Flat modules and Gröbner bases over truncated discrete valuation rings

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Flat modules and Gröbner bases over truncated discrete valuation rings

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Abstract

We present basic properties of Gröbner bases of submodules of a free module of finite rank over a polynomial ring with coefficients in a tdvr (:= truncated discrete valuations ring). As an application, we give a criterion for an algebra of finite type over a tdvr to be flat and prove the existence of a flat lifting of a flat algebra over a tdvr.

1 Introduction

A truncated discrete valuation ring (abbreviated as tdvr in the following) is a commutative ring which is isomorphic to a quotient of finite length of a discrete valuation ring (equivalently, it can be defined to be an Artinian local ring whose maximal ideal is generated by one element). In this paper, we study Gröbner bases over tdvr's and their applications. In particular, we provide a flatness criterion for modules over a tdvr and prove the following:

Theorem 1.1. *Let $\mathcal{O}$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and let a be a positive integer. Then any flat $\mathcal{O}/\mathfrak{m}^a$-algebra of finite type lifts to a flat $\mathcal{O}$-algebra of finite type.*

Such a result has applications in the study of ramification theory of tdvr's ([3], [4]).

Our theory is in fact developed for modules (rather than algebras) over tdvr's. In Section 2, we recall (following [1] and [5]) the general theory of Gröbner bases for submodules of a free module over a polynomial ring with an arbitrary Noetherian coefficient ring A. It is refined in Section 3 in the case where A is a tdvr, and we obtain a flatness criterion for modules over a tdvr (Th. 3.2). As an application, we prove in Section 4 the liftability of

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flat modules over a tdvr A to a "longer" tdvr $\hat{A}$ (Cor. 4.2), which implies the above theorem.

Throughout this paper, all rings are commutative.

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2 Gröbner bases over Noetherian rings

We recall the theory of Gröbner bases for submodules of a free module of finite rank over a polynomial ring with coefficients in a Noetherian ring following Chapter 4 of [1] and [5].

Let A be a Noetherian ring and $R := A[x_1, \ldots, x_n]$ the ring of polynomials in n variables with coefficients in A. Let L be a free R-module of rank $r \geq 1$. Fix an R-basis $\boldsymbol{e} = (e_1, \ldots, e_r)$ of L. Let

$$\begin{aligned} \mathbb{X} &:= \{x_1^{m_1} \cdots x_n^{m_n} \mid (m_1, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^n\}, \\ \mathbb{X}_{\boldsymbol{e}} &:= \{x_1^{m_{l,1}} \cdots x_n^{m_{l,n}} e_l \mid 1 \leqslant l \leqslant r,\ (m_{l,1}, ..., m_{l,n}) \in (\mathbb{Z}_{\geq 0})^n\} \end{aligned}$$

be the sets of all *power products* (or *monomials*) in R and L, respectively. Choose and fix a *term order* $<$ on $\mathbb{X}_{\boldsymbol{e}}$; thus it is a total order on $\mathbb{X}_{\boldsymbol{e}}$ satisfying
(1) $X < ZX$ for any $X \in \mathbb{X}_{\boldsymbol{e}}$ and any $Z \neq 1$ in $\mathbb{X}$;
(2) If $X < Y$ in $\mathbb{X}_{\boldsymbol{e}}$, then $ZX < ZY$ for any $Z \in \mathbb{X}$.

Note that the term order $<$ on $\mathbb{X}_{\boldsymbol{e}}$ induces a term order on $\mathbb{X}$, which we denote by the same notation $<$. It is known ([5], Prop. 3) that any term order makes $\mathbb{X}_{\boldsymbol{e}}$ a well-ordered set. For $X, Y \in \mathbb{X}_{\boldsymbol{e}}$, we write $X \mid Y$ if there exists $Z \in \mathbb{X}$ such that $Y = ZX$.

Any non-zero element f of L can be written uniquely as

$$f \ = \ a_1X_1 + \cdots + a_sX_s \quad \text{ with } a_i \in A \smallsetminus \{0\} \text{ and } X_1 > \cdots > X_s.$$

Then we set $\mathrm{lp}(f) = X_1$, $\mathrm{lc}(f) = a_1$ and $\mathrm{lt}(f) = a_1X_1$; these are called the *leading power product* (or *leading monomial*), *leading coefficient* and *leading term* of f, respectively.

Definition 2.1. Let f, h be two elements of L and $F = \{f_1, \ldots, f_s\}$ a finite subset of $L \smallsetminus \{0\}$.

(i) We write $f \xrightarrow{F} h$ if there exist $a_1, \dots, a_s \in A$, and $X_1, \dots, X_s \in \mathbb{X}_{\boldsymbol{e}}$ satisfying

- $h = f - (a_1 X_1 f_1 + \cdots + a_s X_s f_s)$,
- $\operatorname{lp}(f) = X_i \operatorname{lp}(f_i)$ for all i such that $a_i \neq 0$, and
- $\operatorname{lt}(f) = a_1 X_1 \operatorname{lt}(f_1) + \cdots + a_s X_s \operatorname{lt}(f_s)$.

(ii) We say that f *reduces to* h *modulo* F, and write $f \overset{F}{\Rrightarrow} h$, if there exist finitely many elements $h_1, \dots, h_t \in L$ such that

$$f \xrightarrow{F} h_1 \xrightarrow{F} h_2 \xrightarrow{F} \cdots \xrightarrow{F} h_t \xrightarrow{F} h.$$

We say that f reduces *strictly* to h if $f \overset{F}{\Rrightarrow} h$ and $\operatorname{lp}(h) < \operatorname{lp}(f)$.

Remark. (1) If $f \overset{F}{\Rrightarrow} h$ for $f, h \in L$, then we have $f \equiv g \bmod \langle F \rangle$, where $\langle F \rangle$ denotes the R-submodule of L generated by F.

(2) For any f and F as above, there exists a "minimal reduction" of f modulo F, by which we mean an element $r \in L$ to which f reduces modulo F and which does not reduce strictly any more. This can be proved by induction on $\operatorname{lp}(f)$, upon noticing the fact that the set $\mathbb{X}_{\boldsymbol{e}}$ is well-ordered with respect to $<$.

For a subset $G \subset L \smallsetminus \{0\}$, we denote by $\operatorname{Lt}(G)$ the submodule of L generated by the leading terms of all elements in G.

The following theorem is fundamental for our purposes:

Theorem 2.2 ([5], Th. 14, Cor. 15). *Let M be a non-zero R-submodule of L. Then there exists a finite subset $G = \{g_1, \dots, g_t\}$ of $M \smallsetminus \{0\}$ satisfying the following equivalent conditions:*

(a) $\operatorname{Lt}(G) = \operatorname{Lt}(M)$.

(b) *For any $f \in L$, we have $f \in M$ if and only if $f \overset{G}{\Rrightarrow} 0$.*

(c) *For any $f \in M$, there exist $h_1, \dots, h_t \in R$ such that $f = h_1 g_1 + \cdots + h_t g_t$ with* $\operatorname{lp}(f) = \max_{1 \leqslant i \leqslant t}(\operatorname{lp}(h_i) \operatorname{lp}(g_i))$.

Definition 2.3. Let M be a non-zero R-submodule of L. A finite subset G of $M \smallsetminus \{0\}$ as in the above theorem is called a *Gröbner basis for M*. A finite subset G of $L \smallsetminus \{0\}$ is called a *Gröbner basis* (in L) if it is a Gröbner basis for some non-zero R-submodule of L.

It follows immediately from the theorem that we have $M = \langle G \rangle$ if G is a Gröbner basis for a non-zero R-submodule M in L.

A Gröbner basis not unique for a given submodule. Indeed, if G is a Gröbner basis for M, then any finite subset of $M \smallsetminus \{0\}$ containing G is again a Gröbner basis for M (this follows from (a) of Theorem 2.2).

Definition 2.4 ([1], Exer. 4.1.9)**.** A Gröbner basis G in L is said to be *minimal* if no $g \in G$ can be strictly reduced with respect to $G \smallsetminus \{g\}$.

The minimality of a Gröbner basis $G = \{g_1, \dots, g_t\}$ implies in particular that the leading terms $\mathrm{lt}(g_1), \dots, \mathrm{lt}(g_t)$ are all different.

Every Gröbner basis contains a minimal Gröbner basis. Indeed, if there is an element $g \in G$ which can be strictly reduced with respect to $G \smallsetminus \{g\}$, we have $\mathrm{lt}(g) \in \mathrm{Lt}(G \smallsetminus \{g\})$ (Def. 2.1). Hence $\mathrm{Lt}(G) = \mathrm{Lt}(G \smallsetminus \{g\})$, and $G \smallsetminus \{g\}$ is also a Gröbner basis. Repeating this process finite times, we reach a minimal Gröbner basis.

In Section 4, we need to consider Gröbner bases over polynomial rings with different coefficient rings. Let $\hat{A} \to A$ be a surjective homomorphism of Noetherian rings. Let $\hat{R} = \hat{A}[x_1, \dots, x_n]$, and choose a lifting $\hat{L}$ of L over $\hat{R}$, by which we mean a free $\hat{R}$-module $\hat{L}$ of rank r equipped with an $\hat{R}$-basis $\hat{\boldsymbol{e}} = (\hat{e}_1, \dots, \hat{e}_r)$ and a surjective $\hat{R}$-module homomorphism $\varphi : \hat{L} \to L$ sending $\hat{e}_l$ to e_l.

Lemma 2.5. *For any non-zero submodule M of L, there exist Gröbner bases $G = \{g_1, \dots, g_t\}$ for M and $\hat{G} = \{\hat{g}_1, \dots, \hat{g}_t\}$ in $\hat{L}$ such that $\varphi(\hat{g}_j) = g_j$.*

Proof. Choose a Gröbner basis $G_0 = \{g_1, \dots, g_s\}$ for M, and write $g_j = \sum_X c_{j,X} X$ with $c_{j,X} \in A$. Choose a lift $\hat{c}_{j,X} \in \hat{A}$ of $c_{j,X}$ such that $\hat{c}_{j,X} = 0$ if $c_{j,X} = 0$. Set $\hat{g}_j := \sum_X \hat{c}_{j,X} X \in \hat{L}$. Let $\hat{M}$ be the $\hat{R}$-submodule of $\hat{L}$ generated by $\hat{g}_1 \dots, \hat{g}_s$. If $\hat{G}_1$ is a Gröbner basis for $\hat{M}$, then so is $\hat{G} := \{\hat{g}_1, \dots, \hat{g}_s\} \cup \hat{G}_1$. Put $G := \{\varphi(\hat{g}) \mid \hat{g} \in \hat{G}\}$. Then G and $\hat{G}$ have the required properties. □

In the rest of this section, let M be a non-zero R-submodule of L. Choose a Gröbner basis $G = \{g_1, \dots, g_t\}$ for M. For each j, write $c_j = \mathrm{lc}(g_j)$ and $X_j = \mathrm{lp}(g_j)$. For each $X \in \mathbb{X}_{\boldsymbol{e}}$, let I_X denote the ideal of A generated by c_j for all j such that $X_j | X$ (if there are no such j, put $I_X = 0$), and choose a complete set of coset representatives C_X for A/I_X containing 0. Define the set T_G of *totally reduced* vectors by

$$T_G \ := \ \left\{ \sum_{X \in \mathbb{X}_e} c_X X \ \middle| \ c_X \in C_X \right\} \qquad \subset \ L.$$

The set T_G depends on the choice of G and $(C_X)_{X \in \mathbb{X}_e}$. We write this set T_G formally as

$$T_G \ = \ \bigoplus_{X \in \mathbb{X}_e} C_X X.$$

Theorem 2.6 (*cf.* [1], Th. 4.3.3)**.** *The projection map $L \to L/M$ induces a bijection of sets*

$$\rho : \ T_G \ \xrightarrow{\simeq} \ L/M.$$

Proof. First we prove the injectivity of ρ. Let f and h be two different elements of T_G. Then any non-zero term of $f - h$ has coefficient $\not\equiv 0 \pmod{I_X}$, and hence $f - h$ cannot be reduced strictly modulo G any more. By Theorem 2.2, we have $f - h \notin M$.

Next we prove the surjectivity of ρ. Given an $f \in L$, we shall find an $r \in T_G$ such that $f \overset{G}{\Rrightarrow} r$ by induction on $\mathrm{lp}(f)$. This is trivial if $f = 0$. Suppose $f \neq 0$. Let $c = \mathrm{lc}(f)$ and $X = \mathrm{lp}(f)$. Then there exists $c_X \in C_X$ such that $c \equiv c_X \pmod{I_X}$. Write $c = c_X + \sum_j d_j c_j$ with $d_j \in A$, the sum begin over those j such that $X_j \mid X$. Write $X = Y_j X_j$ with $Y_j \in \mathbb{X}$ for such j. Then we have

$$f \ = \ c_X X + \sum_j d_j Y_j g_j + f_1$$

with some $f_1 \in L$ such that $\mathrm{lp}(f_1) < \mathrm{lp}(f)$. Thus the induction proceeds. □

3 Flatness of modules over a tdvr

First we recall some basic notions on tdvr's from [3] and [4]. A *tdvr* is an Artinian local ring whose maximal ideal is generated by one element. The *length* of a tdvr A is the length of A as an A-module. If $\mathcal{O}$ is a complete discrete valuation ring and $\mathfrak{m}$ is its maximal ideal, then $\mathcal{O}/\mathfrak{m}^a$ is a tdvr for any integer $a \geq 1$. Conversely, it is known that any tdvr is a quotient of a complete discrete valuation ring ([3], Prop. 2.2).

A complete discrete valuation ring may naturally be thought of as a tdvr of length ∞. By abuse of terminology, however, we call also a discrete valuation ring which *may not necessarily be complete* a tdvr of length ∞, because the theory below applies to any discrete valuation ring as well. It is known that a tdvr A is principal, and any ideal is of the form $\mathfrak{m}^i$ for some $i \geq 0$, where $\mathfrak{m}$ is the maximal ideal of A. Any generator π of the maximal ideal $\mathfrak{m}$ is said to be a *uniformizer* of A. Any element x of a tdvr A of length a can be written as $x = u\pi^i$ with $u \in A^\times$, π a uniformizer of A, and $0 \leqslant i \leqslant a$ (with the convention $0^0 = 1$ if $a = 1$). The exponent $i \leqslant a$ is unique (we regard $0 = \pi^\infty$ if $a = \infty$); we denote it by $v(x)$ and call it the *valuation* of x. Thus we have a map $v : A \to \{0, \ldots, a\}$; we have $v(x) = 0$ if and only if x is a unit of A, $v(x) = 1$ if and only if x is a uniformizer of A, and $v(x) = a$ if and only if $x = 0$.

In the rest of this paper, let A be a tdvr of length $a \leqslant \infty$, $\mathfrak{m}$ its maximal ideal, π a uniformizer and k its residue field. Let R, L, $\boldsymbol{e}$ and $\mathbb{X}_{\boldsymbol{e}}$ have the same meaning as in the previous section with the coefficient ring specialized to the tdvr A; thus $R = A[x_1, \ldots, x_n]$ is the ring of polynomials in n variables with coefficients in A, L is a free R-module of rank r with a basis $\boldsymbol{e} = (e_1, \ldots, e_r)$, and $\mathbb{X}_{\boldsymbol{e}}$ is the set of all power products $x_1^{m_{j,1}} \cdots x_n^{m_{j,n}} e_j$ in L furnished with a fixed term order $<$.

Let M be a non-zero R-submodule of L. We fix a Gröbner basis $G = \{g_1, \ldots, g_t\}$ for M. Write $c_j = \mathrm{lt}(g_j)$ and $X_j = \mathrm{lp}(g_j)$ as in the previous section. For each $X \in \mathbb{X}_{\boldsymbol{e}}$, let I_X be the ideal of A generated by c_j for all j such that $X_j \mid X$. Denote by m_X the length of A/I_X as an A-module. In other words, the ideal I_X is generated by π^{m_X}. Then we have

$$m_X = \begin{cases} \min\{v(c_j) \mid X_j \text{ divides } X\}, & \text{if there exists a } j \text{ with } X_j \mid X, \\ a, & \text{otherwise.} \end{cases}$$

Since the coefficient ring A is a tdvr, we can choose a compete set of coset representatives C_X containing 0 specifically as follows: For each $0 \leqslant i < a$, let C_i be a complete set of coset representatives in A of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ which contains 0 (so this does not depend on X). We define

$$C_X := \bigoplus_{0 \leqslant i < m_X} C_i := \left\{ \sum_{0 \leqslant i < m_X} c_i \;\middle|\; c_i \in C_i \right\} \tag{1}$$

(If $m_X = 0$, then put $C_X = \{0\}$). With this C_X, we have the following formal presentation of T_G:

$$(2) \qquad T_G \;=\; \bigoplus_{X\in\mathbb{X}_{\boldsymbol{e}}} \left(\bigoplus_{0\leqslant i<m_X} C_i \right) X.$$

Let $\mathrm{gr}(A) := \bigoplus_{0\leqslant i<a} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ be the graded ring of A and, for any $m \in \mathbb{Z}_{\geq 0}$, we define the quotient $\mathrm{gr}^{<m}(A) := \bigoplus_{0\leqslant i<m} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ of the graded ring $\mathrm{gr}(A)$. The isomorphisms $C_i \xrightarrow{\simeq} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ of sets induce an isomorphism

$$(3) \qquad T_G \;\xrightarrow{\simeq}\; \bigoplus_{X\in\mathbb{X}_{\boldsymbol{e}}} \mathrm{gr}^{<m_X}(A)X$$

of sets. The right-hand side of (3) is naturally a $\mathrm{gr}(A)$-module and it induces a structure of a $\mathrm{gr}(A)$-module on T_G, which we denote by $\mathrm{gr}(T_G)$. Thus we have an isomorphism of $\mathrm{gr}(A)$-modules

$$(4) \qquad \mathrm{gr}(T_G) \;\xrightarrow{\simeq}\; \bigoplus_{X\in\mathbb{X}_{\boldsymbol{e}}} \mathrm{gr}^{<m_X}(A),$$

and, for each i, an isomorphism of graded pieces

$$(5) \qquad \mathrm{gr}^i(T_G) \;=\; \bigoplus_{X\in\mathbb{X}_{\boldsymbol{e}}} C_i X \;\xrightarrow{\simeq}\; \bigoplus_{X\in\mathbb{X}_{\boldsymbol{e}} \text{ s.t. } m_X>i} \left(\mathfrak{m}^i/\mathfrak{m}^{i+1}\right)$$

as k-vector spaces. The isomorphism (3) implies the equivalence of (a) and (b) in the next proposition.

Proposition 3.1. *The following conditions on the* $\mathrm{gr}(A)$*-module* $\mathrm{gr}(T_G)$ *are equivalent:*
(a) $\mathrm{gr}(T_G)$ *is flat over* $\mathrm{gr}(A)$*;*
(b) $\{m_X \mid X \in \mathbb{X}_{\boldsymbol{e}}\} \subset \{0, a\}$*;*
(c) *For any* j*, there exists* j' *such that* $c_{j'}$ *is a unit element in* A *and* $X_{j'} \mid X_j$*.*
Furthermore, we have

$$\mathrm{rank}_{\mathrm{gr}(A)}(\mathrm{gr}(T_G)) \;=\; \#\{X \in \mathbb{X}_{\boldsymbol{e}} \mid m_X = a\}$$

if $\mathrm{gr}(T_G)$ *is flat over* $\mathrm{gr}(A)$*.*

Proof. It is enough to show the equivalence of the conditions (b) and (c). Note that m_{X_j} cannot be a for any j because $c_{j'} \neq 0$ for all j'. Then the assumption (b) implies that $m_{X_j} = 0$ for any j. This together with the definition of m_{X_j} implies (c). Conversely, assume (c) and let $X \in \mathbb{X}_{\boldsymbol{e}}$. If $m_X \neq a$, then we have $X_j \mid X$ for some j. By the assumption, there exists j' such that $c_{j'}$ is a unit and $X_{j'} \mid X_j$, and hence $m_X = v(c_{j'}) = 0$. □

The bijection $\rho : T_G \xrightarrow{\simeq} L/M$ in Theorem 2.6 induces a bijection $\rho|_{\mathfrak{m}^i T_G} : \mathfrak{m}^i T_G \xrightarrow{\simeq} \mathfrak{m}^i(L/M)$, where we put

$$\mathfrak{m}^i T_G \;:=\; \bigoplus_{X \in \mathbb{X}_{\boldsymbol{e}}} \left(\bigoplus_{i \leqslant i' < m_X} C_{i'} \right) X.$$

Since the map ρ is induced by the projection map $L \to L/M$, we have also isomorphisms

$$\begin{aligned} \mathrm{gr}^i(\rho) : \; \mathrm{gr}^i(T_G) &\xrightarrow{\simeq} \mathrm{gr}^i(L/M) \quad \text{and} \\ \mathrm{gr}(\rho) : \; \mathrm{gr}(T_G) &\xrightarrow{\simeq} \mathrm{gr}(L/M) \end{aligned}$$

of k-vector spaces and $\mathrm{gr}(A)$-modules respectively, where $\mathrm{gr}(L/M)$ is the graded $\mathrm{gr}(A)$-module associated with the module L/M with $\mathfrak{m}$-adic filtration. It is known ([2], Chap. III, Sect. 5.2; [6], Th. 22.3) that the A-module L/M is flat over A if and only if the graded $\mathrm{gr}(A)$-module $\mathrm{gr}(L/M)$ is flat over $\mathrm{gr}(A)$. Hence we have

Theorem 3.2. *The following conditions on the A-module L/M are equivalent:*
(a) *L/M is flat over A;*
(b) *$\{m_X \mid X \in \mathbb{X}_{\boldsymbol{e}}\} \subset \{0, a\}$;*
(c) *For any j, there exists j' such that $c_{j'}$ is a unit element in A and $X_{j'} \mid X_j$.*
Furthermore, we have

$$\mathrm{rank}_A(L/M) \;=\; \#\{X \in \mathbb{X}_{\boldsymbol{e}} \mid m_X = a\}$$

if L/M is flat over A.

If we further assume that the Gröbner basis G is minimal, then there are no divisibility relations between the leading terms $\mathrm{lt}(g_j) = c_j X_j$, and hence the the condition (c) in the above theorem means that all c_j are units. Thus we deduce the following corollary:

Corollary 3.3. *If the Gröbner basis G for M is minimal, then L/M is flat over A if and only if c_j is a unit element in A for every $1 \leqslant j \leqslant t$.*

4 Flat liftings of flat modules over a tdvr

Let $\hat{A}$ be a tdvr of length $\hat{a} \leqslant \infty$ and $\hat{\mathfrak{m}}$ its maximal ideal. Let A be a quotient ring $\hat{A}/\hat{\mathfrak{m}}^a$ of $\hat{A}$ with $0 < a \leqslant \hat{a}$; it is a tdvr of length a. Let L be a free R-module of rank r, with a fixed basis $\boldsymbol{e} = (e_1, \ldots, e_r)$, over the polynomial ring $R = A[x_1, \ldots, x_n]$. Let M be a non-zero R-submodule of L. Let $\hat{R} = \hat{A}[x_1, \ldots, x_n]$, and choose a lifting $\hat{L}$ of L over $\hat{R}$ together with an $\hat{R}$-basis $\hat{\boldsymbol{e}} = (\hat{e}_1, \ldots, \hat{e}_r)$ such that $\hat{e}_l \mapsto e_l$. Let M be a non-zero R-submodule of L. Recall that we can find a Gröbner basis G of M together with a lifting $\hat{G} \subset \hat{L}$ which is a Gröbner basis in $\hat{L}$ (Lem. 2.5).

Proposition 4.1. *Let $M, G, \hat{G}$ be as above, and let $\hat{M}$ be the $\hat{R}$-submodule of $\hat{L}$ generated by $\hat{G}$. Then the quotient module $\hat{L}/\hat{M}$ is flat over $\hat{A}$ if and only if L/M is flat over A.*

Proof. Let $\hat{I}_X$ be the ideal of $\hat{A}$ defined from $\hat{G}$ in the same way as I_X was defined from G, and put $\hat{m}_X = \mathrm{length}_{\hat{A}}(\hat{A}/\hat{I}_X)$. Then the properties of G and $\hat{G}$ imply that

$$\hat{m}_X \;=\; \begin{cases} m_X & \text{if } \; m_X < a, \\ \hat{a} & \text{if } \; m_X = a. \end{cases}$$

Hence the assertion follows from Theorem 3.2. □

Any finite R-module is of the form L/M for some n and $M \subset L$. Hence the proposition implies immediately the following:

Corollary 4.2. *Any finite R-module which is flat over A lifts to a finite $\hat{R}$-module which is flat over $\hat{A}$.*

It is known ([6], Th. 7.10) that a flat module over a local ring with *nilpotent* maximal ideal is free (so the corollary follows directly from this fact). However, the proposition is in fact about the lifting of R-submodules of L, and is useful when L has additional structures. Let us apply it to the case of A-algebras. Recall that an A-algebra of finite type is isomorphic to a quotient R/I of $R = A[x_1, \ldots, x_n]$ for some n by an ideal I of R. Proposition 4.1 implies:

Corollary 4.3. *Any flat A-algebra of finite type lifts to a flat $\hat{A}$-algebra of finite type.*

Theorem 1.1 is a special case of this corollary.

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