THE MOTION OF A TRANSITION LAYER FOR A BISTABLE REACTION DIFFUSION EQUATION WITH HETEROGENEOUS ENVIRONMENT

栄, 伸一郎
九州大学大学院数理学研究院

松澤, 寛
沼津工業高等専門学校

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SHIN-ICHIRO EI AND HIROSHI MATSUZAWA

Abstract. In this paper we study the dynamics of a single transition layer of solution to a spatially inhomogeneous bistable reaction diffusion equation in one space dimension. The spatial inhomogeneity is given by a function $a(x)$. In particular, we consider the case when $a(x)$ is identically zero on an interval I and study the dynamics of transition layer on I. In this case the dynamics of the transition layer on I becomes so-called very slow dynamics. In order to analyze such a dynamics, we construct an attractive local invariant manifold giving the dynamics of transition layer and we derive the equation describing the flow on the manifold. We also give applications of our results to well known two nonlinearities of bistable type.

1. Introduction

Reaction diffusion equations have been widely treated in order to study the mechanism of pattern formation for various phenomena. In particular, solutions to some class of equations give rise to sharp transition layers when diffusion coefficients are very small. These solutions appear in these reaction diffusion equations describing important phenomena.

In this paper we consider the following scalar reaction diffusion equation

$$u_t = \varepsilon^2 u_{xx} + f(x,u) \quad x \in \Omega, t > 0, \tag{1.1}$$

in which ε is positive small parameter and $\Omega \subset \mathbb{R}$. In particular, we assume that the nonlinear function f depends on the space variable x. The spatial inhomogeneity is seen as an environmental effect in the phenomena which the equation describes.

In [3, 16], they consider Neumann boundary value problem on a bounded interval Ω and showed that if f does not depend on the space variable x, then there does not exist nonconstant stable stationary solution. On the other hand if f depends on space variable x, stable spatially inhomogeneous patterns may appear and the set of stationary solutions may have a rich structure. Hence the dynamics of solutions to corresponding parabolic problem may become more complicated.

In this paper we assume that the nonlinear function f is so-called *bistable* nonlinearity. This means f has three zeros, say $u = \beta_- < 0 < \beta_+$ of u and f_u is negative at $u = \beta_\pm$. Such a kind of equations appear in various model such as phase transition, chemical reaction and population genetics.

We suppose for (1.1) that:

Assumption . (1) The function f has the following form

$$f(x,u) := f_0(u) + \eta a(x) g(u),$$

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where η is positive small constant.

(2) The function f_0 satisfies

$$f_0(\beta^-) = f_0(0) = f_0(\beta^+),\ f_0'(\beta^\pm) < 0,\ f_0'(0) > 0,$$
$$f_0'(\beta^-) = f_0'(\beta^+), \int_{\beta_-}^{\beta^+} f(u)du = 0.$$

(3) The function $a(x)$ is bounded C^1 function on $\mathbb{R}$ and there exists $L > 0$ such that $a(x) = 0$ on $I = [-L, L]$ and $a(x)$ behaves near $x = \pm L$ as follows:

$$a(x) = A_-|x+L|^{1+\alpha_-} + O(|x+L|^{2+\alpha_-}) \quad \text{as } x \to -L-0,$$
$$a(x) = A_+|x-L|^{1+\alpha_+} + O(|x-L|^{2+\alpha_+}) \quad \text{as } x \to L+0,$$

where $A_\pm$ are nonzero constants and $\alpha_\pm$ are positive constants.

(4) The function g satisfies $g(\beta^\pm) = 0$.

For simplicity we treat only the case

(2)′ f_0 is odd function with $\beta^\pm = \pm 1$, $f_0'(\pm 1) = -1$.

We note that general types of f_0 satisfying (1) to (4) can be treated similarly. Typical examples of $f(x,u)$ are $f(x,u) = \frac{1}{2}(u - \eta a(x))(1-u^2)$ and $f(x,u) = \frac{1}{2}(1+\eta a(x))u(1-u^2)$. In the former case, we have $f_0(u) = \frac{1}{2}u(1-u^2)$ and $g(u) = -\frac{1}{2}(1-u^2)$ and the corresponding problem is studied in [2, 4, 5, 11, 12, 14, 18, 22]. In [2, 4, 5, 14, 18, 22], a stationary problem is studied for the case that Ω is a bounded interval. Roughly speaking, it is shown that there exists a stationary solution with a transition layer near a nondegenerate zero of the function $a(x)$. A dynamics of the transition layer was studied in [11] when $\Omega = \mathbb{R}$. In the latter case, we have $f_0(u) = g(u) = \frac{1}{2}u(1-u^2)$ which is studied in [17, 19, 20, 21]. In particular, Nakashima [20] considered a bounded interval Ω and show that there exists a stable stationary solution with a transition layer near a nondegenerate local minimum point and unstable solution with a transition layer near nondegenerate local maximum point of $a(x)$, respectively. These results are treated from the dynamical point of view as applications of our results to these nonlinearity, which will be stated in Section 2. In case where $a(x)$ is nondegenerate at any local minimum or maximum point, we can derive the dynamics of a single transition layer of a solution to corresponding problem by constructing comparison functions(see Proposition 2.1).

In this paper we suppose $\Omega = \mathbb{R}$ and study the dynamics of a single transition layer on the interval $I = [-L, L]$ where $a(x)$ is identically zero on I. In this case, the motion of transition layer depends on a behavior of the function $a(x)$ at $x = \pm L$ and the speed is $O(e^{-A/\varepsilon})$. Namely, the motion of transition layer is governed by the *very slow dynamics*. Alikakos, Fusco and Kowalczyk [1] dealt homogeneous Allen-Cahn equation on a domain in $\mathbb{R}^2$ which consists of a rectangular part with two attachments on its sides. They show the existence of stationary solutions with nearly flat interfaces intersecting orthogonally the boundary of the domain at its rectangular part and the stability of these stationary solution depends on geometries of the corners of the rectangular part of the domain. They also show that motion of this interfaces depends on such geometries of the domain and the speed is $O(e^{-A/\varepsilon})$, that is, this dynamics is also very slow dynamics.

Our problem is as follows:

$$u_t = \varepsilon^2 u_{xx} + f(x,u), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.2}$$

where $f(x,u)$ satisfies Assumption (1) to (4) and (2)′ with odd function f_0.

At first we prepare an approximate solution. Let $\Phi(x)$ be a function satisfying

$$\begin{cases} \Phi_{xx} + f_0(\Phi) = 0 & \text{on } \mathbb{R}, \\ \Phi(\pm\infty) = \pm 1, \Phi(0) = 0. \end{cases}$$

Φ is uniquelly determined. The function Φ has following properties

Lemma 1.1. *Function Φ satisfies:*

$$\Phi(x) = \begin{cases} 1 - \gamma e^{-x} + O(e^{-2x}) & \text{as } x \to \infty, \\ -1 + \gamma e^{x} + O(e^{2x}) & \text{as } x \to -\infty, \end{cases}$$

$$\Phi_x(x) = \begin{cases} \gamma e^{-x} + O\left(e^{-2x}\right) & \text{as } x \to \infty, \\ \gamma e^{x} + O\left(e^{2x}\right) & \text{as } x \to -\infty, \end{cases}$$

for some $\gamma > 0$.

The reader should keep in mind that the special example $f_0(u) = \frac{1}{2}u(1-u^2)$ leads $\Phi(x) = \tanh(x/2)$.

Let $S(x) = \Phi(x/\varepsilon)$. $S(x)$ satisfies

$$\begin{cases} \varepsilon^2 S_{xx} + f_0(S) = 0 & \text{on } \mathbb{R}, \\ S(\pm\infty) = \pm 1, S(0) = 0. \end{cases}$$

We use $S(x)$ as an approximate solution. From Lemma 1.1 we have:

$$S(x) = \begin{cases} 1 - \gamma e^{-x/\varepsilon} + O(e^{-2x/\varepsilon}) & \text{as } x/\varepsilon \to \infty, \\ -1 + \gamma e^{x/\varepsilon} + O(e^{2x/\varepsilon}) & \text{as } x/\varepsilon \to -\infty, \end{cases} \tag{1.3}$$

$$S_x(x) = \begin{cases} \dfrac{\gamma}{\varepsilon} e^{-x/\varepsilon} + O(\dfrac{1}{\varepsilon} e^{-2x/\varepsilon}) & \text{as } x/\varepsilon \to \infty, \\ \dfrac{\gamma}{\varepsilon} e^{x/\varepsilon} + O\left(\dfrac{1}{\varepsilon} e^{2x/\varepsilon}\right) & \text{as } x/\varepsilon \to -\infty. \end{cases} \tag{1.4}$$

Let us define a nonlinear operator $\mathcal{L}(x,u)$ and $\mathcal{L}_0(u)$ as follows:

$$\mathcal{L}(x,u) := \varepsilon^2 u_{xx} + f(x,u),\ \mathcal{L}_0(u) := \varepsilon^2 u_{xx} + f_0(u).$$

Let $\Xi(l)$ be the transition operator defined by $(\Xi(l)v)(x) = v(x-l)$ for $v \in L^2(\mathbb{R})$. Moreover, define the quantity

$$\delta(l) := \sup_{x\in\mathbb{R}} |\mathcal{L}(x, \Xi(l)S(x))| (= \sup_{x\in\mathbb{R}} |\mathcal{L}(x+l, S(x))|),$$

the set

$$\mathcal{M} := \{S(x-l) | l \in [-L+d, L-d]\}$$

for a constant $d > 0$ and the function

$$H_0(l) := \langle \mathcal{L}(\cdot + l, S(\cdot)), S_x(\cdot) \rangle_{L^2}.$$

For convenience we denote $I_d = [-L+d, L-d]$. Following lemmas hold for $\delta(l)$ and $H_0(l)$.

Lemma 1.2. $\delta(l) = O(\varepsilon^{1+\alpha_-} e^{-(L+l)/\varepsilon} + \varepsilon^{1+\alpha_+} e^{-(L-l)/\varepsilon})$ *(as $\varepsilon \to 0$) holds for $l \in I_d$.*

Lemma 1.3. *For $l \in I_d$, the function $H_0(l)$ satisfies*

$$H_0(l) = \eta\gamma^2 \{A_- g'(-1)\varepsilon^{1+\alpha_-}(\Gamma_- + O(\varepsilon))e^{-2(L+l)/\varepsilon} - A_+ g'(1)\varepsilon^{1+\alpha_+}(\Gamma_+ + O(\varepsilon))e^{-2(L-l)/\varepsilon}\}$$
$$+ O(e^{-c_0/\varepsilon} e^{-2(l+L)/\varepsilon} + e^{-c_0/\varepsilon} e^{-2(L-l)/\varepsilon})$$

for some $c_0 > 0$, where $\Gamma_\pm = \Gamma(2+\alpha_\pm)/2^{2+\alpha_\pm}$ with standard gamma function Γ.

Our main theorems are as follows:

Theorem A. *Fix the constant $\eta > 0$ arbitrarily small enough. Then there exist positive constants ε_0, C_0 and a neighborhood U of $\mathcal{M}$ in $H^2(\mathbb{R})$ such that if $0 < \varepsilon < \varepsilon_0$ and $u(0,\cdot) \in U$, there exist a function $l(t) \in \mathbb{R}$ such that*

$$\|u(t,\cdot) - \Xi(l(t))S(\cdot)\|_\infty \le C_0\delta(l(t)) \tag{1.5}$$

holds as long as $l(t) \in I_d$, where $u(t,x)$ is a solution to (1.2). The function $l(t) \in \mathbb{R}$ satisfies

$$\dot{l} = -\frac{\varepsilon H_0(l)}{\|\Phi'\|_{L^2}^2} + O(\varepsilon\delta^2(l(t))). \tag{1.6}$$

Remark . By Lemma 1.2, we have $\delta(l)^2 = O(\varepsilon^{2(1+\alpha_-)}e^{-2(L+l)/\varepsilon} + \varepsilon^{2(1+\alpha_+)}e^{-2(L-l)/\varepsilon})$. On the other hand by Lemma 1.3 we see that $H_0(l)$ in (1.6) is $O(\varepsilon^{1+\alpha_-}e^{-2(L+l)/\varepsilon} + \varepsilon^{1+\alpha_+}e^{-2(L-l)/\varepsilon})$ which means $H_0(l)$ in (1.6) is necessarily dominant as long as $l \in I_d$ for ε is sufficiently small $\varepsilon > 0$.

Let $\overline{H}_0(l) := \eta\gamma^2\{A_-g'(-1)\Gamma_-\varepsilon^{1+\alpha_-}e^{-2(L+l)/\varepsilon} - A_+g'(1)\Gamma_+\varepsilon^{1+\alpha_+}e^{-2(L-l)/\varepsilon}\}$ and consider the ordinary differential equation consisting of the principal part of (1.6)

$$\dot{l} = -\frac{\varepsilon\overline{H}_0(l)}{\|\Phi'\|_{L^2}^2}. \tag{1.7}$$

Theorem B. *There exists a positive constant C_1 such that if (1.7) has an equilibrium $l_\varepsilon^* \in I_d$, then there exists a stationary solution u of* (1.2) *such that*

$$\|u(\cdot) - \Xi(l_\varepsilon^*)S(\cdot)\|_\infty \le C_1\delta(l_\varepsilon^*).$$

The location $l_\varepsilon^\dagger$ of the transition layer of u satisfies

$$l_\varepsilon^\dagger = l_\varepsilon^* + O(\varepsilon^{1+\alpha_-} + \varepsilon^{1+\alpha_+}).$$

In particular l_ε^* is given by

$$l_\varepsilon^* = \frac{\alpha_- - \alpha_+}{2}\varepsilon\log\varepsilon + \frac{\varepsilon}{2}\log\left|\frac{A_-g'(-1)\Gamma_-}{A_+g'(1)\Gamma_+}\right|.$$

Moreover this stationary solution is stable if $\overline{H}_0'(l_\varepsilon^*) > 0$ and unstable if $\overline{H}_0'(l_\varepsilon^*) < 0$.

Remark . We can easily show that there exists $l_\varepsilon^{**} \in I_d$ such that $H_0(l_\varepsilon^{**}) = 0$ and $l_\varepsilon^{**} = l_\varepsilon^* + O(\varepsilon^2)$.

Theorem B says that a location of the transition layer of the stationary solution approaches the middle point of I as $\varepsilon \to 0$ due to the special hypothesis $f_0'(1) = f_0'(-1)$ in $(2)'$.

The paper is organized in the following way: In Section 2 we apply our main theorems to two examples of $f(x,u) = \frac{1}{2}(u - \eta a(x))(1-u^2)$ and $f(x,u) = \frac{1}{2}(1+\eta a(x))u(1-u^2)$ and analyze the dynamics of a transition layer. We will show that our results give some extensions of well known results from the viewpoint of dynamical system (for example [2, 4, 5, 14, 19, 20]). In Section 3 we proof Lemmas 1.2 and 1.3. In Section 4 is devoted to the proof our main theorems. The main tools for the proofs are the invariant manifold theory used in [6]. The main idea to apply the methods in [6] to our problem with spatial inhomogeneity is that we can see the spatial inhomogeneity as a perturbation of the spatially homogeneous problem in our setting.

We can apply our method to the dynamics of pulse like localaized patterns in reaction diffusion systems such as Gierer-Meinhardt system. Applications to reaction diffusion systems will be mentioned in the forthcoming paper [9].

2. Applications

In this section we apply our main theorems to the problems with well known nonlinearities $f(x,u) = \frac{1}{2}(u - \eta a(x))(1-u^2)$ and $f(x,u) = \frac{1}{2}(1+\eta a(x))u(1-u^2)$.

2.1. The case when $f(x,u) = \frac{1}{2}(u - \eta a(x))(1-u^2)$.

In this subsection we assume $f(x,u) = \frac{1}{2}(u - \eta a(x))(1-u^2)$. In this case, this equation appears in Fisher's model of the propagation of genetic composition in a population [10, 13]. The problem has been well studied for a bounded domain Ω. In this model, each individual of the population living in the habitat Ω belongs to one of three possible genotypes `aa`, `aA` and `AA` and a solution $u(x,t) \in [-1,1]$ denotes the frequency of the allele `a` in the population at the point x at time $t \geq 0$. The bistability of nonlinearity represents that a population in which the fitness of the heterozygote `Aa` is inferior to the fitness of the homozygotes `aa` and `AA`, who are competing with each other for territories in the habitat. In this setting we consider the case when the habitat has nonuniform environment which is expressed by the function $\eta a(\cdot) : \Omega \to (-1,1)$ in the nonlinearity f. We shall assume that there are two regions Ω_+, $\Omega_- \subset \Omega$ such that in the region Ω_+ the genotype `aa` has a selective advantage over the other genotypes and in Ω_- the genotype `AA` has a selective advantage over the other genotypes. The set Ω_+ (or Ω_-) where `aa` (or `AA`) is favored is readily described in terms of the function $a(x)$:

$$\Omega_- := \{x \in \Omega | a(x) < 0\}, \quad \Omega_+ := \{x \in \Omega | a(x) > 0\}.$$

The stationary problem of this problem on bounded interval $(0,1)$ is well studied. Let $\Omega_0 = \{x \in \Omega | a(x) = 0\}$. Angenent, Mallet-Paret and Peletier [2] proved the existence of stable solution u_ε which possess a single transition layer near an $x_0 \in \Omega_0$ with $u_\varepsilon'(x_0)a'(x_0) < 0$ when ε is small and u_ε close to -1 and 1 on compact subsets of Ω_+ and Ω_-. They used a sub and supersolution method to construct the solution u_ε. Dancer and Yan [4], constructed solutions having transition layers near the set Ω_0 in higher dimensional spaces by using variational method. Hale and Sakamoto [14] discussed about an unstable solution u_ε which possesses a single transition layer near an $x_0 \in \Omega_0$ with $u'(x_\varepsilon)a'(x_0) > 0$. Dancer and Yan [5] also constructed an unstable radially symmetric stationary solution u_ε in n dimensional ball. In [11], Fife and Hsiao studied the dynamics of the transition layer when $\Omega = \mathbb{R}$. They showed that the motion of the transition layer $l(t)$ is approximately governed by the equation

$$\frac{dl}{dt} = \varepsilon \eta a(l).$$

We note that in this case we do not need the smallness of η. In [7], Ei, Kuwamura and Morita derived this equation formally by using variational method when Ω is bounded interval.

In these results they assume that $a'(x_0) \neq 0$ at any $x_0 \in \Omega_0$. Although Dancer and Yan [4] considered the case when the set Ω_0 contains an interval and constructed a stationary solution with transition layers near the set Ω_0, the precise configuration of the transition layer have not been known. From the population genetics point of view, it is natural to consider a setting that the set Ω_0 contains an interval. Hence it is important to decide the configuration of transition layers of stationary solutions as well as the dynamics in this setting.

In this case $f_0(u) = \frac{1}{2}u(1-u^2)$ and $g(u) = -\frac{1}{2}(1-u^2)$. Since $g'(\pm 1) = \pm 1$, we have

$$\overline{H}_0(l) = \eta\gamma^2 \{-A_-\Gamma_-\varepsilon^{1+\alpha_-} e^{-2(L+l)/\varepsilon} - A_+\Gamma_+\varepsilon^{1+\alpha_+} e^{-2(L-l)/\varepsilon}\}.$$

The dynamics of the transition layer on the interval I_d is essentially governed by the equation $\dot{l} = -\varepsilon \overline{H}_0(l)/\|\Phi'\|_{L^2}^2$ which is determined by signs of constants A_- and A_+.

At first we can see that if $A_-A_+ > 0$, then $H_0(l)$ has constant sign on I_d. Hence, problem (1.2) does not have stationary solution with transition layer on I_d. In fact, the case where $A_- > 0$ and $A_+ > 0$, $H_0(l) < 0$ on I_d holds. Since $\dot{l} > 0$ in this case, the transition layer move to the right. From the view point of the potential function $W(x,u) = -\int_{-1}^{u} f(x,s)ds$, the state $u = -1$ is more stable than $u = 1$ near the set I_d. In this case, the transition layer of the solution moves to right to take values close to -1 as much as possible in a large region near the interval I_d. We can analyze similar situation in the case where $A_- < 0$ and $A_+ < 0$.

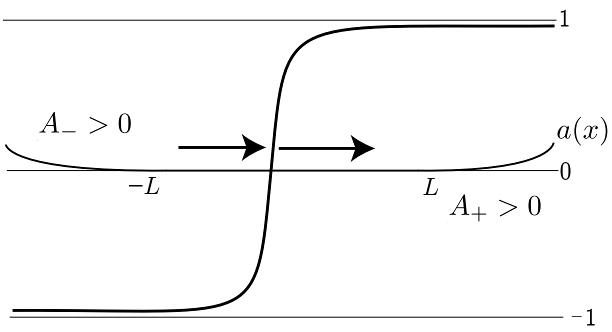

FIGURE 1. The dynamics of transition layer: Case $A_- > 0$, $A_+ > 0$.

When if $A_-A_+ < 0$, a simple calculation leads that the zero l_ε^* of $\overline{H}_0(l)$ is given by

$$l_\varepsilon^* := \frac{\alpha_- - \alpha_+}{2}\varepsilon \log \varepsilon + \frac{\varepsilon}{2}\log\left(-\frac{A_-\Gamma_-}{A_+\Gamma_+}\right).$$

From Theorem B, we can say that there exists a stationary solution u_ε of (1.2) such that

$$\|u_\varepsilon(\cdot) - \Xi(l_\varepsilon^*)S(\cdot)\|_\infty \le C_1\delta(l_\varepsilon^*)$$

and their stability is determined by the sign of $\overline{H}_0'(l_\varepsilon^*)$. If $A_- > 0$ and $A_+ < 0$, then we can see easily that $\overline{H}_0'(l_\varepsilon^*) > 0$ and the stationary solution is stable. More precisely, since $\overline{H}_0(l) < 0$ for $l < l_\varepsilon^*$ and $\overline{H}_0(l) > 0$ for $l > l_\varepsilon^*$. we have $\dot{l} > 0$ for $l < l_\varepsilon^*$ and $\dot{l} < 0$ for $l > l_\varepsilon^*$. Hence the dynamics of the transition layer is as in the Figure 2. Similary we can analyze

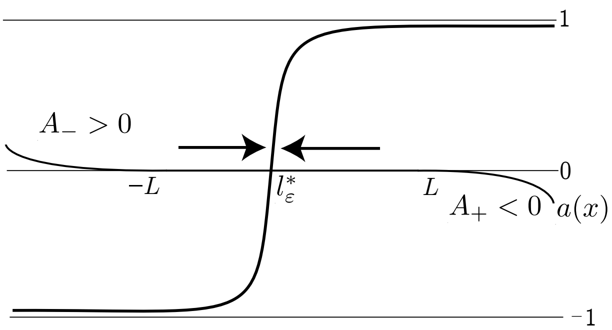

FIGURE 2. The dynamics of transition layer: Case $A_- > 0$, $A_+ < 0$.

the dynamics in the case where $A_- < 0$ and $A_+ > 0$ (see Figure 3).

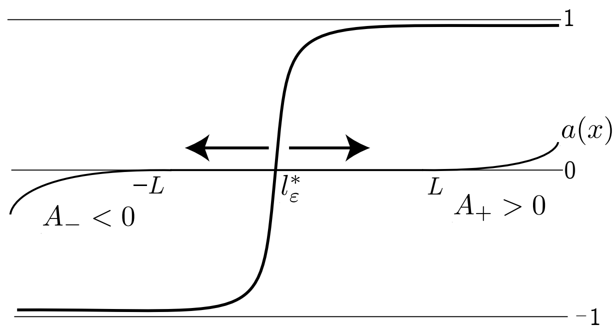

FIGURE 3. The dynamics of transition layer: Case $A_- < 0$, $A_+ > 0$.

2.2. **The case when $f(x,u) = \frac{1}{2}(1+\eta a(x))u(1-u^2)$.** In this subsection we assume $f(x,u) = \frac{1}{2}(1+\eta a(x))u(1-u^2)$. In this case $f_0(u) = g(u) = \frac{1}{2}u(1-u^2)$. This equation appears in phase transition problem. The stationary problem on a bounded interval is well studied in [17, 19, 20, 21]. In [20] Nakashima showed that there exists of a stable solution with single transition layer near a nondegenerate local minimum point of $a(x)$ and an unstable solution with multiple transition layers near a nondegenerate local maximum point of $a(x)$. In case where $a(x)$ does not degenerate at any local minimum and maximum point, the motion of a transition layer is given by the following proposition.

Proposition 2.1. *Let $b(x) = 1+\eta a(x)$. If $u(0,x)$ is close to $\Phi(\sqrt{b(l)}(x-l)/\varepsilon)$ for some $l \in \mathbb{R}$. Then there exists a function $l(t)$ of $t \geq 0$ such that the solution $u(t,x)$ to (1.1) is close to $\Phi(\sqrt{b(l(t))}(x-l(t))/\varepsilon)$ for $x \in \mathbb{R}$ and $t > 0$ and $l(t)$ satisfies*

$$\frac{dl}{dt} = \varepsilon^2 \frac{d}{dl}(\log b(l)) + O(\varepsilon^3).$$

We note that these results do not need the smallness of η. This proposition is shown in [8] by constructing suitable sub- and supersolution. This results says that the transition layer moves toward the direction where the value of the function $a(x)$ is smaller.

Now we apply our results to this nonlinearity. Since $g'(\pm 1) = -1$, we have

$$\overline{H}_0(l) = -\eta\gamma^2\{A_-\Gamma_-\varepsilon^{1+\alpha_-}e^{-2(L+l)/\varepsilon} + A_+\Gamma_+\varepsilon^{1+\alpha_+}e^{-2(L-l)/\varepsilon}\}.$$

The dynamics of a transition layer on the interval I_d is governed by the equation $\dot{l} = -\varepsilon\overline{H}_0(l)/\|\Phi'\|^2_{L^2}$ and we see this is determined by signs of constants A_- and A_+.

At first we can see that if $A_-A_+ < 0$, then $\overline{H}_0(l)$ has constant sign on I_d. Hence, problem (1.2) does not have stationary solution with transition layer on I_d. The case where $A_- > 0$ and $A_+ < 0$, $H_0(l) < 0$ holds on I_d. Since $\dot{l} > 0$ in this case, the transition layer moves to the right (see Figure 4).

When $A_-A_+ > 0$, a simple calculation similar to subsection 2.1 leads that the zero l^*_ε of $\overline{H}_0(l)$ is given by

$$l^*_\varepsilon := \frac{\alpha_- - \alpha_+}{2}\varepsilon\log\varepsilon + \frac{\varepsilon}{2}\log\frac{A_-\Gamma_-}{A_+\Gamma_+}.$$

From Theorem B, we can say that there exists stationary solution u_ε of (1.2) such that

$$\|u_\varepsilon(\cdot) - \Xi(l^*_\varepsilon)S(\cdot)\|_\infty \leq C_1\delta(l^*_\varepsilon)$$

and the stability is determined by the sign of $\overline{H}'_0(l^*_\varepsilon)$. If $A_- > 0$ and $A_+ > 0$ (resp $A_- < 0$ and $A_+ < 0$), we can see easily that $\overline{H}'_0(l^*_\varepsilon) > 0$ (resp $\overline{H}'_0(l^*_\varepsilon) < 0$) and the stationary

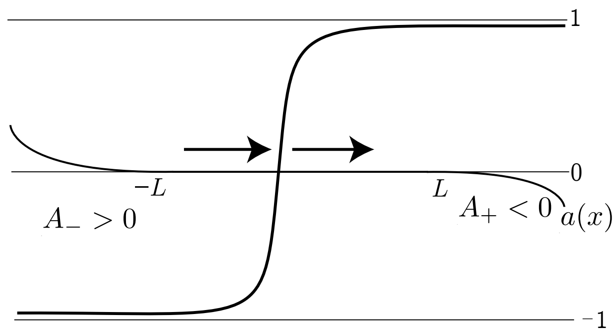

FIGURE 4. The dynamics of transition layer: Case $A_- > 0$, $A_+ < 0$.

solution is stable(resp. unstable). When $A_- > 0$ and $A_+ > 0$ (resp. $A_- < 0$ and $A_+ < 0$) we note that the function $a(x)$ takes local minimum(resp. maximum) on the interval I and our result gives an extension of the result in [20]. More precisely, since $\overline{H}_0(l) < 0$ for $l < l_\varepsilon^*$ and $\overline{H}_0(l) > 0$ for $l > l_\varepsilon^*$ when $A_- > 0$ and $A_+ > 0$, we have $\dot{l} > 0$ for $l < l_\varepsilon^*$ and $\dot{l} < 0$ for $l > l_\varepsilon^*$. Hence the dynamics of the transition layer is as in Figure 5. We can analyze similar situation in the case where $A_- < 0$ and $A_+ < 0$ (see Figure 6).

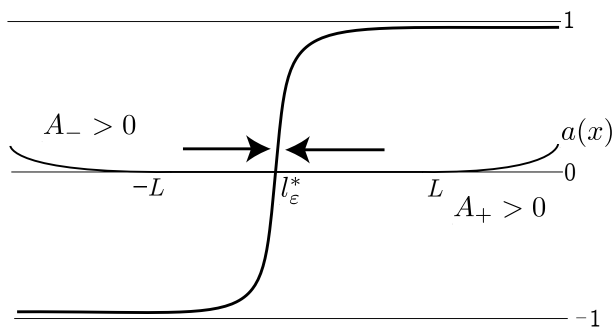

FIGURE 5. The dynamics of transition layer: Case $A_- > 0$, $A_+ > 0$.

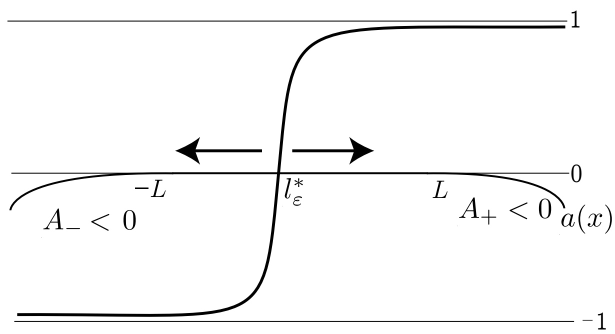

FIGURE 6. The dynamics of transition layer: Case $A_- < 0$, $A_+ < 0$.

3. Proofs of Lemmas 1.2 and 1.3

Proof of Lemma 1.2. Since $\mathcal{L}(x,u) = \mathcal{L}_0(u) + \eta a(x)g(u)$ and $\mathcal{L}_0(S(x)) = 0$, $\mathcal{L}(x+l, S(x)) = \eta a(x+l)g(S(x))$ holds. At first we note that we have

$$g(S) = g'(\pm 1)(S \mp 1) + O((S \mp 1)^2). \tag{3.1}$$

by since $g(\pm 1) = 0$. Since $a(x+l) = 0$ on $[-L-l, L-l]$ we estimete $|\eta a(x+l)g(S(x))|$ on $(-\infty, -(L+l)]$ and $[L-l, +\infty)$. We estimate only on $(-\infty, -(L+l)]$ because we can estimate this on $[L-l, +\infty)$ similar way. We note that for some $r > 0$ the function $a(x+l)$ satisfies $|a(x+l)| \le C|x+l+L|^{1+\alpha_-}$ on $[-(L+l+r), -(L+l)]$. Using this, (1.3), (1.4) and (3.1) we find that on $[-(L+l+r), -(L+l)]$

$$\begin{aligned} |a(x+l)g(S)| &\le C|x+l+L|^{1+\alpha_-}\{|g'(-1)|(S+1) + O((S+1)^2)\} \\ &\le C'|x+l+L|^{1+\alpha_-}e^{x/\varepsilon}. \end{aligned}$$

Hence

$$\begin{aligned} \sup_{x\in[-(L+l+r),-(L+r)]} |a(x+l)g(S)| &\le C' \sup_{x\in[-(L+l+r),-(L+l)]} |x+l+L|^{1+\alpha_-}e^{x/\varepsilon} \\ &= C' \sup_{t\in[0,r]} t^{1+\alpha_-}e^{-t/\varepsilon}e^{-(L+l)/\varepsilon} \\ &= C''\varepsilon^{1+\alpha_-}e^{-(L+l)/\varepsilon}. \end{aligned}$$

On the other hand we have

$$\sup_{x\in(-\infty,-(L+l+r)]} |a(x+l)g(S)| \le Ce^{-(L+l)/\varepsilon}e^{-r/\varepsilon}.$$

We can estimate $|a(x+l)g(S)|$ on $[L-l, +\infty)$ similarly and this complete the proof of Lemma 1.2. □

Proof of Lemma 1.3. Since $H_0(l) = \langle \mathcal{L}(\cdot + l, S(\cdot)), S_x(\cdot)\rangle = \langle \eta a(\cdot + l)g(S(\cdot)), S_x(\cdot)\rangle_{L^2}$, we compute $\langle \eta a(x+l)g(S(x)), S_x(x)\rangle_{L^2}$ as follows.

$$\begin{aligned} &\langle \eta a(\cdot + l)g(S(\cdot)), S_x(\cdot)\rangle_{L^2} \\ = &\int_{-\infty}^{\infty} \eta a(x+l)g(S(x))S_x(x)dx \\ = &-\int_{-\infty}^{-L-l} a(x+l)g(S(x))S_x(x)dx - \int_{L-l}^{\infty} \eta a(x+l)g(S(x))S_x(x)dx \\ = &\int_{-\infty}^{-L-l} a(x+l)\{g'(-1)(S+1) + O((S+1)^2)\}S_x(x)dx \\ &\qquad + \int_{L-l}^{\infty} a(x+l)\{g'(1)(S-1) + O((S-1)^2)\}S_x(x)dx. \end{aligned}$$

We decompose above two integrals as

$$\begin{aligned} \int_{-\infty}^{-L-l} + \int_{L-l}^{\infty} &= \int_{-\infty}^{-L-l-r} + \int_{-L-l-r}^{-L-l} + \int_{L-l}^{L-l+r} + \int_{L-l+r}^{\infty} \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

for small fixed $r>0$. First we estimate integrals I_1 and I_4. Using asymptotic forms for S and S_x (see (1.3), (1.4)), we have

$$\begin{aligned} g(S(x))S_x(x) &= \{\gamma g'(-1)e^{x/\varepsilon}+O(e^{2x/\varepsilon})\}\left\{\frac{\gamma}{\varepsilon}e^{x/\varepsilon}+O\left(\frac{1}{\varepsilon}e^{2x/\varepsilon}\right)\right\} \\ &= \frac{\gamma^2}{\varepsilon}g'(-1)e^{2x/\varepsilon}+O\left(\frac{1}{\varepsilon}e^{3x/\varepsilon}\right). \end{aligned} \tag{3.2}$$

Substituting (3.2) into the integral I_1, we have

$$I_1 = -\int_{-\infty}^{-L-l-r}\eta a(x+l)\left\{\frac{\gamma^2}{\varepsilon}g'(-1)e^{2x/\varepsilon}+O\left(\frac{1}{\varepsilon}e^{3x/\varepsilon}\right)\right\}dx.$$

Since $a(x)$ is a bounded function, we can estimate

$$|I_1|\le\frac{C}{\varepsilon}\int_{-\infty}^{-L-r}e^{2(x-l)/\varepsilon}dx=Ce^{2(-L-l-r)/\varepsilon}=Ce^{-2(L+l)/\varepsilon}e^{-2r/\varepsilon}.$$

Similarly we can show

$$|I_4|\le Ce^{-2(L-l)/\varepsilon}e^{-2r/\varepsilon}.$$

Next we compute I_2 and I_3. Substituting (3.2) into the integral I_2 and using the assumption on $a(x)$, we obtain

$$I_2 = \int_{-L-l-r}^{-L-l}\eta\left\{A_-|x+L+l|^{1+\alpha_-}+O(|x+L+l|^{2+\alpha_-})\right\}\left\{\frac{\gamma^2}{\varepsilon}e^{2x/\varepsilon}+O\left(\frac{1}{\varepsilon^2}e^{3x/\varepsilon}\right)\right\}dx.$$

The substitution $-L-l-x=t$ to the integral I_2, leads

$$\begin{aligned} I_2 &= -\int_r^0\eta\left\{A_-t^{1+\alpha_-}+O(t^{2+\alpha_-})\right\}\left\{\frac{\gamma^2}{\varepsilon}g'(-1)e^{-2(L+l)/\varepsilon}e^{-2t/\varepsilon}+O\left(\frac{1}{\varepsilon^2}e^{-3(l+L)/\varepsilon}e^{-3t/\varepsilon}\right)\right\}dt \\ &= \int_0^r\eta\left\{A_-t^{1+\alpha_-}+O(t^{2+\alpha_-})\right\}\left\{\frac{\gamma^2}{\varepsilon}g'(-1)e^{-2(l+L)/\varepsilon}e^{-2t/\varepsilon}+O\left(\frac{1}{\varepsilon^2}e^{-3(l+L)/\varepsilon}e^{-3t/\varepsilon}\right)\right\}dt \end{aligned}$$

Next the substitution $2t=\varepsilon s$ and note that $l\in I_d$ implies

$$\begin{aligned} I_2 &= \int_0^{2r/\varepsilon}\eta\left\{A_-\left(\frac{\varepsilon s}{2}\right)^{1+\alpha_-}+O\left(\left(\frac{\varepsilon s}{2}\right)^{2+\alpha_-}\right)\right\} \\ &\qquad\times\left\{\frac{\gamma^2}{\varepsilon}g'(-1)e^{-2(l+L)/\varepsilon}e^{-s}+O\left(\frac{1}{\varepsilon}e^{-3(l+L)/\varepsilon}e^{-3s/2}\right)\right\}\frac{\varepsilon}{2}ds \\ &= \eta\frac{\gamma^2}{2^{2+\alpha_-}}A_-g'(-1)e^{-2(l+L)/\varepsilon}\varepsilon^{1+\alpha_-}\left(\int_0^{2r/\varepsilon}s^{1+\alpha_-}e^{-s}ds+O(\varepsilon)\right)+O(e^{-2(l+L)/\varepsilon})e^{-c_0/\varepsilon} \\ &= \eta\frac{\gamma^2}{2^{2+\alpha_-}}A_-g'(-1)e^{-2(l+L)/\varepsilon}\varepsilon^{1+\alpha_-}(\Gamma(2+\alpha_-)+O(\varepsilon))+O(e^{-2(L+l)/\varepsilon})e^{-c_0/\varepsilon} \end{aligned}$$

for some $c_0>0$. Similarly, it is shown that

$$I_3=-\eta A_+g'(1)\frac{\gamma^2}{2^{2+\alpha_+}}e^{-2(L-l)/\varepsilon}\varepsilon^{1+\alpha_+}(\Gamma(2+\alpha_+)+O(\varepsilon))+O(e^{-2(L-l)/\varepsilon})e^{-c_0/\varepsilon}$$

Thus we obtain

$$\begin{aligned} &\langle a(\cdot+l)g(S(\cdot)),S_x(\cdot)\rangle_{L^2} \\ =\ &\eta\gamma^2\Bigg\{A_-g'(-1)\left(\frac{\Gamma(2+\alpha_-)}{2^{2+\alpha_-}}+O(\varepsilon)\right)\varepsilon^{1+\alpha_-}e^{-2(l+L)/\varepsilon} \end{aligned}$$

$$-A_+g'(1)\left(\frac{\Gamma(2+\alpha_+)}{2^{2+\alpha_+}}+O(\varepsilon)\right)\varepsilon^{1+\alpha_+}e^{-2(L-l)/\varepsilon}\Bigg\}$$
$$+O(e^{-c_0/\varepsilon}e^{-(l+L)/\varepsilon}+e^{-c_0/\varepsilon}e^{-(L-l)/\varepsilon}).$$

This implies Lemma 1.3. □

4. Proofs of Main Theorems

In this section, we prove Theorems A and B.

Let a differential operator $\mathfrak{L}$ be

$$\mathfrak{L}v=\varepsilon^2v_{xx}+f_0'(S(x))v$$

for $v\in H^2(\mathbb{R})$. It is well known that the followng propositon holds.

Proposition 4.1. *The spectrum $\Sigma(\mathfrak{L})$ of $\mathfrak{L}$ is given by $\Sigma(\mathfrak{L})=\Sigma_0\cup\{0\}$, where 0 is a simple eigenvalue and there exists a positive constant $\rho_0>0$ (independent of ε) such that $\Sigma_0\subset(-\infty,-\rho_0)$. Moreover the eigenfunction ϕ of 0 eigenvalue is denoted by $\phi(x)=c_1\varepsilon S_x$ for some constant c_1.*

We fix constant c_1 so that $\langle S_x,\phi\rangle_{L^2}=1$ holds.

Let E be the eigenspace corresponding to 0 eigenvalue of the operator of $\mathfrak{L}$. Let operetors Q and R be the projection from X to E and $R=Id-Q$ respectively, where Id is the identity on X. Let $E^\perp=RX$. Note that Q is represented by

$$Qv=\frac{\langle v,S_x\rangle_{L^2}}{\langle S_x,S_x\rangle_{L^2}}S_x$$

and $E^\perp$ is characterized by

$$E^\perp=\{v\in X:\langle v,S_x\rangle_{L^2}=0\}.$$

Let X^ω be the space with the norm $\|\cdot\|_\omega$ defined by the fractional power $\mathfrak{L}^\omega$ of $\mathfrak{L}$ for $\omega\in[0,1)$. Hereafter, we fix ω in $\frac{3}{4}<\omega<1$ such that X^ω is imbedded into $BU^1(\mathbb{R})$ (see [15]), where $BU^k(\mathbb{R})$ is the space consisting of uniformly continuous and bounded function on $\mathbb{R}$ up to their k th order derivatives.

We set $X_0:=QX^\omega$ and $X_1:=RX^\omega$.

Lemma 4.2. *There exists a neighborhood U of $\mathcal{M}$ in X^ω such that any $v\in U$ is represented by*

$$v=\Xi(l)\{S(x)+w\}$$

for $l\in I_d$ and $w\in E^\perp$.

Proof. Fix $l_0\in I_d$ arbitrarily and put $v_0=\Xi(l_0)S(x)$. We will show that for sufficiently small v, there exist $l\in I_d$ near l_0 and $w\in E^\perp$ near 0 such that

$$v+v_0=\Xi(l)\{S(\cdot)+w\}.$$

For this, it suffices to show

$$\Xi(-l)(v+v_0)-S(\cdot)\in E^\perp.$$

This is equivalent to

$$\begin{aligned}0&=\langle\Xi(l)(v+v_0)-S(\cdot),\phi\rangle_{L^2}\\&=\langle v+v_0-\Xi(l)S(\cdot),\Xi(l)\phi\rangle_{L^2}\end{aligned}$$

Hence we define

$$V(l,v) = \langle v + v_0 - \Xi(l)S(\cdot), \Xi(l)\phi\rangle_{L^2}$$

and apply the implicit function theorem to the map V. First we note $V(l_0, 0) = 0$ holds. On the other hand, Proposition 4.1 shows $\langle S_x, \phi\rangle_{L^2} = 1$. By this and the fact $\Xi'(l) = -\Xi(l)\partial_x$, we have

$$\frac{\partial V}{\partial l}(l_0, 0) = \langle \Xi(l_0)S_x(\cdot), \Xi(l_0)\phi(\cdot)\rangle_{L^2} = \langle S_x, \phi\rangle_{L^2} = 1.$$

By implicit function theorem, there exist $l = l(v)$ for small v such that $V(l(v), v) = 0$. $\square$

We transform the equation (1.2) of u to that of (w,l) by

$$u(t,x) = \Xi(l)\{S(x) + w\}$$

for $l \in \mathbb{R}$, $w \in E^\perp$. Since $\Xi'(l) = -\Xi(l)\partial_x$ holds, we have

$$u_t = \dot{l}\Xi'(l)\{S(x) + w\} + \Xi(l)w_t = -\dot{l}\Xi(l)\{S_x(x) + w_x\} + \Xi(l)w_t$$

and

$$\begin{aligned}
\mathcal{L}(x,u) &= \mathcal{L}(x, \Xi(l)\{S(x)+w\}) = \mathcal{L}_0(\Xi(l)\{S(x)+w\}) + a(x)g(\Xi(l)\{S(x)+w\}) \\
&= \Xi(l)\mathcal{L}_0(S(x)+w) + \Xi(l)\eta a(x+l)g(S(x)+v) \\
&= \Xi(l)\mathfrak{L}w + +\Xi(l)\eta a(x+l)g(S(x)) + \Xi(l)a(x+l)g'(S(x))w + \Xi(l)G(w) \\
&= \Xi(l)\{\mathfrak{L}w + \eta a(x+l)g(S(x)) + \eta a(x+l)g'(S(x)) + G(w)\},
\end{aligned}$$

where

$$\begin{aligned}
G(w) &= f_0(S(x)+w) - f_0(S(x)) - f_0'(S(x))w \\
&\quad + \eta a(x+l)\{g(S(x)+w) - g(S(x)) - g'(S(x))w\}
\end{aligned}$$

and G satisfies

$$|G(w)| \le C|w|^2 \tag{4.1}$$

for some positive constant C. Hence, it follows that

$$-\dot{l}\{S_x(x) + w_x\} + w_t = \mathfrak{L}w + \eta a(x+l)g(S(x)) + \eta a(x+l)g'(S(x))w + G(w)$$

and we have

$$Q[-\dot{l}\{S_x + w_x\}] = Q\{\mathfrak{L}w + \eta a(\cdot+l)g(S) + \eta a(\cdot+l)g'(S)w + G(w)\} \tag{4.2}$$

and

$$R[-\dot{l}\{S_x + w_x\}] + w_t = R\{\mathfrak{L}w + \eta a(\cdot+l)g(S) + \eta a(\cdot+l)g'(S)w + G(w)\}. \tag{4.3}$$

Put

$$\widetilde{W}(D_1) = \{w(\cdot) \in C(I_d; X_0); \|w(l)\|_\omega \le D_1\delta(l)\}.$$

We determine D_1 later. First we consider (4.2). It follows that if $\|w\|_\omega \le D_1\delta(l)$, then we have

$$\begin{aligned}
&-\dot{l}\{\langle S_x, S_x\rangle_{L^2} + \langle w_x, S_x\rangle_{L^2}\} \qquad (4.4)\\
&\qquad = \langle \mathfrak{L}w, S_x\rangle_{L^2} + \eta\langle a(\cdot+l)g(S), S_x\rangle_{L^2} + \eta\langle a(\cdot+l)g'(S)w, S_x\rangle_{L^2} + \langle G(w), S_x\rangle_{L^2}.
\end{aligned}$$

Since $\langle \mathfrak{L}w, S_x\rangle_{L^2} = 0$ and $H_0(l) = \eta\langle a(\cdot+l)g(S), S_x\rangle_{L^2}$ we have

$$\dot{l} = -\frac{H_0(l) + \eta\langle a(\cdot+l)g'(S)w, S_x\rangle_{L^2} + \langle G(w), S_x\rangle_{L^2}}{\langle S_x, S_x\rangle_{L^2} + \langle w_x, S_x\rangle_{L^2}}. \tag{4.5}$$

We set $H_0(l,w)$ right-hand-side of (4.5) and we can easily seen

$$H_0(l,w) = -\frac{\varepsilon H_0(l)}{\|\Phi\|_{L^2}^2} + O(\eta|w|\delta\varepsilon + \varepsilon^2|w|\delta + \varepsilon|w|^2). \tag{4.6}$$

Similarly, it follows from (4.3) that

$$w_t = \mathfrak{L}w + \widetilde{G}(l,w) \tag{4.7}$$

with $\|\widetilde{G}\| = O(\delta(l))$ for $l \in I_d$ and $w \in \widetilde{W}(D_1)$, where

$$\widetilde{G}(l,w) = R\{\eta a(\cdot + l)g(S) + \eta a(\cdot + l)g'(S)w + G(w)\} + H_0(l,w)Rw_x.$$

For convenience, we define folowing quantity

$$\delta_{p_1,p_2,q}(l) = \varepsilon^{p_1}e^{-\frac{q(l+L)}{\varepsilon}} + \varepsilon^{p_2}e^{-\frac{q(L-l)}{\varepsilon}}.$$

for $p_i > 0$ $(i = 1,2)$, $q > 0$ and $l \in I_d$. We note that

$$\delta(l) = O(\delta_{1+\alpha_-,1+\alpha_+,1}(l)),\ H_0(l) = O(\delta_{1+\alpha_-,1+\alpha_+,2}(l))$$

hold.

We can easily see there exist $C_3 > 0$ (independent of D_1) and $C_4 = C_4(D_1)$ (dependent on D_1) in (4.6) and (4.7) that

$$|H_0(l,w)| \le C_3\delta(l), \tag{4.8}$$

$$\begin{aligned}&|H_0(l,w) - H_0(k,v)| \\ &\qquad \le C_4\{\delta_{1+\alpha_-,1+\alpha_+,2}(l) + \delta_{1+\alpha_-,1+\alpha_+,2}(k)\}|l-k| + \delta(l)\|w-v\|_\omega,\end{aligned} \tag{4.9}$$

$$\|\widetilde{G}(l,w)\| \le C_3\delta(l)\{\eta D_1 + D_1^2\delta(l) + 1\}, \tag{4.10}$$

$$\|\widetilde{G}(l,w) - \widetilde{G}(k,v)\| \le C_4\left\{\frac{\delta(l)+\delta(k)}{\varepsilon}|l-k| + \eta\|w-v\|_\omega\right\} \tag{4.11}$$

hold for $l,\ k \in I_d$ and $v,\ w \in X^\omega$ with $\|v\|_\omega,\ \|w\|_\omega \le D_1\delta(l)$. We extend $\delta(l)$ for $l \notin I_d$ such that $\delta(l) \le \delta^* = \max_{t\in I_d}\delta(t) = O((\varepsilon^{1+\alpha_-} + \varepsilon^{1+\alpha_+})e^{-d/\varepsilon})$ and also extend H_0 and G approproately to the outside of I_d so that (4.8) to (4.11) hold for ant $l,\ k \in \mathbb{R}$.

We shall construct an attractive invariant manifold of

$$\begin{cases} l_t &= H_0(l,w), \\ w_t &= \mathfrak{L}w + \widetilde{G}(l,w) \end{cases} \tag{4.12}$$

for $l \in \mathbb{R}$ and $w \in \widetilde{W}(D_1)$. Although the method of construction of the invariant manifold is similar to the method in [6], we state proof in detail for reader's convenience. Define

$$\begin{aligned}\widetilde{W}(D_1,D_2) &= \{w \in C(\mathbb{R};X_1); \|w(l)\| \le D_1\delta(l), \\ &\qquad \|w(l) - w(k)\| \le D_2\frac{\delta(l)+\delta(k)}{\varepsilon}|l-k|, \text{ for } l,k \in \mathbb{R}\}\end{aligned}$$

for a positive constant D_2. For $\sigma \in \widetilde{W}(D_1,D_2)$, define $l(t) = l(t;\xi,\sigma)$ by the solution of

$$\begin{cases} l_t &= H_0(l,\sigma(l)), \\ l(0) &= \xi \in \mathbb{R} \end{cases} \tag{4.13}$$

and define $T(t,s)$ by the evolution operator of

$$w_t = \mathfrak{L}w. \tag{4.14}$$

We have following lemma about the evolution oparator $T(t,s)$.

Lemma 4.3. *There exist positive constants C_5 and γ_1 independent of D_1 such that $\|T(t,s)w\|_\omega \le C_5 \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)}\|w\|$ holds for $w \in E^\perp$.*

Proof. See Theorem 7.4.2 of [15]. □

For $l(\cdot) \in C^1(\mathbb{R};\mathbb{R})$, consider a bounded solution of

$$w_t = \mathfrak{L}w + \widetilde{G}(l(t), w), \ -\infty < t < \infty. \tag{4.15}$$

Lemma 4.4. *For small $\eta > 0$, there exists a constant D_1 such that a bounded solution of (4.15), say $w(t, l(\cdot)))$ uniquely exists satisfying*

$$\|w(t, l(\cdot))\|_\omega \le D_1 \delta(l(t)).$$

Proof. Let $v(t)$ be a function satisfying

$$\|v(t)\|_\omega \le D_1\delta(l(t)), \quad -\infty < t < \infty$$

and consider a bounded solution of

$$w_t = \mathfrak{L}w + \widetilde{G}(l(t), v(t)). \tag{4.16}$$

Solution of (4.16) is represented as

$$w(t) = T(t,s)w(s) + \int_s^t T(t,\zeta)\widetilde{G}(l(\zeta), v(\zeta))d\zeta.$$

Since $\|w(t)\|_\omega$ is bounded as $t \to -\infty$, $w(t)$ satisfies

$$w(t) = \int_{-\infty}^t T(t,s)\widetilde{G}(l(s), v(s))ds. \tag{4.17}$$

Let $W(t; v(\cdot))$ be the right hand side of (4.17). Then we have from Lemma 4.3

$$\begin{aligned}
&\|W(t;v(\cdot))\|_\omega \\
&\le \int_{-\infty}^t C_5 \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)} \|\widetilde{G}(l(s), v(s))\| ds \\
&\le \int_{-\infty}^t C_5 \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)} C_3 \delta(l(s))\{\eta D_1 + D_1^2\delta(l(s))) + 1\} \\
&= \int_{-\infty}^t C \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)} \delta(l(s))ds \cdot \{\eta D_1 + D_1^2\delta^* + 1\} \\
&= C\int_0^\infty \max\{s^{-\omega}, 1\} e^{-\gamma_1 s}\delta(l(t-s))ds \cdot \{\eta D_1 + D_1^2\delta^* + 1\} \\
&= C\int_0^\infty \max\{s^{-\omega}, 1\} e^{-\gamma_1' s}\{e^{-(\gamma_1-\gamma_1')s}\delta(l(t-s))ds \cdot \{\eta D_1 + D_1^2\delta^* + 1\} \\
&\le C\int_0^\infty \max\{s^{\omega}, 1\} e^{-\gamma_1' s} ds \cdot \delta(l(t))\{\eta D_1 + D_1^2\delta^* + 1\}
\end{aligned} \tag{4.18}$$

for a positive constant γ_1' with $0 < \gamma_1' < \gamma_1$. Here, we used the monotone decrement of $e^{-(\gamma_1-\gamma_1')s}\delta(l(t-s))$ with respect to s, which is due to $\frac{d}{ds}\delta(l(t-s)) = O(\delta^2(l(t-s))) \le \delta(l(t-s))O(\delta^*)$. Hence, we take D_1 so large and ε and η so small that

$$C(\eta D_1 + D_1^2\delta^* + 1) \le D_1$$

and we have

$$\|W(t; v(\cdot))\|_\omega \le D_1\delta(l(t)). \tag{4.19}$$

Let $\widetilde{W}(D_1) = \{w \in C(\mathbb{R}; X_1); \|w(t)\|_\omega \le D_1\delta(l(t))\}$. Then, (4.19) shows W is a map from $\widetilde{W}(D_1)$ into $\widetilde{W}(D_1)$.

Now, we shall show W is a contraction on $\widetilde{W}(D_1)$.

$$\begin{aligned}
&\|W(t; w(\cdot)) - W(t; v(\cdot))\|_\omega \\
&\int_{-\infty}^{t} \|T(t,s)\{\widetilde{G}(l(s), w(s)) - \widetilde{G}(l(s), v(s))\}\|_\omega ds \\
&\le C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)} \|\widetilde{G}(l(s), w(s)) - \widetilde{G}(l(s), v(s))\|_\omega ds \\
&\le C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)} \eta \|w(s) - v(s)\|_\omega ds \\
&\le C\eta \int_{0}^{\infty} \max\{s^{-\omega}, 1\} e^{-\gamma_1 s} ds \cdot \sup_t \|w(t) - v(t)\|_\omega \\
&\le C\eta \sup_t \|w(t) - v(t)\|_\omega.
\end{aligned}$$

This shows W is a contraction on $\widetilde{W}(D_1)$ if η is small enough, which completes the proof. ☐

Fix D_1 such that Lemma 4.4 holds.

Define

$$J(\sigma)(\xi) = w(0; l(\cdot; \xi, \sigma))$$

for $\xi \in \mathbb{R}$ and $\sigma \in \widetilde{W}(D_1, D_2)$. Then

$$\|J(\sigma)(\xi)\|_\omega \le D_1\delta(l(0; \xi, \sigma)) = D_1\delta(\xi) \tag{4.20}$$

holds by the definition and Lemma 4.4.

We shall estimate $\|J(\sigma)(\xi_2) - J(\sigma)(\xi_1)\|_\omega$ for $\xi_1, \xi_2 \in \mathbb{R}$ and $\sigma \in \widetilde{W}(D_1, D_2)$.

Lemma 4.5. *If $l_1, l_2 \in C^1(\mathbb{R}; \mathbb{R})$ satisfy*

$$|l_2(t) - l_1(t)| \le \zeta e^{\gamma_2 \delta^* |t|}$$

for positive constants ζ and γ_2, then

$$\|w(t; l_2(\cdot)) - w(t; l_1(\cdot))\|_\omega \le \frac{\delta(l_1(t)) + \delta(l_2(t))}{\varepsilon} C\zeta e^{\gamma_2 \delta^* |t|}$$

holds.

Proof. Let $w_j(t) = w(t; l_j(\cdot))$ $(j = 1, 2)$. Since w_j satisfy

$$w_j(t) = \int_{-\infty}^{t} T(t,s)\widetilde{G}(l_j(s), w_j(s)) ds,$$

we have by using similar arguments to (4.3)

$$\begin{aligned}
&\|w_2(t) - w_1(t)\|_\omega \\
&\le \int_{-\infty}^{t} \|T(t,s)\{\widetilde{G}(l_2(s), w_2(s)) - \widetilde{G}(l_1(s), w_1(s))\}\|_\omega ds \\
&\le C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma_1(t-s)} \|\widetilde{G}(l_2(s), w_2(s)) - \widetilde{G}(l_1(s), w_1(s))\|_\omega ds,
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\\
&\quad\left[\left\{\frac{\delta(l_1(t-s))+\delta(l_2(t-s))}{\varepsilon}\right\}|l_2(s)-l_1(s)|+\eta\|w_2(s)-w_1(s)\|_\omega\right]ds\\
&\le C\frac{\delta(l_1(t))+\delta(l_2(t))}{\varepsilon}\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1'(t-s)}\zeta e^{\gamma_2\delta^*|s|}ds\\
&\quad+C\eta\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}\|w_2(s)-w_1(s)\|_\omega\\
&\le C\frac{\delta(l_1(t))+\delta(l_2(t))}{\varepsilon}\zeta e^{\gamma_2\delta^*|t|}\int_{0}^{\infty}\max\{s^{-\omega},1\}e^{-(\gamma_1'-\gamma_2\delta^*)s}ds\\
&\quad+C\eta\int_{-\infty}^{t}\max\{(t-s)^{-\omega},1\}e^{-\gamma_1(t-s)}e^{\gamma_2\delta^*|s|}ds\cdot\sup_t(e^{-\gamma_2\delta^*|t|}\|w_2(t)-w_1(t)\|_\omega)\\
&\le C\frac{\delta(l_1(t))+\delta(l_2(t))}{\varepsilon}\zeta e^{\gamma_2\delta^*|t|}\int_{0}^{\infty}\max\{s^{-\omega},1\}e^{-(\gamma_1'-\gamma_2\delta^*)s}ds\\
&\quad+C\eta e^{\gamma_2\delta^*|t|}\int_{0}^{\infty}\max\{s^{-\omega},1\}e^{-(\gamma_1-\gamma_2\delta^*)s}ds\cdot\sup_t(e^{-\gamma_2\delta^*|t|}\|w_2(t)-w_1(t)\|_\omega)\\
&\le Ce^{\gamma_2\delta^*|t|}\left\{\frac{\delta(l_1(t))+\delta(l_2(t))}{\varepsilon}\zeta+\eta\sup_t(e^{-\gamma_2\delta^*|t|}\|w_2(t)-w_1(t)\|_\omega)\right\}
\end{aligned}
$$

for a positive constant $0<\gamma_1'<\gamma_1$. This yields

$$\sup_t(e^{-\gamma_2\delta^*|t|}\|w_2(t)-w_1(t)\|_\omega)\le C\left\{\frac{\delta(l_1(t))+\delta(l_2(t))}{\varepsilon}\zeta+\eta\sup_t(e^{-\gamma_2\delta^*|t|}\|w_2(t)-w_1(t)\|_\omega)\right\},$$

which complete proof by taking η small enough. □

Let $l_j(t)=l(t;\xi_j,\sigma)$ $(j=1,2)$ for $\xi_j\in\mathbb{R}$ and $\sigma\in\widetilde{W}(D_1,D_2)$. Let $w_j(t)=\sigma(l_j(t))$.

Lemma 4.6. *$l_j(t)$ $(j=1,2)$ defined above satisfy*

$$|l_2(t)-l_1(t)|\le e^{\delta^*C_4(1+2\frac{\delta^*}{\varepsilon}D_2)|t|}|\xi_2-\xi_1|.$$

Proof. Let $l_3(t)=l_2(t)-l_1(t)$. From (4.9), we have

$$\left|\frac{d}{dt}l_3\right|\le\delta^*C_4\{|l_3(t)|+\|w_2(t)-w_1(t)\|_\omega\}.$$

Since

$$
\begin{aligned}
\|w_2(t)-w_1(t)\|_\omega &= \|\sigma(l_2(t))-\sigma(l_1(t))\|_\omega\\
&\le \frac{2\delta^*}{\varepsilon}D_2|l_3(t)|
\end{aligned}
$$

holds, it follows that

$$\left|\frac{d}{dt}l_3\right|\le\delta^*C_4\left(1+\frac{2\delta^*}{\varepsilon}D_2\right)|l_3|.$$

This gives the proof. □

From Lemmas 4.5 and 4.6,

$$\|w(t;l_2(\cdot))-w(t;l_1(\cdot))\|_\omega\le\frac{\delta(l_2(t))+\delta(l_1(t))}{\varepsilon}C|\xi_2-\xi_1|e^{\delta^*C_4(1+\frac{2\delta^*}{\varepsilon}D_2)|t|}$$

holds. Therefore, we have

$$\begin{aligned}\|J(\sigma)(\xi_2)-J(\sigma)(\xi_1)\|_\omega &= \|w(0;l_2(\cdot))-w(0;l_1(\cdot))\|_\omega, \\ &\le \frac{\delta(\xi_2)+\delta(\xi_1)}{\varepsilon}C|\xi_2-\xi_1|, \\ &\le D_2\frac{\delta(\xi_2)+\delta(\xi_1)}{\varepsilon}|\xi_2-\xi_1|,\end{aligned} \tag{4.21}$$

by taking D_2 appropriately large. (4.20) and (4.21) imply that

$$J(\sigma):\widetilde{W}(D_1,D_2)\to\widetilde{W}(D_1,D_2). \tag{4.22}$$

We shall show J is a contraction map on $\widetilde{W}(D_1,D_2)$. For given σ_1, $\sigma_2\in\widetilde{W}(D_1,D_2)$ and $\xi\in\mathbb{R}^N$, we let $l_j(t)=l(t;\xi,\sigma_j)$ and $w_j(t)=\sigma(l_j(t))$ $(j=1,2)$.

Lemma 4.7.

$$|l_2(t)-l_1(t)|\le\frac{1}{1+2\frac{\delta^*}{\varepsilon}D_2}|||\sigma_2-\sigma_1|||e^{\delta^*C_4(1+2\frac{\delta^*}{\varepsilon}D_2)|t|}$$

holds, where $|||\cdot|||$ *denotes* $|||\sigma|||=\sup_{\xi\in\mathbb{R}^N}\|\sigma(\xi)\|_\omega$ *for* $\sigma\in\widetilde{W}(D_1,D_2)$.

Proof. Let $l_3(t)=l_2(t)-l_1(t)$. Then, from (4.9), we have

$$\begin{aligned}\left|\frac{d}{dt}l_3\right| &\le C_4\delta^*\{|l_3|+\|w_2(t)-w_1(t)\|_\omega\} \\ &= C_4\delta^*\{|l_3|+\|\sigma_2(l_2(t))-\sigma_1(l_1(t))\|_\omega\}.\end{aligned} \tag{4.23}$$

Here

$$\begin{aligned}|||\sigma_2(l_2(t))-\sigma_1(l_1(t))||| &\le |||\sigma_1(l_2(t))-\sigma_1(l_1(t))|||_\omega \\ &\quad +|||\sigma_2(l_2(t))-\sigma_1(l_2(t))|||_\omega \\ &\le D_2\frac{\delta(l_2(t))+\delta(l_1(t))}{\varepsilon}|l_2-l_1|+|||\sigma_2-\sigma_1||| \\ &\le \frac{2\delta^*}{\varepsilon}D_2|l_2-l_1|+|||\sigma_2-\sigma_1|||\end{aligned}$$

hold. Substituting this into (4.23), we have

$$\left|\frac{d}{dt}l_3\right|\le C_4\delta^*\left(1+\frac{2\delta^*}{\varepsilon}\right)|l_3|+C_4\delta^*|||\sigma_2-\sigma_1|||.$$

By using Gronwall's inequality, this yields

$$\begin{aligned}|l_3(t)| &\le \int_0^{|t|}e^{C_4\delta^*\left(1+\frac{2\delta^*}{\varepsilon}D_2\right)(|t|-s)}ds\cdot C_4\delta^*|||\sigma_2-\sigma_1||| \\ &\le \frac{1}{1+\frac{2\delta^*}{\varepsilon}D_2}\left(e^{C_4\delta^*\left(1+\frac{2\delta^*}{\varepsilon}D_2\right)|t|}-1\right)|||\sigma_2-\sigma_1||| \\ &\le \frac{1}{1+\frac{2\delta^*}{\varepsilon}D_2}|||\sigma_2-\sigma_1|||e^{C_4\delta^*\left(1+\frac{2\delta^*}{\varepsilon}D_2\right)|t|}.\end{aligned}$$

□

Lemmas 4.5 and 4.7 imply that

$$\|w(t;l_2(\cdot))-w(t;l_1(\cdot))\|_\omega\le\frac{\frac{2\delta^*}{\varepsilon}C}{1+2\frac{\delta^*}{\varepsilon}D_2}|||\sigma_2-\sigma_1|||.$$

This directly shows

$$\begin{aligned}\|J(\sigma_2)(\xi)-J(\sigma_1)(\xi)\|_\omega &= \|w(0;l_2(\cdot))-w(0;l_1(\cdot))\| \\ &\le \frac{\frac{2\delta^*}{\varepsilon}C}{1+2\frac{\delta^*}{\varepsilon}D_2}|||\sigma_2-\sigma_1||| \\ &\le \frac{2\delta^*}{\varepsilon}C|||\sigma_2-\sigma_1|||.\end{aligned}$$

Hence, J is a contraction and there uniquely exists $\widehat{\sigma}\in\widetilde{W}(D_1,D_2)$ satisfying $J(\widehat{\sigma})=\widehat{\sigma}$.

Let $\widehat{\mathcal{M}}=\{\Xi(l)[S(x)+\widehat{\sigma}(l)];l\in I_d\}$. Then, from the construction of $\widehat{\sigma}$, we can easily show that $\widehat{\mathcal{M}}$ is positively invariant with respect to the flow of (1.2) as long as $l(t)\in I_d$, where $l(t)$ is as solution of (4.13) with $\sigma=\widehat{\sigma}$. Smoothness and an exponential attractivity of $\widehat{\mathcal{M}}$ together with the asymptotic phase are also shown in quite similar manner to Section 9 of [15]. Here, we note that the attractivity is determined only by the estimate of semigroup $e^{-\mathfrak{L}t}$ (see Proposition 4.1). That is, we have following theorem.

Theorem 4.8. *There exist positive constants ρ_1, γ_3, η_1 and M_3 such that any $0<\eta<\eta_1$ and any $u(0,\cdot)$ with $\operatorname{dist}(u(0,\cdot),\widehat{\mathcal{M}})<\rho_1$, there exist function $l(t)$ which is solution of (4.12) with $w=\widehat{\sigma}(l)$ such that*

$$\|u(t,\cdot)-\Xi(l(t))\{S(\cdot)+\widehat{\sigma}(l(t))\}\|_\omega\le M_3e^{-\gamma_3t}\operatorname{dist}\{u(0,\cdot),\widehat{\mathcal{M}}\}$$

holds as long as $l(t)\in I_d$, where $u(t,x)$ is a solution of (1.2).

By using this theorem, we show Theorems A and B.

Proof of Theorem A. From Theorem 4.8 we have

$$\|u(t,\cdot)-\Xi(l(t))\{S(\cdot)+\widehat{\sigma}(l(t))\}\|_\omega\le M_3e^{-\gamma_3t}\operatorname{dist}\{u(0,\cdot),\widehat{\mathcal{M}}\}.$$

Using this we obtain

$$\begin{aligned}\|u(t,\cdot)-\Xi(l(t))S(\cdot)\| &\le M_3e^{-\gamma_3t}\operatorname{dist}\{u(0,\cdot),\widehat{\mathcal{M}}\}+\|\Xi(l(t))\widehat{\sigma}(l(t))\|_\omega \\ &\le M_3e^{-\gamma_3t}\operatorname{dist}\{u(0,\cdot),\widehat{\mathcal{M}}\}+C\delta(l(t)) \\ &\le C\delta(l(t))\end{aligned}$$

for sufficiently large $t>0$, because $\widehat{\sigma}\in W(D_1,D_2)$. Since we may assume $C\delta(l(t))\le C\delta^*<\rho_1$, the solution $u(t,\cdot)$ is in an attractive region of $\widehat{\mathcal{M}}$. This completes the proof. □

Proof of Theorem B. Let l_ε^* be the equilibrium of $\dot{l}=-\varepsilon\overline{H}_0(l)/\|\Phi'\|_{L^2}^2$ with $\overline{H}_0'(l_\varepsilon^*)>0$. Let $H_0(l,\widehat{\sigma}(l))$ denote as follows:

$$H_0(l,\widehat{\sigma}(l))=-\frac{\varepsilon H_0(l)}{\|\Phi'\|_{L^2}^2}+\widetilde{H}_0(l,\widehat{\sigma}(l)).$$

We can easily see that $\varepsilon H_0(l)$ and $\tilde{H}_0(l,\widehat{\sigma}(l))$ satisfies

$$\begin{aligned}&\varepsilon H_0(l)=O(\delta_{2+\alpha_-,2+\alpha_+,2}(l)),\\ &\varepsilon|H_0(l)-H_0(k)|\le C(\delta_{1+\alpha_-,1+\alpha_+,2}(l)+\delta_{1+\alpha_-,1+\alpha_+,2}(k))|l-k|,\\ &\widetilde{H}_0(l,\widehat{\sigma}(l))=O(\delta_{3+2\alpha_-,3+2\alpha_+,2}(l)),\\ &|\widetilde{H}_0(l,\widehat{\sigma}(l))-\widetilde{H}_0(k,\widehat{\sigma}(k))|\le C(\delta_{2+\alpha_-,2+\alpha_+,2}(l)+\delta_{2+2\alpha_-,2+2\alpha_+,2}(k))|l-k|.\end{aligned}$$

From this it is easily seen that $H(l, \widehat{\sigma}(l))$ has a stable equilibrium $l_\varepsilon^\dagger$ such that

$$l_\varepsilon^\dagger = l_\varepsilon^* + O(\varepsilon^{1+\alpha_-} + \varepsilon^{1+\alpha_+}),$$

Define

$$u(x) = S(x - l_\varepsilon^\dagger) + \Xi(l_\varepsilon^\dagger)\widehat{\sigma}(l_\varepsilon^\dagger)$$

and this gives a stationary solution of (1.2). Stability of this stationary solution implies from the construction of the invariant manifold.

We can similarly show that the existence of an unstable stationary solution, if there exists an equilibrium l_ε^* of $\dot{l} = -\varepsilon\overline{H}_0(l)/\|\Phi'\|_{L^2}^2$ with $\overline{H}_0'(l_\varepsilon^*) < 0$. $\square$

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(S-I. Ei) Faculty of Mathematics, Kyushu University, Hakozaki 6-10-1, Higashiku, Fukuoka-city, Fukuoka 812-8560, Japan

E-mail address: ichiro@math.kyushu-u.ac.jp

(H. Matsuzawa) Numazu National College of Technology, Ooka 3600, Numazu-city, Shizuoka 410-8501, Japan

E-mail address: hmatsu@numazu-ct.ac.jp