisfies*

$$\widehat{\Pi}(\xi')\left[\partial_{x_n}u\right] = -\left(\partial_{y_n}\widehat{\Pi}(\xi')\right)[u], \quad \widehat{\Pi}^{(0)}\left[\partial_{x_n}u\right] = 0,$$

$$\widehat{\Pi}_j^{(1)}\left[\partial_{x_n}u\right] = -\left(\partial_{y_n}\widehat{\Pi}_j^{(1)}\right)[u]\,, \quad \widehat{\Pi}^{(2)}(\xi')\left[\partial_{x_n}u\right] = -\left(\partial_{y_n}\widehat{\Pi}^{(2)}(\xi')\right)[u]\,.$$

(iii) *It holds*

$$\partial_{x_n}\widehat{\Pi}^{(0)}u = 0, \quad \partial_{x_n}\widehat{\Pi}_j^{(1)}\widetilde{Q}u = 0.$$

*Furthermore, if* $Q_0u|_{x_n=0,1} = 0$*, then*

$$\partial_{x_n}\widehat{\Pi}_j^{(1)}\left[\partial_{x_n}u\right] = 0.$$

**Proof.** The proof of (i) is the same as that in the proof of [3, Theorem 3.3]. The assertion (ii) can be also proved in a similar manner to the proof of [3, Theorem 3.3] by integration by parts. As for (iii), it is clear that $\partial_{x_n}\widehat{\Pi}^{(0)} = 0$. Concerning the properties of $\widehat{\Pi}_j^{(1)}$, we note that

$$\widehat{\Pi}_j^{(1)} = -\widehat{\Pi}^{(0)}\widehat{L}_j^{(1)}\widehat{S} - \widehat{S}\widehat{L}_j^{(1)}\widehat{\Pi}^{(0)}.$$

Since $\widehat{\Pi}^{(0)}\widetilde{Q} = 0$, $\partial_{x_n}\widehat{\Pi}^{(0)} = 0$, we see that $\partial_{x_n}\widehat{\Pi}_j^{(1)}\widetilde{Q}u = 0$. If $Q_0u|_{x_n=0,1} = 0$, then $\widehat{\Pi}^{(0)}[\partial_{x_n}u] = 0$, which implies $\partial_{x_n}\widehat{\Pi}_j^{(1)}\left[\partial_{x_n}u\right] = 0$. □

We are now in a position to prove Proposition 4.2.

**Proof of Proposition 4.2.** By Theorem 5.2, $U_0(t)u_0$ is written as

$$U_0(t)u_0 = \mathcal{F}^{-1}\left[\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\chi^{(0)}(\xi')(\lambda + \widehat{L}_{\xi'})^{-1}\,d\lambda\right],$$

where $\Gamma = \{\lambda = \eta + se^{\pm i\theta};\ s \geq 0\}$ with some $\eta > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$.

By Theorems 5.2 and 5.3, we can deform the contour $\Gamma$ into $\Gamma_0 \cup \widetilde{\Gamma}$ and a suitable circle around 0, where

$$\Gamma_0 = \{\lambda = -\eta_0 + is;\ |s| \leq s_0\}, \quad \widetilde{\Gamma} = \{\lambda = \eta + se^{\pm i\theta};\ s \geq \widetilde{s}_0\}.$$

Here we choose positive numbers $s_0$ and $\widetilde{s}_0$ so that $\Gamma_0$ connects with $\widetilde{\Gamma}$ at the end points of $\Gamma_0$. It then follows from Theorems 5.3, 5.4 and the residue theorem that $U_0(t)u_0$ is written as

$$U_0(t)u_0 = \mathcal{U}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0,$$

where

$$\mathcal{U}^{(0)}(t)u_0 = \mathcal{F}^{-1}\left[\chi^{(0)}(\xi')e^{\lambda_0(\xi')t}\widehat{\Pi}(\xi')\widehat{u}_0\right]$$

and

$$\mathcal{R}^{(0)}(t)u_0 = \mathcal{F}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_0\cup\widetilde{\Gamma}} e^{\lambda t}\chi^{(0)}(\xi')(\lambda+\widehat{L}_{\xi'})^{-1}\widehat{u}_0\,d\lambda\right].$$

By Theorem 5.2, one can see that $\mathcal{R}^{(0)}(t)u_0$ has the desired estimate in Proposition 4.2.

Let us consider $\mathcal{U}^{(0)}(t)u_0$. We write it as

$$\mathcal{U}^{(0)}(t)u_0 = G_t *_{x'} \Pi^{(0)}u_0 + \mathcal{U}^{(0)}_1(t)u_0 + \mathcal{U}^{(0)}_2(t)u_0 + \mathcal{U}^{(0)}_3(t)u_0 + \mathcal{U}^{(0)}_4(t)u_0,$$

where

$$G_t *_{x'} \Pi^{(0)}u_0 = \mathcal{F}^{-1}\left[e^{-(\frac{i}{2}\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(0)}\widehat{u}_0\right],$$

$$\mathcal{U}^{(0)}_1(t)u_0 = \mathcal{F}^{-1}\left[(\chi^{(0)}(\xi')-1)e^{-(\frac{i}{2}\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(0)}\widehat{u}_0\right],$$

$$\mathcal{U}^{(0)}_2(t)u_0 = \mathcal{F}^{-1}\left[\chi^{(0)}(\xi')e^{-(\frac{i}{2}\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(1)}(\xi')\widehat{u}_0\right],$$

$$\mathcal{U}^{(0)}_3(t)u_0 = \mathcal{F}^{-1}\left[\chi^{(0)}(\xi')e^{-(\frac{i}{2}\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t}\widehat{\Pi}^{(2)}(\xi')\widehat{u}_0\right]$$

and

$$\mathcal{U}^{(0)}_4(t)u_0 = \mathcal{F}^{-1}\left[\chi^{(0)}(\xi')(e^{\lambda_0(\xi')t} - e^{-(\frac{i}{2}\xi_1+\kappa_1\xi_1^2+\kappa_2|\xi''|^2)t})\widehat{\Pi}(\xi')\widehat{u}_0\right]$$

with

$$\widehat{\Pi}^{(1)}(\xi') = \sum_{j=1}^{n-1}\xi_j\widehat{\Pi}^{(1)}_j.$$

The desired estimates for $\mathcal{U}^{(0)}(t)u_0$ now follows from Theorems 5.3 and 5.4. This completes the proof. □

## 6. Proof of Proposition 4.3

In this section we investigate problem (5.1) for $|\xi'| \geq r > 0$ and prove Proposition 4.3. We also give a proof of the estimate given in Theorem 3.1.

As mentioned before, for each fixed $\xi' \in \mathbf{R}^{n-1}$, the operator $-\widehat{L}_{\xi'}$ generates an analytic semigroup $e^{-t\widehat{L}_{\xi'}}$ on $H^1(0,1)\times L^2(0,1)$. This implies that $u(t) = e^{-t\widehat{L}_{\xi'}}u_0 = {}^T(\phi(t),\psi(t))$ gives a unique solution of (5.1), and we have

$$\frac{1}{\gamma^2}\partial_t\phi + \frac{1}{\gamma^2}i\xi_1x_n\phi + i\xi'\cdot\psi' + \partial_{x_n}\psi^n = 0, \tag{6.1}$$

$$\partial_t\psi' + \nu(|\xi'|^2-\partial_{x_n}^2)\psi' - i\widetilde{\nu}\xi'(i\xi'\cdot\psi'+\partial_{x_n}\psi^n) + i\xi'\phi + i\xi_1x_n\psi' + \psi^n\boldsymbol{e}'_1 = 0, \tag{6.2}$$

(6.3) $\partial_t\psi^n + \nu(|\xi'|^2 - \partial_{x_n}^2)\psi^n - \widetilde{\nu}\partial_{x_n}(i\xi'\cdot\psi' + \partial_{x_n}\psi^n) + \partial_{x_n}\phi + i\xi_1 x_n\psi^n = 0,$

$$\psi|_{x_n=0,1} = 0 \tag{6.4}$$

for $t > 0$, and

$$u|_{t=0} = u_0 = {}^T(\phi_0, \psi_0). \tag{6.5}$$

We introduce some notations. For $u = {}^T(\phi,\psi)$ we define $E_0[u]$ and $\widetilde{E}_0[u]$ by

$$E_0[u] = |u|_2^2, \quad \widetilde{E}_0[u] = \frac{1}{\gamma^2}|\phi|_2^2 + |\psi|_2^2.$$

For $v = \phi$, $v = \psi = {}^T(\psi_1, \cdots, \psi^n)$ or $v = {}^T(\phi,\psi)$, we define $D_0[v]$ by

$$D_0[v] = |\xi'|^2|v|_2^2 + |\partial_{x_n}v|_2^2,$$

and, for $\psi = {}^T(\psi_1, \cdots, \psi^n)$, we define $\widetilde{D}_0[\psi]$ by

$$\widetilde{D}_0[\psi] = \nu D_0[\psi] + \widetilde{\nu}|i\xi'\cdot\psi' + \partial_{x_n}\psi^n|_2^2.$$

We also introduce the quantity $E_1[u]$ that is defined by

$$E_1[u] = 2\left(1 + \frac{2n\gamma^2}{\nu}\right)(1+|\xi'|^2)\widetilde{E}_0[u] + \widetilde{D}_0[\psi] - 2\mathrm{Re}\,(\phi, i\xi'\cdot\psi' + \partial_{x_n}\psi^n).$$

Note that there holds the estimate

$$\left(2 + \frac{n\gamma^2}{\nu}\right)(1+|\xi'|^2)\widetilde{E}_0[u] + \frac{1}{2}\widetilde{D}_0[\psi] \le E_1[u] \le \left(2 + \frac{6n\gamma^2}{\nu}\right)(1+|\xi'|^2)\widetilde{E}_0[u] + \frac{3}{2}\widetilde{D}_0[\psi].$$

We denote the *material derivative* $\partial_t\phi + i\xi_1 x_n\phi$ by $\dot{\phi}$, i.e.,

$$\dot{\phi} = \partial_t\phi + i\xi_1 x_n\phi.$$

In what follows $u(t) = (\phi(t), \psi(t))$ will denote the unique solution of problem (6.1)–(6.5).

**Proposition 6.1.** *There exists constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $(\nu + \widetilde{\nu})/\gamma^2 \le 1/\gamma_0^2$, then for any $r > 0$ there exists a constant $a = a(r) > 0$ such that the estimate*

$$E_0[u](t) + D_0[u](t) \le Ce^{-a(t-1)}\{E_0[u](1) + D_0[u](1)\} \tag{6.6}$$

*holds uniformly for $t \ge 1$, provided that $|\xi'| \ge r$.*

**Proposition 6.2.** *There holds the following estimate uniformly for* $0 < t \le 1$ *and* $\xi' \in \mathbf{R}^{n-1}$*:*

$$E_0[u](t) + D_0[u](t) \le C\left\{(1+|\xi'|^2)|u_0|_2^2 + |\partial_{x_n}\phi_0|_2^2 + t^{-1}|\psi_0|_2^2\right\}. \tag{6.7}$$

Proposition 4.3 is an immediate consequence of Propositions 6.1 and 6.2. The estimate in Theorem 3.1 also follows from Proposition 6.2.

We will first give several estimates which prove (6.6) for $|\xi'| \ge R$ with some $R > 0$, and, then prove (6.6) for $0 < r \le |\xi'| \le R$. (See Propositions 6.8 and 6.10.) To prove (6.6) for $|\xi'| \ge R$, we employ the Fourier transformed version of Matsumura-Nishida's energy method. As for the proof for $r \le |\xi'| \le R$ we employ the argument in [9]. Proposition 6.2 will be proved in the end of this section.

In what follows the letters $C_j$, $j = 1, 2, \cdots$, denote constants which can be taken uniformly for $\nu$, $\widetilde{\nu}$ and $\gamma$ in their specified range.

**Proposition 6.3.** *If* $\nu \ge 4$*, then the following estimate holds:*

$$\frac{d}{dt}\widetilde{E}_0[u](t) + \widetilde{D}_0[\psi] + \frac{\nu+\widetilde{\nu}}{4n\gamma^4}|\dot{\phi}|_2^2 \le 0. \tag{6.8}$$

**Proof.** Taking the inner product of (6.1), (6.2) and (6.3) with $\phi$, $\psi'$ and $\psi^n$, respectively, and integrating by parts, we have

$$\frac{1}{\gamma^2}(\partial_t\phi, \phi) + \frac{1}{\gamma^2}i\xi_1(x_n\phi, \phi) + (i\xi'\cdot\psi' + \partial_{x_n}\psi^n, \phi) = 0, \tag{6.9}$$

$$\begin{aligned}(\partial_t\psi', \psi') &+ \nu D_0[\psi'] - i\widetilde{\nu}\xi'(i\xi'\psi' + \partial_{x_n}\psi^n, \psi') + (i\xi'\phi, \psi') \\ &+ i\xi_1(x_n\psi', \psi') + (\psi\boldsymbol{e}_1', \psi') = 0\end{aligned} \tag{6.10}$$

$$\begin{aligned}(\partial_t\psi^n, \psi^n) &+ \nu D_0[\psi^n] - \widetilde{\nu}\partial_{x_n}(i\xi'\psi' + \partial_{x_n}\psi^n, \psi^n) + (\partial_{x_n}\phi, \psi^n) \\ &+ i\xi_1(x_n\psi^n, \psi^n) = 0.\end{aligned} \tag{6.11}$$

We observe that

$$(i\xi'\cdot\psi' + \partial_{x_n}\psi^n, \phi) + (i\xi'\phi, \psi') + (\partial_{x_n}\phi, \psi^n) = 2i\mathrm{Im}\,(i\xi'\cdot\psi' + \partial_{x_n}\psi^n, \phi),$$

and,

$$(x_n\phi, \phi), \quad (x_n\psi', \psi'), \quad (x_n\psi^n, \psi^n) \in \mathbf{R}.$$

Then, adding (6.9)–(6.11) and taking the real part of the resulting identity we obtain

$$\frac{1}{2}\frac{d}{dt}\widetilde{E}_0[u] + \widetilde{D}_0[\psi] + \mathrm{Re}\,(\psi^n, \psi^1) = 0. \tag{6.12}$$

By the Poincaré inequality we have $|(\psi^n, \psi^1)| \leq |\partial_{x_n}\psi|_2^2$. It then follows from (6.12) that

$$\frac{d}{dt}\widetilde{E}_0[u] + \frac{3}{4}\widetilde{D}_0[\psi] \leq 0, \tag{6.13}$$

provided that $\nu \geq 2$. Furthermore, by (6.1), we have

$$\dot{\phi} = -\gamma^2(i\xi' \cdot \psi' + \partial_{x_n}\psi^n),$$

and, hence,

$$(\nu + \widetilde{\nu})|\dot{\phi}|_2^2 \leq n\gamma^4\widetilde{D}_0[\psi].$$

This, together with (6.13), gives the desired estimate (6.8). This completes the proof. □

**Proposition 6.4.** *If $\nu \geq 4$, then the following estimate holds for any $\eta > 0$:*

$$\begin{aligned} &\frac{d}{dt}E_1[u](t) + (1 + |\xi'|^2)\widetilde{D}_0[\psi] + \frac{\nu + \widetilde{\nu}}{2n\gamma^4}(1 + |\xi'|^2)|\dot{\phi}|_2^2 + |\partial_t\psi|_2^2 \\ &\quad \leq \frac{\eta}{\nu + \widetilde{\nu}}|\xi'|^2|\phi|_2^2 + \frac{n}{\eta}\widetilde{D}_0[\psi]. \end{aligned} \tag{6.14}$$

**Proof.** We take the inner product of (6.2) and (6.3) with $\partial_t\psi'$ and $\partial_t\psi^n$, respectively, to obtain

$$\begin{aligned} &|\partial_t\psi|_2^2 + \nu\{|\xi'|^2(\psi, \partial_t\psi) + (\partial_{x_n}\psi, \partial_t\partial_{x_n}\psi)\} + \widetilde{\nu}(i\xi' \cdot \psi' + \partial_{x_n}\psi^n, \partial_t(i\xi' \cdot \psi' + \partial_{x_n}\psi^n)) \\ &\quad -(\phi, \partial_t(i\xi' \cdot \psi' + \partial_{x_n}\psi^n)) + i\xi_1(x_n\psi, \partial_t\psi) + (\psi^n, \partial_t\psi^1) = 0. \end{aligned}$$

Taking the real part, we have

$$|\partial_t\psi|^2 + \frac{1}{2}\frac{d}{dt}\widetilde{D}_0[\psi] = \mathrm{Re}\,\{(\phi, \partial_t(i\xi' \cdot \psi' + \partial_{x_n}\psi^n)) - i\xi_1(x_n\psi, \partial_t\psi) - (\psi^n, \partial_t\psi^1)\}. \tag{6.15}$$

Since (6.1) is written as

$$\partial_t\phi = -i\xi_1 x_n\phi - \gamma^2(i\xi' \cdot \psi' + \partial_{x_n}\psi^n),$$

we have

$$(\phi, \partial_t(i\xi' \cdot \psi' + \partial_{x_n}\psi^n))$$

$$\begin{aligned}
&= \frac{d}{dt}(\phi, i\xi' \cdot \psi' + \partial_{x_n}\psi^n) - (\partial_t\phi, i\xi' \cdot \psi' + \partial_{x_n}\psi^n) \\
&= \frac{d}{dt}(\phi, i\xi' \cdot \psi' + \partial_{x_n}\psi^n) \\
&\quad + i\xi_1(x_n\phi, i\xi' \cdot \psi' + \partial_{x_n}\psi^n) + \gamma^2|i\xi' \cdot \psi' + \partial_{x_n}\psi^n|_2^2.
\end{aligned} \tag{6.16}$$

From (6.15) and (6.16) we find that

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\widetilde{D}_0[\psi] - \mathrm{Re}\,(\phi, i\xi' \cdot \psi' + \partial_{x_n}\psi^n)\right\} + |\partial_t\psi|_2^2 \\
&\quad \leq \frac{1}{2}|\partial_t\psi|_2^2 + D_0[\psi] + (|\xi'||\phi|_2 + \gamma^2|i\xi' \cdot \psi' + \partial_{x_n}\psi^n|_2)|i\xi' \cdot \psi' + \partial_{x_n}\psi^n|_2.
\end{aligned}$$

Since $|i\xi' \cdot \psi' + \partial_{x_n}\psi^n|_2^2 \leq \frac{n}{\nu+\widetilde{\nu}}\widetilde{D}_0[\psi]$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\widetilde{D}_0[\psi] - 2\mathrm{Re}\,(\phi, i\xi' \cdot \psi' + \partial_{x_n}\psi^n)\right\} + |\partial_t\psi|_2^2 \\
&\quad \leq 2D_0[\psi] + \frac{\eta}{\nu+\widetilde{\nu}}|\xi'|^2|\phi|_2^2 + n\left(\frac{2\gamma^2}{\nu+\widetilde{\nu}} + \frac{1}{\eta}\right)\widetilde{D}_0[\psi].
\end{aligned} \tag{6.17}$$

Adding $2(1+\frac{2n\gamma^2}{\nu})(1+|\xi'|^2) \times$ (6.8) to (6.17), we obtain the desired result of Proposition 6.4. This completes the proof. $\square$

**Proposition 6.5.** *If $\nu \geq 4$, then the following estimate holds for any $\eta > 0$:*

$$\begin{aligned}
&\frac{d}{dt}E_2[u](t) + \frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2 + \left(1 + \frac{1}{\nu+\widetilde{\nu}}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi] \\
&+ \left(1 + \frac{1}{\nu+\widetilde{\nu}}\right)|\partial_t\psi|_2^2 + \frac{2(\nu+\widetilde{\nu})}{n\gamma^4}(1+|\xi'|^2)|\dot{\phi}|_2^2 + \frac{\nu+\widetilde{\nu}}{4\gamma^4}|\partial_{x_n}\dot{\phi}|_2^2 \\
&\leq 4\left(\frac{\nu+\widetilde{\nu}}{\gamma^4} + \frac{2\eta}{\nu+\widetilde{\nu}}\right)|\xi'|^2|\phi|_2^2 + \frac{8n}{\eta}\widetilde{D}_0[\psi].
\end{aligned} \tag{6.18}$$

*Here*

$$E_2[u] = 4\left(1 + \frac{1}{\nu+\widetilde{\nu}}\right)E_1[u] + \frac{1}{\gamma^2}|\partial_{x_n}\phi|_2^2.$$

**Proof.** Differentiating (6.1) in $x_n$ we have

$$\frac{1}{\gamma^2}\partial_t\partial_{x_n}\phi + \frac{1}{\gamma^2}i\xi_1 x_n\partial_{x_n}\phi + \frac{1}{\gamma^2}i\xi_1\phi + i\xi' \cdot \partial_{x_n}\psi' + \partial_{x_n}^2\psi^n = 0. \tag{6.19}$$

We rewrite (6.3) as

$$\partial_{x_n}\phi - (\nu+\widetilde{\nu})\partial_{x_n}^2\psi^n = -\{\partial_t\psi^n + \nu|\xi'|^2\psi^n - i\widetilde{\nu}\xi'\cdot\partial_{x_n}\psi' + i\xi_1 x_n\psi^n\}. \tag{6.20}$$

By adding (6.19) and $\frac{1}{\nu+\widetilde{\nu}}\times$ (6.20) we have

$$\frac{1}{\gamma^2}\partial_t\partial_{x_n}\phi + \frac{1}{\gamma^2}i\xi_1 x_n\partial_{x_n}\phi + \frac{1}{\nu+\widetilde{\nu}}\partial_{x_n}\phi + \frac{1}{\gamma^2}i\xi_1\phi = -\frac{1}{\nu+\widetilde{\nu}}M[\psi]. \tag{6.21}$$

Here

$$M[\psi] = \partial_t\psi^n + \widetilde{M}[\psi],$$
$$\widetilde{M}[\psi] = \nu|\xi'|^2\psi^n + i\nu\xi'\cdot\partial_{x_n}\psi' + i\xi_1 x_n\psi^n.$$

Taking the inner product of (6.21) with $\partial_{x_n}\phi$, we have

$$\begin{aligned}&\frac{1}{\gamma^2}(\partial_t\partial_{x_n}\phi, \partial_{x_n}\phi) + \frac{1}{\gamma^2}i\xi_1(x_n\partial_{x_n}\phi, \partial_{x_n}\phi) + \frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2\\ &\quad = -\frac{1}{\gamma^2}i\xi_1(\phi, \partial_{x_n}\phi) - \frac{1}{\nu+\widetilde{\nu}}(M[\psi], \partial_{x_n}\phi).\end{aligned} \tag{6.22}$$

Since $\nu \geq 2$, we have

$$|M[\psi]|_2 \leq |\partial_t\psi|_2 + \nu\sqrt{(1+|\xi'|^2)D_0[\psi]}.$$

The real part of (6.22) then yields

$$\begin{aligned}&\frac{1}{2\gamma^2}\frac{d}{dt}|\partial_{x_n}\phi|_2^2 + \frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2\\ &\leq \frac{|\xi'|}{\gamma^2}|\phi|_2|\partial_{x_n}\phi|_2 + \frac{1}{\nu+\widetilde{\nu}}|M[\psi]|_2|\partial_{x_n}\phi|_2\\ &\leq \frac{1}{4}\frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2 + \frac{\nu+\widetilde{\nu}}{\gamma^4}|\xi'|^2|\phi|_2^2 + \frac{1}{\nu+\widetilde{\nu}}|\partial_t\psi|_2^2\\ &\quad + \frac{\nu^2}{\nu+\widetilde{\nu}}(1+|\xi'|^2)D_0[\psi].\end{aligned}$$

We thus obtain

$$\begin{aligned}&\frac{1}{\gamma^2}\frac{d}{dt}|\partial_{x_n}\phi|_2^2 + \frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2\\ &\leq \frac{2(\nu+\widetilde{\nu})}{\gamma^4}|\xi'|^2|\phi|_2^2 + \frac{2}{\nu+\widetilde{\nu}}|\partial_t\psi|_2^2 + 2(1+|\xi'|^2)\widetilde{D}_0[\psi].\end{aligned} \tag{6.23}$$

It then follows from $4(1+\frac{1}{\nu+\widetilde{\nu}})\times$ (6.14) and (6.23) that

$$
\begin{aligned}
&\frac{d}{dt}E_2[u](t)+\frac{3}{2}\frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2+2\left(1+\frac{1}{\nu+\widetilde{\nu}}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi]\\
&\quad+\ 2\left(1+\frac{1}{\nu+\widetilde{\nu}}\right)|\partial_t\psi|_2^2+\frac{2(\nu+\widetilde{\nu})}{n\gamma^4}(1+|\xi'|^2)|\dot{\phi}|_2^2 \qquad (6.24)\\
&\qquad\le\left(\frac{2(\nu+\widetilde{\nu})}{\gamma^4}+\frac{8\eta}{\nu+\widetilde{\nu}}\right)|\xi'|^2|\phi|_2^2+\frac{8n}{\eta}\widetilde{D}_0[\psi].
\end{aligned}
$$

We next rewrite (6.21) as

$$\frac{1}{\gamma^2}\partial_{x_n}\dot{\phi}+\frac{1}{\nu+\widetilde{\nu}}\partial_{x_n}\phi=-\frac{1}{\nu+\widetilde{\nu}}M[\psi].$$

This gives

$$
\begin{aligned}
|\partial_{x_n}\dot{\phi}|_2^2 &= -\frac{\gamma^2}{\nu+\widetilde{\nu}}(\partial_{x_n}\phi,\partial_{x_n}\dot{\phi})-\frac{\gamma^2}{\nu+\widetilde{\nu}}(M[\psi],\partial_{x_n}\dot{\phi})\\
&\le \frac{1}{2}|\partial_{x_n}\dot{\phi}|_2^2+\frac{\gamma^4}{(\nu+\widetilde{\nu})^2}\left\{|\partial_{x_n}\phi|_2^2+|\partial_t\psi|_2^2+\nu^2(1+|\xi'|^2)D_0[\psi]\right\},
\end{aligned}
$$

and hence,

$$\frac{\nu+\widetilde{\nu}}{\gamma^4}|\partial_{x_n}\dot{\phi}|_2^2\le\frac{2}{\nu+\widetilde{\nu}}\left\{|\partial_{x_n}\phi|_2^2+|\partial_t\psi|_2^2+\nu^2(1+|\xi'|^2)D_0[\psi]\right\}. \qquad (6.25)$$

The desired estimate (6.18) now follows from (6.24) and (6.25), provided that $\nu\ge 4$. This completes the proof. □

To control $|\xi'|^2|\phi|$ we make use of an estimate for the Stokes system. Consider the Fourier transformed Stokes system

$$
\left\{
\begin{aligned}
&i\xi'\cdot v'+\partial_{x_n}v^n=F^0,\\
&(|\xi'|^2-\partial_{x_n}^2)v'+i\xi'p=F',\\
&(|\xi'|^2-\partial_{x_n}^2)v^n+\partial_{x_n}p=F^n,\\
&v|_{x_n=0,1}=0.
\end{aligned}
\right. \qquad (6.26)
$$

We have the following estimate when $|\xi'|$ is large.

**Lemma 6.6.** *There exists a constant $R_0>0$ such that if $|\xi'|\ge R_0$, then the following estimate holds for the solution $(p,v)$ $(v=(v',v^n))$ of (6.26):*

$$|p|_2^2+D_0[p]+|v|_2^2+D_0[v]+\sum_{j+k=2}\left||\xi'|^j\partial_{x_n}^k v\right|_2^2\le C\left\{|F^0|_2^2+D_0[F^0]+|F'|_2^2+|F^n|_2^2\right\}.$$

**Proposition 6.7.** *Let $R = \max\{R_0, 1\}$. If $|\xi'| \ge R$, then there holds the estimate*

$$
\begin{aligned}
&\frac{1}{\nu^2}\left(|\phi|_2^2 + D_0[\phi]\right) + \left(|\psi|_2^2 + D_0[\psi] + \sum_{j+k=2}\left||\xi'|^j\partial_{x_n}^k\psi\right|_2^2\right) \\
&\quad \le C_1\left\{\frac{(\nu+\widetilde{\nu})^2}{\gamma^4}\frac{1}{\nu^2}\left(|\dot{\phi}|_2^2 + D_0[\dot{\phi}]\right) + \frac{1}{\nu^2}|\partial_t\psi|_2^2 + \frac{1}{\nu^2}D_0[\psi]\right\}.
\end{aligned}
\tag{6.27}
$$

**Proof.** By (6.1)–(6.4) we have

$$
\begin{cases}
i\xi'\cdot\psi' + \partial_{x_n}\psi^n = F^0, \\
(|\xi'|^2 - \partial_{x_n}^2)\psi' + i\xi'\left(\frac{1}{\nu}\phi\right) = F', \\
(|\xi'|^2 - \partial_{x_n}^2)\psi^n + \partial_{x_n}\left(\frac{1}{\nu}\phi\right) = F^n, \\
\psi|_{x_n=0,1} = 0,
\end{cases}
$$

with

$$
F^0 = -\frac{1}{\gamma^2}\dot{\phi},
$$

$$
F' = -\frac{1}{\nu}\left(\partial_t\psi' + i\xi_1 x_n\psi' + \psi^n\boldsymbol{e}_1'\right) - \frac{\widetilde{\nu}}{\nu\gamma^2}i\xi'\dot{\phi},
$$

$$
F^n = -\frac{1}{\nu}\left(\partial_t\psi^n + i\xi_1 x_n\psi^n\right) - \frac{\widetilde{\nu}}{\nu\gamma^2}\partial_{x_n}\dot{\phi}.
$$

Applying Lemma 6.6 gives the desired estimate. This completes the proof. □

We are ready to prove Proposition 6.1 for $|\xi'| \ge R$.

**Proposition 6.8.** *There exists constants $\nu_0 > 0$ and $\gamma_0 > 0$ such that if $\nu \ge \nu_0$ and $(\nu + \widetilde{\nu})/\gamma^2 \le 1/\gamma_0^2$, then the estimate* (6.6) *in Poroposition* 6.1 *holds for $|\xi'| \ge R$.*

**Proof.** Let $|\xi'| \ge R$ and set $C_2 = \min\{1/(8C_1), 1/(nC_1)\}$. We see from

$\frac{8n}{\eta} \times (6.8) + (6.18) + \frac{C_2\nu^2}{\nu+\widetilde{\nu}} \times (6.27)$ with $\eta = \frac{C_2}{16}$ that

$$\begin{aligned} &\frac{d}{dt}E_3[u](t) + \frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2 + \left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi] \\ &+ \quad \left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)|\partial_t\psi|_2^2 + \frac{(\nu+\widetilde{\nu})}{n\gamma^4}(1+|\xi'|^2)|\dot{\phi}|_2^2 + \frac{\nu+\widetilde{\nu}}{8\gamma^2}|\partial_{x_n}\dot{\phi}|_2^2 \\ &+ \quad \frac{C_2}{2(\nu+\widetilde{\nu})}\left(|\phi|_2^2 + D_0[\phi]\right) + \frac{C_2}{\nu+\widetilde{\nu}}\left(|v|_2^2 + D_0[v] + \sum_{j+k=2}\left||\xi'|^j\partial_{x_n}^k v\right|_2^2\right) \\ &\le \frac{4(\nu+\widetilde{\nu})}{\gamma^4}|\xi'|^2|\phi|_2^2. \end{aligned}$$

Here

$$E_3[u] = E_2[u] + \frac{8n}{\eta}\widetilde{E}_0[u] \quad \text{with} \quad \eta = \frac{C_2}{16}.$$

Therefore, if $\frac{\nu+\widetilde{\nu}}{\gamma^2} \le \frac{\sqrt{C_2}}{4}$, we have

$$\begin{aligned} &\frac{d}{dt}E_3[u](t) + \frac{1}{\nu+\widetilde{\nu}}|\partial_{x_n}\phi|_2^2 + \left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi] \\ &+ \quad \left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)|\partial_t\psi|_2^2 + \frac{(\nu+\widetilde{\nu})}{n\gamma^4}(1+|\xi'|^2)|\dot{\phi}|_2^2 + \frac{\nu+\widetilde{\nu}}{8\gamma^2}|\partial_{x_n}\dot{\phi}|_2^2 \\ &+ \quad \frac{C_2}{4(\nu+\widetilde{\nu})}\left(|\phi|_2^2 + D_0[\phi]\right) + \frac{C_2}{\nu+\widetilde{\nu}}\left(|v|_2^2 + D_0[v] + \sum_{j+k=2}\left||\xi'|^j\partial_{x_n}^k v\right|_2^2\right) \\ &\le 0. \end{aligned}$$

The desired result follows from this inequality by setting $\nu_0 = 4$ and $\gamma_0^2 = \frac{4}{\sqrt{C_2}}$, since $E_3[u]$ is equivalent to $E_0[u] + D_0[u]$. This completes the proof. $\square$

To prove Proposition 6.1 for $r \le |\xi'| \le R$, we decompose $\phi$ into

$$\phi = \phi^{(0)} + \phi^{(1)},$$

where

$$\phi^{(0)} = \langle\phi\rangle.$$

Note that this is an orthogonal decomposition of $\phi$ in $L^2(0,1)$. We have

$$(\phi^{(0)}, \phi^{(1)}) = 0, \quad |\phi|_2^2 = |\phi^{(0)}|_2^2 + |\phi^{(1)}|_2^2,$$

and furthermore, by the Poincaré inequality,

$$|\phi^{(1)}|_2 \le |\partial_{x_n}\phi^{(1)}|_2 = |\partial_{x_n}\phi|_2.$$

**Proposition 6.9.** *There holds the estimate*

(6.28)
$$\frac{1}{\gamma^2}\frac{d}{dt}|\phi|_2^2 + \frac{\widetilde{c}_0(\xi')}{\nu}|\phi^{(0)}|_2^2 \le \frac{(1+c_0)}{\nu}|\partial_{x_n}\phi|_2^2 + \left(\nu + \frac{4}{c_0\nu}\right)|\partial_{x_n}\psi|_2^2 + \frac{4\widetilde{\nu}^2}{c_0\nu}|i\xi'\cdot\psi' + \partial_{x_n}\psi^n|_2^2.$$

*Here* $\widetilde{c}_0(\xi') = c_0\min\{1, |\xi'|^2\}$ *with some constant* $c_0 > 0$.

**Proof.** We define an operator $A$ on $L^2(0,1)$ with domain $D(A)$ by $A\varphi = -\partial_{x_n}^2\varphi$ for $\varphi \in D(A) = H^2(0,1)\cap H_0^1(0,1)$. By (6.2) we have

$$\nu(|\xi'|^2 + A)\psi' = -i\xi'\phi - h',$$

where

$$h' = \partial_t\psi' - i\widetilde{\nu}\xi'(i\xi'\cdot\psi' + \partial_{x_n}\psi^n) + i\xi_1 x_n\psi' + \psi^n\boldsymbol{e}_1'.$$

It follows that

$$\psi' = -\frac{i\xi'}{\nu}(|\xi'|^2 + A)^{-1}\phi - \frac{1}{\nu}(|\xi'|^2 + A)^{-1}h'. \tag{6.29}$$

Substituting (6.29) into (6.1) we have

$$\frac{1}{\gamma^2}\partial_t\phi + \frac{1}{\gamma^2}i\xi_1 x_n\phi + \frac{1}{\nu}|\xi'|^2(|\xi'|^2 + A)^{-1}\phi = h^0, \tag{6.30}$$

where

$$h^0 = -\partial_{x_n}\psi^n + \frac{i\xi'}{\nu}\cdot(|\xi'|^2 + A)^{-1}h'.$$

We take the inner product of (6.30) with $\phi$. Then the real part of the resulting identity yields

$$\frac{1}{2\gamma^2}\frac{d}{dt}|\phi|_2^2 + \frac{1}{\nu}|\xi'|^2|(|\xi'|^2 + A)^{-\frac{1}{2}}\phi|_2^2 = \mathrm{Re}\,(h^0, \phi). \tag{6.31}$$

Since

$$|(|\xi'|^2 + A)^{-1}f|_2 \le \frac{1}{|\xi'|^2 + 1}|f|_2,$$

we see that for any $\varepsilon_1 > 0$
(6.32)

$$\begin{aligned}
&\left|\left(\frac{i\xi'}{\nu}\cdot(|\xi'|^2+A)^{-1}h',\phi\right)\right| \\
&\quad\le \frac{1}{\nu}|\phi|_2\frac{|\xi'|}{|\xi'|^2+1}\{|\partial_t\psi'|_2+\widetilde{\nu}|\xi'||i\xi'\cdot\psi'+\partial_{x_n}\psi^n|_2+|\xi_1||\psi|_2+|\psi|_2\} \\
&\quad\le \frac{\varepsilon_1}{\nu}(1\wedge|\xi'|^2)|\phi^{(0)}|_2^2+\frac{\varepsilon_1}{\nu}|\partial_{x_n}\phi|_2^2 \\
&\qquad +\frac{1}{\varepsilon_1\nu}\{|\partial_t\psi|_2^2+2|\partial_{x_n}\psi|_2^2+\widetilde{\nu}^2|i\xi'\cdot\psi'+\partial_{x_n}\psi^n|_2^2\}.
\end{aligned}$$

Here $(1\wedge|\xi'|^2)=\min\{1,|\xi'|^2\}$. Since

$$|\xi'|^2(|\xi'|^2+A)^{-1}f\to f \text{ in } L^2(0,1) \text{ as } |\xi|\to\infty,$$

$$(|\xi'|^2+A)^{-1}=A^{-1}-A^{-2}|\xi'|^2\sum_{N=0}^{\infty}(-1)^N||\xi'|^{2N}A^{-N} \text{ for } |\xi'|<1,$$

and since $|(\mu+A)^{-\frac{1}{2}}\cdot 1|_2^2$ is continuous in $\mu\ge 0$, we see that there exists a constant $c_0>0$ such that

$$|\xi'|^2\left|(|\xi'|^2+A)^{-\frac{1}{2}}\cdot 1\right|_2^2\ge c_0(1\wedge|\xi'|^2). \tag{6.33}$$

By integration by parts and the Poincaré inequality, we have

$$|(\partial_{x_n}\psi^n,\phi)|\le\frac{1}{2\nu}|\partial_{x_n}\phi|_2^2+\frac{\nu}{2}|\partial_{x_n}\psi|_2^2. \tag{6.34}$$

The desired inequality now follows from (6.31)–(6.34) by taking $\varepsilon_1=c_0/2$. This completes the proof. □

We are now in a position to prove Proposition 6.2 for $r\le|\xi'|\le R$.

**Proposition 6.10.** *There exist constants $\nu_0>0$ and $\gamma_0>0$ such that if $\nu\ge\nu_0$ and $(\nu+\widetilde{\nu})/\gamma^2\le 1/\gamma_0^2$, then for any fixed $r>0$ the estimate* (6.6) *in Poroposition* 6.1 *holds provided that $r\le|\xi'|\le R$.*

**Proof.** Let $r\le|\xi'|\le R$ and set $C_3=\min\{1/4,1/2(1+c_0),c_0/8\}$. We see from (6.18) + $\frac{C_3\nu}{\nu+\widetilde{\nu}}\times$ (6.28) that

$$
\begin{aligned}
&\frac{d}{dt}E_4[u](t)+\frac{1}{2(\nu+\widetilde{\nu})}|\partial_{x_n}\phi|_2^2+\frac{1}{2}\left(1+\frac{1}{\nu+\widetilde{\nu}}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi]\\
&+\left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)|\partial_t\psi|_2^2+\frac{C_3\widetilde{c}_0(\xi')}{\nu+\widetilde{\nu}}|\phi^{(0)}|_2^2\\
&\le 4\left(\frac{\nu+\widetilde{\nu}}{\gamma^4}+\frac{2\eta}{\nu+\widetilde{\nu}}\right)|\xi'|^2|\phi|_2^2+\frac{8n}{\eta}\widetilde{D}_0[\psi].
\end{aligned}
\tag{6.35}
$$

Here

$$E_4[u]=E_2[u]+\frac{1}{\gamma^2}\frac{C_3\nu}{\nu+\widetilde{\nu}}|\phi|_2^2.$$

We take $\eta=\frac{c_0C_3}{32}(1\wedge 1/|\xi'|^2)$ and add $\frac{8n}{\eta}(1\vee|\xi'|^2)\times$ (6.8) to (6.35). Here $(1\vee|\xi'|^2)=\max\{1,|\xi'|^2\}$. It follows that

$$
\begin{aligned}
&\frac{d}{dt}E_5[u](t)+\frac{1}{2(\nu+\widetilde{\nu})}|\partial_{x_n}\phi|_2^2+\frac{1}{2}\left(1+\frac{1}{\nu+\widetilde{\nu}}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi]\\
&+\left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)|\partial_t\psi|_2^2+\frac{C_3\widetilde{c}_0(\xi')}{\nu+\widetilde{\nu}}|\phi^{(0)}|_2^2\\
&\le 4\left(\frac{\nu+\widetilde{\nu}}{\gamma^4}+\frac{2\eta}{\nu+\widetilde{\nu}}\right)|\xi'|^2|\phi|_2^2
\end{aligned}
$$

with $\eta=\frac{c_0C_3}{32}(1\wedge 1/|\xi'|^2)$ and

$$E_5[u]=E_4[u]+\frac{8n}{\eta}(1\vee|\xi'|^2)\widetilde{E}_0[u].$$

Therefore, if $\nu\ge 4$ and $\frac{(\nu+\widetilde{\nu})^2}{\gamma^4}\le\min\left\{\frac{1}{32},\frac{1}{32R^2},\frac{c_0C_3}{16}\right\}$, we have

$$
\begin{aligned}
&\frac{d}{dt}E_5[u](t)+\frac{1}{4(\nu+\widetilde{\nu})}|\partial_{x_n}\phi|_2^2+\frac{1}{2}\left(1+\frac{1}{\nu+\widetilde{\nu}}\right)(1+|\xi'|^2)\widetilde{D}_0[\psi]\\
&+\left(1+\frac{1}{2(\nu+\widetilde{\nu})}\right)|\partial_t\psi|_2^2+\frac{C_3\widetilde{c}_0(\xi')}{2(\nu+\widetilde{\nu})}|\phi^{(0)}|_2^2\le 0,
\end{aligned}
$$

from which the desired estimate follows, since $E_5[u]$ is equivalent to $E_0[u]+D_0[u]$ and $\frac{1}{4(\nu+\widetilde{\nu})}|\partial_{x_n}\phi|_2^2+\frac{C_3\widetilde{c}_0(\xi')}{2(\nu+\widetilde{\nu})}|\phi^{(0)}|_2^2\ge\frac{C_5}{\nu+\widetilde{\nu}}(D_0[\phi]+|\phi|_2^2)$ for some constant $C_5=C_5(r,R)>0$. This completes the proof. □

We next prove Proposition 6.2.

**Proposition 6.11.** *There hold the estimates*

$$(1+|\xi'|^2)\widetilde{E}_0[u](t)+\nu\int_0^t(1+|\xi'|^2)D_0[\psi]\,ds\le C(1+|\xi'|^2)\widetilde{E}_0[u_0]\tag{6.36}$$

*and*

$$|\partial_{x_n}\phi(t)|_2^2 \le C\left\{(1+|\xi'|^2)E_0[u_0] + |\partial_{x_n}\phi_0|_2^2\right\} \tag{6.37}$$

*for* $0 \le t \le 1$.

**Proof.** We see from (6.12) that

$$\begin{aligned}(1+|\xi'|^2)\widetilde{E}_0[u](t) + \nu\int_0^t (1+|\xi'|^2)D_0[\psi]\,ds \\ \le (1+|\xi'|^2)\widetilde{E}_0[u_0] + \int_0^t (1+|\xi'|^2)|\psi|_2^2\,ds\end{aligned} \tag{6.38}$$

By Gronwall's inequality we have

$$(1+|\xi'|^2)|\psi(t)|_2^2 \le e^t(1+|\xi'|^2)\widetilde{E}_0[u_0].$$

This, together with (6.38), yields the estimate (6.36) for $0 \le t \le 1$.

Let us next prove the estimate (6.37). We set $a_0 = \frac{\gamma^2}{\nu+\tilde{\nu}}$. By (6.21) we have

$$\partial_t\partial_{x_n}\phi + (i\xi_1 x_n + a_0)\partial_{x_n}\phi = -i\xi_1\phi - a_0\partial_t\psi^n - a_0\widetilde{M}[\psi].$$

It then follows that

$$\begin{aligned}\partial_{x_n}\phi(x_n,t) &= e^{-(i\xi_1 x_n + a_0)t}\partial_{x_n}\phi_0(x_n) - i\xi_1\int_0^t e^{-(i\xi_1 x_n + a_0)(t-s)}\phi(x_n,s)\,ds \\ &\quad -a_0\int_0^t e^{-(i\xi_1 x_n + a_0)(t-s)}\partial_s\psi^n(x_n,s)\,ds \\ &\quad -a_0\int_0^t e^{-(i\xi_1 x_n + a_0)(t-s)}\widetilde{M}[\psi](x_n,s)\,ds.\end{aligned} \tag{6.39}$$

The second and forth terms on the right of (6.39) is bounded by

$$\begin{aligned}\int_0^t e^{-a_0(t-s)}\left\{|\xi'||\phi(s)|_2 + a_0|\xi'|\sqrt{D_0[\psi](s)}\right\}\,ds \\ \le C\left[\sup_{0\le s\le t}|\xi'|^2|\phi(s)|_2^2 + \int_0^t |\xi'|^2 D_0[\psi](s)\,ds\right]^{\frac{1}{2}}.\end{aligned} \tag{6.40}$$

As for the third term on the right of (6.39), by integration by parts, we see that it is written as

$$\begin{aligned}-a_0\left[e^{-(i\xi_1 x_n + a_0)(t-s)}\psi^n(x_n,s)\right]_{s=0}^{s=t} \\ +a_0(i\xi_1 x_n + a_0)\int_0^t e^{-(i\xi_1 x_n + a_0)(t-s)}\psi^n(x_n,s)\,ds.\end{aligned}$$

Therefore, it is estimated by

$$a_0|\psi(t)|_2 + a_0 e^{-a_0 t}|\psi_0|_2 + C\left(\int_0^t D_0[\psi]\,ds\right)^{\frac{1}{2}}. \tag{6.41}$$

The desired estimate (6.37) now follows from (6.36) and (6.39)–(6.41). This completes the proof. □

**Proof of Proposition 6.2.** We write (6.2) and (6.3) as

$$\partial_t \psi + \widehat{T}_{\xi'}\psi = -\widehat{B}_{\xi'}\phi,$$

where Here $\widehat{T}_{\xi'}$ is the operator on $L^2(0,1)$ of the form

$$\widehat{T}_{\xi'} = \begin{pmatrix} \{\nu(|\xi'|^2 - \partial_{x_n}^2) + i\xi_1 x_n\}I_{n-1} + \widetilde{\nu}\xi'^T\xi' & -i\widetilde{\nu}\xi'\partial_{x_n} + \boldsymbol{e}_1' \\ -i\widetilde{\nu}^T\xi'\partial_{x_n} & \nu(|\xi'|^2 - \partial_{x_n}^2) - \widetilde{\nu}\partial_{x_n}^2 + i\xi_1 x_n \end{pmatrix}$$

with domain $D(\widehat{T}_{\xi'}) = H^2(0,1) \cap H_0^1(0,1)$, and

$$\widehat{B}_{\xi'} = \begin{pmatrix} i\xi' \\ \partial_{x_n} \end{pmatrix}.$$

Then $\psi$ is written as

$$\psi(t) = e^{-t\widehat{T}_{\xi'}}\psi_0 - \int_0^t e^{-(t-s)\widehat{T}_{\xi'}}\widehat{B}_{\xi'}\phi(s)\,ds. \tag{6.42}$$

Using the estimates

$$\mathrm{Re}\,(\widehat{T}_{\xi'}\psi, \psi) \geq \widetilde{D}_0[\psi] - |\psi|_2^2$$

and

$$\left|\mathrm{Re}\,\left\{(\widehat{T}_{\xi'}\psi_1, \psi_2) - (\psi_2, \widehat{T}_{\xi'}^*\psi_1)\right\}\right| \leq C\sqrt{D_0[\psi_1]}|\psi_2|_2,$$

one can see that

$$|\partial_{x_n}^l e^{-t\widehat{T}_{\xi'}}\psi_0|_2^2 \leq Ct^{-\frac{l}{2}}|\psi_0|_2^2 \quad (l = 0, 1) \tag{6.43}$$

for $0 < t \leq 1$.

By (6.36) and (6.37) we have

$$\begin{aligned} |\widehat{B}_{\xi'}\phi(t)|_2 &\leq C\{|\xi'||\phi(t)|_2 + |\partial_{x_n}\phi(t)|_2\} \\ &\leq C\{(1 + |\xi'|)|u_0|_2 + |\partial_{x_n}\phi_0|_2\}. \end{aligned} \tag{6.44}$$

It follows from (6.42)–(6.44) that

$$\begin{aligned} |\partial_{x_n}\psi(t)|_2 &\leq Ct^{-\frac{1}{2}}|\psi_0|_2 + C\int_0^t (t-s)^{-\frac{1}{2}}|\widehat{B}_{\xi'}\phi(s)|_2\,ds \\ &\leq Ct^{-\frac{1}{2}}|\psi_0|_2 + C\{(1+|\xi'|)|u_0|_2 + |\partial_{x_n}\phi_0|_2\} \end{aligned}$$

for $0 < t \leq 1$. This, together with (6.36) and (6.37), gives the desired estimate of Proposition 6.2. This completes the proof. □

## 7. The Nonlinear Problem

In this section we prove Thoerem 3.3. We here give an outline of the proof only, since the argument is similar to those in [4, 6, 11, 12].

Theorem 3.3 (i) is proved by showing the local existence of strong solutions and the $H^s$-energy a priori estimate:

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x\phi\|_{H^{s-1}}^2 + \|\partial_x\psi\|_{H^s}^2\,d\tau \leq C\|u_0\|_{H^s}^2 \tag{7.1}$$

uniformly for $t \geq 0$ with $s \geq [n/2]+1$. The local existence is proved by applying the local $H^s$-solvability result in [5]. The $H^s$-energy a priori estimate is obtained by the energy method by Matsumura and Nishida [12]. The conditions $\nu \geq c_1$ and $(2\nu+\nu')/\gamma^2 \leq c_2$ appear in a similar manner to the proof of Proposition 6.8. The decay property $\|u(t)\|_\infty \to 0$ as $t \to \infty$ can be proved in the same way as in [6].

The proof of Theorem 3.3 (ii) is based on the $H^s$-energy bound (7.1) and Theorem 4.1. We will give the necessary estimates for the proof.

We write $u(t)$ as

$$u(t) = \mathcal{U}(t)u_0 + \int_0^t \mathcal{U}(t-\tau)f(\tau)\,d\tau,$$

where $f = {}^T(\gamma^2 f^0, g)$ with $f^0$ and $g$ defined in (2.5) and (2.6). We decompose $f$ into

$$f = Q_0 f + \widetilde{Q}f \quad \text{and} \quad \widetilde{Q}f = f_1 + f_2,$$

where $f_j = {}^T(0, g_j)$, $j = 1, 2$, with

$$\begin{aligned} g_1 &= -\psi\cdot\nabla\psi - \nu\nabla\left(\frac{\phi}{\gamma^2+\phi}\right) : \nabla\psi - \widetilde{\nu}\mathrm{div}\,\psi\nabla\left(\frac{\phi}{\gamma^2+\phi}\right) \\ &\quad + \frac{(P_2(\gamma,\phi)-1)\phi}{\gamma^2+\phi}\nabla\phi, \end{aligned}$$

$$g_2 = -\nu \sum_{j=1}^{n} \partial_{x_j} \left( \frac{\phi}{\gamma^2+\phi} \partial_{x_j} \psi \right) - \widetilde{\nu} \nabla \left( \frac{\phi}{\gamma^2+\phi} \operatorname{div} \psi \right) \equiv \operatorname{div} \widetilde{g}_2.$$

We define $M(t)$ by

$$M(t) = \sup_{0 \le \tau \le t} \sum_{l=0}^{1} (1+\tau)^{\frac{n-1}{4}+\frac{l}{2}} \|\partial_x^l u(\tau)\|_2.$$

By applying the Gagliardo-Nirenberg inequality, Poincaré inequality and (7.1) one can obtain the following estimates. In what follows we will write $s_0 = [n/2] + 1$.

**Lemma 7.1.** *Let $s \ge s_0 + 1$ and assume that $\|u_0\|_{H^s} \le \varepsilon_0$. Then the following inequalities hold.*

(i) $$\|\phi\psi\|_1 + \|Q_0 f\|_1 \le C(1+t)^{-\frac{n-1}{2}-\frac{1}{2}} M(t)^2.$$

(ii) $$\|g_1\|_1 \le C(1+t)^{-\frac{n-1}{2}-\frac{1}{2}} M(t)^2.$$

(iii) $$\|\widetilde{g}_2\|_1 \le C(1+t)^{-\frac{n-1}{2}-\frac{1}{2}} M(t)^2.$$

(iv)
$$\begin{aligned}
\|Q_0 f\|_{H^1} &\le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \|u\|_{H^{s_0+1}} M(t) \quad (n \ge 4), \\
\|Q_0 f\|_{H^1} &\le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \{\|u\|_{H^{s_0+1}} M(t) + \|\partial_x \psi\|_{H^{s_0}}^{\frac{3}{4}} M(t)^{\frac{5}{4}}\} \\
&\quad (n \ge 3), \\
\|Q_0 f\|_{H^1} &\le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \{\|u\|_{H^{s_0+1}} M(t) + \|\partial_x \psi\|_{H^{s_0}}^{\frac{1}{2}} M(t)^{\frac{3}{2}}\} \\
&\quad (n \ge 2).
\end{aligned}$$

(v)
$$\begin{aligned}
\|\widetilde{Q} f\|_{H^1} &\le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \{\|u\|_{H^{s_0+1}} + \|u\|_{H^{s_0+1}}^{1-\alpha} \|\partial_x \psi\|_{H^{s_0+1}}^{\alpha}\} M(t) \\
&\quad (n \ge 4), \\
\|\widetilde{Q} f\|_{H^1} &\le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \big( \{\|u\|_{H^{s_0+1}} + \|u\|_{H^{s_0+1}}^{1-\alpha} \|\partial_x \psi\|_{H^{s_0+1}}^{\alpha}\} M(t) \\
&\qquad + \|\partial_x \psi\|_{H^{s_0}}^{\frac{3}{4}} M(t)^{\frac{5}{4}} + \|\partial_x \psi\|_{H^{s_0+1}}^{\frac{4}{5}} M(t)^{\frac{6}{5}} \big) \quad (n \ge 3), \\
\|\widetilde{Q} f\|_{H^1} &\le C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}} \big( \{\|u\|_{H^{s_0+1}} + \|u\|_{H^{s_0+1}}^{1-\alpha} \|\partial_x \psi\|_{H^{s_0+1}}^{\alpha}\} M(t) \\
&\qquad + \|\partial_x \psi\|_{H^{s_0}}^{\frac{1}{2}} M(t)^{\frac{3}{2}} + \|\partial_x \psi\|_{H^{s_0+1}}^{\frac{2}{3}} M(t)^{\frac{4}{3}} \big) \quad (n \ge 2).
\end{aligned}$$

*Here $\alpha$ is some constant with $\alpha \in (0,1)$.*

Let $\mathcal{U}(t) = \mathcal{U}^{(0)}(t) + \mathcal{U}^{(\infty)}(t)$ be the decomposition of $\mathcal{U}(t)$ given in Theorem 4.1. Then we write $u(t)$ as

$$u(t) = \mathcal{U}(t)u_0 + I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \int_0^{t-1} \mathcal{U}(t-\tau)Q_0 f(\tau)\,d\tau, \quad I_2 = \int_0^{t-1} \mathcal{U}^{(0)}(t-\tau)f_1(\tau)\,d\tau,$$

$$I_3 = \int_0^{t-1} \mathcal{U}^{(0)}(t-\tau)f_2(\tau)\,d\tau, \quad I_4 = \int_0^{t-1} \mathcal{U}^{(\infty)}(t-\tau)\widetilde{Q}f(\tau)\,d\tau$$

and

$$I_5 = \int_{t-1}^{t} \mathcal{U}(t-\tau)f(\tau)\,d\tau.$$

Then applying Theorem 3.1, Theorem 4.1 and Lemma 7.1 and using (7.1) one can obtain the following estimates.

**Proposition 7.2.** *Let $\varepsilon_1 \in (0, \varepsilon_0]$. Assume that $\|u_0\|_{H^s} + \|u_0\|_1 \le \varepsilon_1$. Then the following estimates hold:*

(i) $M(t) \le C\{\varepsilon_1 + \varepsilon_1 M(t) + M(t)^2\}$ $(n \ge 4)$,

$M(t) \le C\{\varepsilon_1 + \varepsilon_1 M(t) + M(t)^2 + \varepsilon_1^{\frac{3}{4}} M(t)^{\frac{5}{4}} + \varepsilon_1^{\frac{4}{5}} M(t)^{\frac{6}{5}}\}$ $(n \ge 3)$,

$M(t) \le C\{\varepsilon_1 + \varepsilon_1 M(t) + M(t)^2 + \varepsilon_1^{\frac{1}{2}} M(t)^{\frac{3}{2}} + \varepsilon_1^{\frac{2}{3}} M(t)^{\frac{4}{3}}\}$ $(n \ge 2)$.

(ii) *Furthermore, if $M(t) \le \widehat{C}$ for all $t \ge 0$ with some constant $\widehat{C} > 0$, then*

$$\|u(t) - \mathcal{U}(t)u_0\|_2 \le C\widehat{C}(1+t)^{\frac{n-1}{4}-\frac{1}{2}}L(t)$$

*for all $t \ge 0$.*

It follows from Proposition 7.2 (i) that $M(t) \le C\varepsilon_1$ for all $t \ge 0$, if $\varepsilon_1 > 0$ is sufficiently small. This yields the decay estimate for $\|u(t)\|_2$ in Theorem 3.3 (ii). Proposition 7.2 (ii), together with Theorem 3.2, then gives the estimate for $\|u(t) - G_t *_{x'} \Pi^{(0)}u_0\|_2$ in Theorem 3.3 (ii).

## Appendix

In this section we will give an outline of the proof of Lemma 6.6. For this purpose we first consider the halp-space problem:

$$\text{(A.1)}\qquad \begin{cases} i\xi'\cdot v' + \partial_{x_n} v^n = F \quad (x_n \in (0,\infty)),\\ \lambda v' + (|\xi'|^2 - \partial_{x_n}^2) v' + i\xi' p = F' \quad (x_n \in (0,\infty)),\\ \lambda v^n + (|\xi'|^2 - \partial_{x_n}^2) v^n + \partial_{x_n} p = F^n \quad (x_n \in (0,\infty)),\\ v|_{x_n=0} = 0. \end{cases}$$

Here $\lambda \in \mathbf{C}$ is a parameter. We define an operator $\widehat{S}_{\xi'}$ in $H^1(0,\infty)\times L^2(0,\infty)$ with domain $D(\widehat{S}_{\xi'})$ by

$$\widehat{S}_{\xi'} u = \begin{pmatrix} 0 & i\,{}^T\xi' & \partial_{x_n} \\ i\xi' & (|\xi'|^2 - \partial_{x_n}^2) I_{n-1} & 0 \\ \partial_{x_n} & 0 & |\xi'|^2 - \partial_{x_n}^2 \end{pmatrix} \begin{pmatrix} p \\ v' \\ v^n \end{pmatrix}$$

for $u = {}^T(p, v', v^n) \in D(\widehat{S}_{\xi'}) = H^1(0,1)\times(H^2(0,1)\cap H_0^1(0,1))$. Then problem (A.1) is written as

$$\text{(A.1)}' \qquad \widehat{S}_{\xi'}(\lambda) u = F,$$

where $u = {}^T(p, v', v^n) \in D(\widehat{S}_{\xi'})$, $F = {}^T(F^0, F', F^n)$ and

$$\widehat{S}_{\xi'}(\lambda) = \lambda \widetilde{Q} + \widehat{S}_{\xi'}.$$

Before proceeding further, we introduce some notations. In the following we will denote the norms of $L^2(0,1)$ and $L^2(0,\infty)$ by the same symbol $|\cdot|_2$ if no confusions occur.

Let $\lambda \in \mathbf{C}\backslash(-\infty, -|\xi'|^2]$, $\lambda \neq 0$ and $\xi' \neq 0$. We denote the principal branch of the square root of $\lambda + |\xi'|^2$ by $\mu_1 = \mu_1(\lambda, \xi')$, i.e.,

$$\mu_1 = \mu_1(\lambda, \xi') = \sqrt{\lambda + |\xi'|^2}$$

with $\mathrm{Re}\,\mu_1 > 0$. For a complex number $\mu$, we define functions $g_\mu^{(\pm)}(x_n, y_n)$ and $h_\mu(x_n)$ by

$$g_\mu^{(\pm)}(x_n, y_n) = \frac{1}{2\mu}\left\{e^{-\mu|x_n - y_n|} \pm e^{-\mu(x_n + y_n)}\right\}$$

and

$$h_\mu(x_n) = \frac{1}{\mu} e^{-\mu x_n}.$$

We also introduce functions $g^{(\pm)}_{\mu_1,|\xi'|}(x_n, y_n)$ and $h_{\mu_1,|\xi'|}(x_n)$ which are defined by

$$g^{(\pm)}_{\mu_1,|\xi'|}(x_n, y_n) = g^{(\pm)}_{\mu_1}(x_n, y_n) - g^{(\pm)}_{|\xi'|}(x_n, y_n)$$

and

$$h_{\mu_1,|\xi'|}(x_n) = h_{\mu_1}(x_n) - h_{|\xi'|}(x_n).$$

We denote by $\delta(x_n)$ the Dirac delta function. As in the case of the interval $(0, 1)$, for a function $K(x_n, y_n)$ on $(0,\infty) \times (0,\infty)$ we will denote by $Kf$ the integral operator $\int_0^\infty K(x_n, y_n)f(y_n)\, dy_n$.

Then as in [2, 7, 8], we will establish estimates for solutions of (A.1)′ by using a solution formula for (A.1)′. Following the argument [2, Section 3] (cf., [7, Appendix]), one can show that the solution $u = S_{\xi'}(\lambda)^{-1}F$ of (A.1)′ is written as

$$u(x_n) = \left[\widehat{S}_{\xi'}(\lambda)^{-1}F\right](x_n) = \int_0^a \widehat{R}(\lambda, \xi', x_n, y_n)f(y_n)\, dy_n,$$

where

$$\widehat{R}(\lambda, \xi', x_n, y_n) = \widehat{G}(\lambda, \xi', x_n, y_n) + \widehat{H}(\lambda, \xi', x_n, y_n)$$

with $\widehat{G}(\lambda, \xi', x_n, y_n)$ being an $(n + 1) \times (n + 1)$ matrix of the form

$$\begin{aligned}
&\widehat{G}(\lambda, \xi', x_n, y_n)\\
&\quad= \delta(x_n - y_n)Q_0\\
&\qquad+ \begin{pmatrix} g^{(+)}_{|\xi'|}(x_n, y_n) & -i^T\xi' g^{(+)}_{|\xi'|}(x_n, y_n) & -\partial_{x_n} g^{(-)}_{|\xi'|}(x_n, y_n) \\ -i\xi' g^{(+)}_{|\xi'|}(x_n, y_n) & 0 & 0 \\ -\partial_{x_n} g^{(+)}_{|\xi'|}(x_n, y_n) & 0 & 0 \end{pmatrix}\\
&\qquad+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & g^{(-)}_{\mu_1}(x_n, y_n)I_{n-1} & 0 \\ 0 & 0 & g^{(-)}_{\mu_1}(x_n, y_n) \end{pmatrix}\\
&\qquad+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\xi'^T\xi'}{\lambda} g^{(+)}_{\mu_1,|\xi'|}(x_n, y_n) & -\frac{i\xi'}{\lambda}\partial_{x_n} g^{(-)}_{\mu_1,|\xi'|}(x_n, y_n) \\ 0 & -\frac{i\xi'}{\lambda}\partial_{x_n} g^{(+)}_{\mu_1,|\xi'|}(x_n, y_n) & -\frac{1}{\lambda}\partial^2_{x_n} g^{(-)}_{\mu_1,|\xi'|}(x_n, y_n) \end{pmatrix},
\end{aligned}$$

and $\widehat{H}(\lambda, \xi', x_n, y_n)$ being defined by

$$
\begin{aligned}
&\widehat{H}(\lambda, \xi', x_n, y_n) \\
&= \begin{pmatrix} 0 & i^T\xi' h_{|\xi'|}(x_n) e^{-\mu_1 y_n} & 0 \\ 0 & -\frac{\xi'^T\xi'}{\lambda} h_{\mu_1,|\xi'|}(x_n) e^{-\mu_1 y_n} & 0 \\ 0 & \frac{i^T\xi'}{\lambda} \partial_{x_n} h_{\mu_1,|\xi'|}(x_n) e^{-\mu_1 y_n} & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & 0 \\ h_{\mu_1}(x_n)\beta_0(y_n) & h_{\mu_1}(x_n)B(y_n) & h_{\mu_1}(x_n)\beta_n(y_n) \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} -i^T\xi' h_{|\xi'|}(x_n)\beta_0(y_n) & -i^T\xi' h_{|\xi'|}(x_n)B(y_n) & -i^T\xi' h_{|\xi'|}(x_n)\beta_n(y_n) \\ \frac{\xi'^T\xi'}{\lambda} h_{\mu_1,|\xi'|}(x_n)\beta_0(y_n) & \frac{\xi'^T\xi'}{\lambda} h_{\mu_1,|\xi'|}(x_n)B(y_n) & \frac{\xi'^T\xi'}{\lambda} h_{\mu_1,|\xi'|}(x_n)\beta_n(y_n) \\ -\frac{i^T\xi'}{\lambda} h_{\mu_1,|\xi'|}(x_n)\beta_0(y_n) & -\frac{i'^T\xi'}{\lambda} \partial_{x_n} h_{\mu_1,|\xi'|}(x_n)B(y_n) & -\frac{i^T\xi'}{\lambda} \partial_{x_n} h_{\mu_1,|\xi'|}(x_n)\beta_n(y_n) \end{pmatrix}.
\end{aligned}
$$

Here $\mu_1 = \mu_1(\lambda, \xi')$; and

$$\beta_0(y_n) = \beta_0(\lambda, \xi', y_n) = \frac{i\lambda\xi'}{|\xi'|(\mu_1 - |\xi'|)} e^{-|\xi'| y_n},$$

$$B(y_n) = B(\lambda, \xi', y_n) = -\frac{\xi'^T\xi'}{|\xi'|(\mu_1 - |\xi'|)} (e^{-\mu_1 y_n} - e^{-|\xi'| y_n}),$$

and

$$\beta_n(y_n) = \beta_n(\lambda, \xi', y_n) = \frac{i|\xi'|\xi'}{|\xi'|(\mu_1 - |\xi'|)} (e^{-\mu_1 y_n} - e^{-|\xi'| y_n}).$$

To estimate $\widehat{S}_{\xi'}(\lambda)^{-1}F$, we prepare some lemmas.

**Lemma A.1.** *For $\mu_1 = \mu_1(\lambda, \xi')$, there hold the following estimates uniformly in $\lambda \in \Sigma(-\frac{|\xi'|^2}{2}, \frac{3}{4}\pi)$, $\xi' \in \mathbf{R}^{n-1}$ with $|\xi'| \geq 1$ and $x_n > 0$:*

(i) $C^{-1}(|\lambda| + |\xi'|^2)^{\frac{1}{2}} \leq \operatorname{Re}\mu_1 \leq C(|\lambda| + |\xi'|^2)^{\frac{1}{2}}$,

(ii) $C^{-1}(|\lambda| + |\xi'|^2)^{\frac{1}{2}} \leq |\mu_1| \leq C(|\lambda| + |\xi'|^2)^{\frac{1}{2}}$,

(iii) $C^{-1}\dfrac{|\lambda|}{(|\lambda| + |\xi'|^2)^{\frac{1}{2}}} \leq |\mu_1 - |\xi'|| \leq C\dfrac{|\lambda|}{(|\lambda| + |\xi'|^2)^{\frac{1}{2}}}$,

(iv) $\displaystyle \left|\frac{1}{\mu_1}-\frac{1}{|\xi'|}\right| \le C\frac{|\lambda|}{|\xi'|(|\lambda|+|\xi'|^2)}$,

(v) $\displaystyle \left|e^{-\mu_1 x_n}-e^{-|\xi'|x_n}\right| \le C|\mu_1-|\xi'||\left(\frac{1}{|\xi'|}e^{-\frac{1}{2}\mathrm{Re}\,\mu_1 x_n}+\frac{1}{\mathrm{Re}\,\mu_1}e^{-\frac{1}{2}|\xi'|x_n}\right)$.

**Proof.** An elementary observation gives the inequalities (i) and (ii). The inequalities (iii) and (iv) can be obtained by writing

$$\mu_1-|\xi'|=\frac{\lambda}{\mu_1+|\xi'|}.$$

As for (v), since $\mathrm{Re}\,\mu_1>0$ and $x_n>0$, we have

$$\begin{aligned}
\left|e^{-\mu_1 x_n}-e^{-|\xi'|x_n}\right| &= \left|(|\xi'|-\mu_1)x_n\int_0^1 e^{-\theta\mu_1 x_n-(1-\theta)|\xi'|x_n}\,d\theta\right| \\
&\le |\mu_1-|\xi'||x_n\left(\int_0^{\frac{1}{2}}+\int_{\frac{1}{2}}^1\right)e^{-\theta\mathrm{Re}\,\mu_1 x_n-(1-\theta)|\xi'|x_n}\,d\theta \\
&\le C|\mu_1-|\xi'||\left(\frac{1}{\mathrm{Re}\,\mu_1}e^{-\frac{1}{2}|\xi'|x_n}+\frac{1}{|\xi'|}e^{-\frac{1}{2}\mathrm{Re}\,\mu_1 x_n}\right).
\end{aligned}$$

This completes the proof. □

We will employ the following well-known inequality.

**Lemma A.2.** *Let $K(x_n,y_n)$ be a function on $(0,\infty)\times(0,\infty)$ satisfying*

$$\sup_{y_n>0}\int_0^\infty |K(x_n,y_n)|\,dx_n \le M_1 \quad\text{and}\quad \sup_{x_n>0}\int_0^\infty |K(x_n,y_n)|\,dy_n \le M_2$$

*for some constants $M_1>0$ and $M_2>0$. Set*

$$KF(x_n)=\int_0^\infty K(x_n,y_n)F(y_n)\,dy_n.$$

*Then it holds*

$$|KF|_2 \le M_1^{\frac{1}{2}}M_2^{\frac{1}{2}}|F|_2.$$

Using Lemmas A.1 and A.2 one can obtain the following estimates for $\widehat{G}F$ and $\widehat{K}F$.

**Lemma A.3.** *There hold the following estimates uniformly in* $\lambda \in \Sigma(-\frac{|\xi'|^2}{2}, \frac{3}{4}\pi) \cap \{\lambda;\, |\lambda| \le 1,\, \lambda \neq 0\}$ *and* $\xi' \in \mathbf{R}^{n-1}$ *with* $|\xi'| \ge 1$*:*

(i)
$$\begin{aligned} |\xi'|^k|\partial_{x_n}^l Q_0\widehat{G}F|_2 \ &\le \ C\big\{|\xi'|^k|\partial_{x_n}^l Q_0F|_2 + |\xi'|^{k+l-2}|Q_0F|_2 \\ &\qquad + |\xi'|^{k+l-1}|\widetilde{Q}F|_2\big\} \quad (0 \le k+l \le 1), \end{aligned}$$

(ii)
$$\begin{aligned} |\xi'|^k|\partial_{x_n}^l \widetilde{Q}\widehat{G}Q_0F|_2 \ &\le \ C|\xi'|^{k+l-1}|\partial_{x_n}^{(l-1)_+}Q_0F|_2 \quad (0 \le k+l \le 2,\ l \le 1), \\ |\partial_{x_n}^2 \widetilde{Q}\widehat{G}Q_0F|_2 \ &\le \ C|\partial_{x_n}Q_0F|_2, \end{aligned}$$

(iii)
$$|\xi'|^k|\partial_{x_n}^l \widetilde{Q}\widehat{G}\widetilde{Q}F|_2 \le C|\xi'|^{k+l-2}|\widetilde{Q}F|_2 \quad (0 \le k+l \le 2),$$

(iv)
$$\begin{aligned} |\xi'|^k|\partial_{x_n}^l Q_0\widehat{H}F|_2 \ &\le \ C\big\{|\xi'|^{k+l}|Q_0F|_2 + |\xi'|^{k+l-1}|\widetilde{Q}F|_2\big\} \\ &\qquad (0 \le k+l \le 1), \end{aligned}$$

(v)
$$|\xi'|^k|\partial_{x_n}^l \widetilde{Q}\widehat{H}Q_0F|_2 \le C|\xi'|^{k+l-1}|Q_0F|_2 \quad (0 \le k+l \le 2,),$$

(vi)
$$|\xi'|^k|\partial_{x_n}^l \widetilde{Q}\widehat{H}\widetilde{Q}F|_2 \le C|\xi'|^{k+l-2}|\widetilde{Q}F|_2 \quad (0 \le k+l \le 2).$$

**Proof.** We first make two observations. By integration by parts we see that $\partial_{x_n} g_{|\xi'|}^{(+)}F^0 = g_{|\xi'|}^{(-)}[\partial_{x_n}F^0]$. For an integer $l$ we set $g_\mu^{(l,\pm)}F = g_\mu^{(\pm)}F$ if $l$ is even and

$$g_\mu^{(l,\pm)}F = -\frac{1}{2\mu}\int_0^\infty \{\mathrm{sgn}\,(x_n - y_n)e^{-\mu|x_n-y_n|} \pm e^{-\mu(x_n+y_n)}\}F(y_n)\,dy_n$$

if $l$ is odd. Then for $l = 1, 2, 3, 4,$ we have

$$\partial_{x_n}^l g_{\mu_1,|\xi'|}^{(\pm)}F = (\mu_1^l g_{\mu_1}^{(l,\pm)} - |\xi'|^l g_{|\xi'|}^{(l,\pm)})F - \delta_{4l}(\mu_1^2 - |\xi'|^2)F.$$

The desired inequalities in Lemma A.3 then follow from Lemmas A.1 and A.2. We omit the details. (Cf., [8, Section 8] and [2, Sections 4–5].) □

We now derive the estimate for the solution $u$ of (A.1)$'$ with $\lambda = 0$.

**Lemma A.4.** *If $|\xi'| \geq 1$, then there exists a unique solution $u = \widehat{S}_{\xi'}^{-1} F = {}^T(p, v)$ with $v = (v', v^n)$ and $u$ satisfies the estimate*

$$
\text{(A.2)} \qquad \begin{aligned} &|p|_2^2 + D_0[p] + |v|_2^2 + D_0[v] + \sum_{j+k=2} \left| |\xi'|^j \partial_{x_n}^k v \right|_2^2 \\ &\leq C \left\{ |F^0|_2^2 + D_0[F^0] + |F'|_2^2 + |F^n|_2^2 \right\}. \end{aligned}
$$

**Proof.** By Lemma A.3 we see that if $\lambda \in \Sigma(-\frac{|\xi'|^2}{2}, \frac{3}{4}\pi) \cap \{\lambda; |\lambda| \leq 1, \ \lambda \neq 0\}$ and $|\xi'| \geq 1$, then $\widehat{S}_{\xi'}(\lambda)$ has a bounded inverse in $H^1(0,\infty) \times L^2(0,\infty)$ and $u = \widehat{S}_{\xi'}(\lambda)^{-1} F$ satisfies the estimate

$$
\text{(A.3)} \qquad \begin{aligned} &|Q_0 \widehat{S}_{\xi'}(\lambda)^{-1} F|_2^2 + D_0[Q_0 \widehat{S}_{\xi'}(\lambda)^{-1} F] \\ &+ |\widetilde{Q} \widehat{S}_{\xi'}(\lambda)^{-1} F|_2^2 + D_0[\widetilde{Q} \widehat{S}_{\xi'}(\lambda)^{-1} F] + \sum_{j+k=2} \left| |\xi'|^j \partial_{x_n}^k \widetilde{Q} \widehat{S}_{\xi'}(\lambda)^{-1} F \right|_2^2 \\ &\leq C \left\{ |F^0|_2^2 + D_0[F^0] + |F'|_2^2 + |F^n|_2^2 \right\}. \end{aligned}
$$

uniformly for $\lambda \in \Sigma(-\frac{|\xi'|^2}{2}, \frac{3}{4}\pi) \cap \{\lambda; |\lambda| \leq 1, \ \lambda \neq 0\}$ and $|\xi'| \geq 1$. Since

$$
\widehat{S}_{\xi'} u = -\lambda \widetilde{Q} + \widehat{S}_{\xi'}(\lambda) = \left[ I - \lambda \widetilde{Q} \widehat{S}_{\xi'}(\lambda)^{-1} \right] \widehat{S}_{\xi'}(\lambda),
$$

we see from (A.3) that $\widehat{S}_{\xi'}$ has a bounded inverse $\widehat{S}_{\xi'}^{-1} = \widehat{S}_{\xi'}(\lambda)^{-1} \left[ I - \lambda \widetilde{Q} \widehat{S}_{\xi'}(\lambda)^{-1} \right]^{-1}$ (with suitably small $\lambda$) in $H^1(0,\infty) \times L^2(0,\infty)$ and $u = \widehat{S}_{\xi'}^{-1} F = {}^T(p, v)$ satisfies the estimate (A.2). This completes the proof. $\square$

We are now in a position to prove Lemma 6.6.

**Proof of Lemma 6.6.** Let $(p, v', v^n)$ be a solution of (6.26) and let $\chi$ be a $C^\infty$ funciton with $\operatorname{supp} \chi \subset [0,1)$ or $\operatorname{supp} \chi \subset (0,1]$. Then $(\chi p, \chi v', \chi v^n)$ is a solution of the following problem on the half spaces $[0,\infty)$ or $(-\infty, 1]$:

$$
\begin{cases} i\xi' \cdot (\chi v') + \partial_{x_n}(\chi v^n) = \widetilde{F}^0, \\ (|\xi'|^2 - \partial_{x_n}^2)(\chi v') + i\xi'(\chi p) = \widetilde{F}', \\ (|\xi'|^2 - \partial_{x_n}^2)(\chi v^n) + \partial_{x_n}(\chi p) = \widetilde{F}^n, \\ \chi v|_{x_n=0} = 0 \ \text{ or } \ \chi v|_{x_n=1} = 0. \end{cases}
$$

Here

$$
\widetilde{F}^0 = \chi F^0 + (\partial_{x_n} \chi) v^n,
$$

$$\widetilde{F}' = \chi F' - 2(\partial_{x_n}\chi)\partial_{x_n} v' - (\partial_{x_n}^2 \chi) v',$$
$$\widetilde{F}^n = \chi F^n - 2(\partial_{x_n}\chi)\partial_{x_n} v^n - (\partial_{x_n}^2 \chi) v^n + (\partial_{x_n}\chi) p.$$

It follows from Lemma A.4 that

$$\begin{aligned} &|\chi p|_2^2 + D_0[\chi p] + |\chi v|_2^2 + D_0[\chi v] + \sum_{j+k=2} \left| |\xi'|^j \partial_{x_n}^k (\chi v) \right|_2^2 \\ &\le C \left\{ |\widetilde{F}^0|_2^2 + D_0[\widetilde{F}^0] + |\widetilde{F}'|_2^2 + |\widetilde{F}^n|_2^2 \right\} \\ &\le C \left\{ |F^0|_2^2 + D_0[F^0] + |F'|_2^2 + |F^n|_2^2 + |v|_2^2 + D_0[v] + |p|_2^2 \right\}. \end{aligned} \tag{A.4}$$

We take $C^\infty$ functions $\chi_1$ and $\chi_2$ with $\chi_1 + \chi_2 = 1$ on $[0,1]$ with $\mathrm{supp}\,\chi_1 \subset [0,1)$ and $\mathrm{supp}\,\chi_2 \subset (0,1]$. Applying (A.4) to $(\chi_j p, \chi_j v)$ $(j=1,2)$, we obtain

$$\begin{aligned} &|p|_2^2 + D_0[p] + |v|_2^2 + D_0[v] + \sum_{j+k=2} \left| |\xi'|^j \partial_{x_n}^k v \right|_2^2 \\ &\le C \left\{ |F^0|_2^2 + D_0[F^0] + |F'|_2^2 + |F^n|_2^2 + |v|_2^2 + D_0[v] + |p|_2^2 \right\}. \end{aligned} \tag{A.5}$$

The desired estimate of Lemma 6.6 now follows from (A.5) if $|\xi'|^2 \ge 2C$. This completes the proof. □

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