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STOPPING RULES FOR SEQUENTIAL DENSITY ESTIMATION

By

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1. Introduction.

There have been several papers on the sequential estimation of a probability density function (p. d. f.) $f(x)$ and stopping rules. Yamato [10], Wegman and Davies [9] and Isogai [6] proposed recursive density estimators. On the other hand, Davies and Wegman [4] treated the problem of sequential stopping rules. Carroll [2] developed two classes of stopping rules for estimating $f(x)$, one of which yields a confidence interval for $f(x)$ of fixed-width and prescribed coverage probability.

In this paper we consider the problem of estimating $f(x)$ at a given point x, where $f(x)$ is a (unknown) p. d. f. on the p-dimensional Euclidean space R^p with respect to Lebesgue measure. We then use the recursive estimator $f_n(x)$ proposed by Isogai [6] and give a class of stopping rules for estimating $f(x)$ which is similar to that of Carroll [2]. One of the differences between our paper and Carroll's [2] is that the recursive density estimators which appear in specifications of stopping rules are different.

In Section 2 we shall present the recursive density estimator $f_n(x)$ and auxiliary results obtained by Isogai [6] needed for Section 3. In Section 3 we shall give a class of stopping rules for estimating $f(x)$ and examine the asymptotic behaviors of the rules; the stopping rules yield a confidence interval for $f(x)$ of fixed-width $2d$ and prescribed coverage probability $1-\alpha$. This asymptotic behavior can be obtained by the use of the asymptotic normality of $f_n(x)$ and the verification of Anscombe's condition. Moreover we shall discuss the asymptotic normality of the stopping rules.

2. Preliminaries.

In this section we shall make preparations for Section 3. Let $K(y)$ be a real-valued Borel function on R^p satisfying

(K1) $K(y)\geqq 0$ on R^p and $\int K(y)dy=1$,

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(K2) $\|K\|_\infty = \sup_{y \in R^p} K(y) < \infty$,

(K3) $\lim_{\|y\| \to \infty} \|y\|^p K(y) = 0$

and

(K4) $\int \|y\| K(y) dy < \infty$,

where $\|\cdot\|$ denotes the Euclidean norm and the domain of integral is R^p unless otherwise specified. Let $\{a_n\}$ be a sequence of positive numbers defined by

(A) $$a_n = \frac{a}{n} \quad \text{with} \quad \frac{1}{2} < a \leqq 1 \qquad \text{for all} \quad n \geqq 1 .$$

Let $\{h_n\}$ be a sequence of positive numbers. In Section 3, on this sequence we shall impose some or all of the following conditions:

(H1) $\lim_{n \to \infty} h_n = 0$,

(H2) $h_{n_0} \geqq h_{n_0+1} \geqq \cdots$ for some $n_0 \geqq 1$,

(H3) $\lim_{n \to \infty} n h_n^p = \infty$,

(H4) $n_0 h_{n_0}^p \leqq (n_0+1) h_{n_0+1}^p \leqq \cdots$ for some $n_0 \geqq 1$,

(H5) $\lim_{n \to \infty} \frac{h_n}{h_{n+1}} = 1$,

(H6) $\lim_{n \to \infty} (n h_n^p)^{-1/2} \log n = 0$,

(H7) $\sum_{n=1}^{\infty} (n^2 h_n^p)^{-1} < \infty$.

We should note that (H6) implies (H3). Define $K_n(x, y)$ by

$$K_n(x, y) = h_n^{-p} K\left(\frac{x-y}{h_n}\right) \qquad \text{for all} \quad x, y \in R^p, \ n=1, 2, \cdots .$$

The following recursive density estimator is proposed by the author [6].

(F) $$f_0(x) = K(x) \qquad \text{for all} \quad x \in R^p$$

$$f_n(x) = (1-a_n) f_{n-1}(x) + a_n K_n(x, X_n) \qquad \text{for all} \quad x \in R^p, \ n=1, 2, \cdots ,$$

where $X_1, X_2, X_3, \cdots$ is a sequence of independent identically distributed p-dimensional random vectors with the common (unknown) p. d. f. f on some probability space $(\Omega, \mathcal{B}, P)$, and the conditions (K1)~(K4), (A) and (H1) are assumed to be satisfied. In what follows, for the estimator f_n we shall assume the conditions (K1)~(K4), (A) and (H1) without restating them repeatedly. Throughout this paper $C_1, C_2, \cdots$ denote positive constants, and for any function g $C(g)$ stands for the set of all points of continuity of g.

REMARK. It is easy to see that $f_n(x)$ $(n=1, 2, \cdots)$ are probability density functions. We shall, now, introduce some notations. Let

$$\beta_{mn}=\prod_{k=m+1}^{n}(1-a_k) \qquad \text{if} \quad 0\leqq m<n$$

$$=1 \qquad \text{if} \quad m=n\geqq 0,$$

$$\gamma_0=\gamma_1=1$$

and

$$\gamma_n=\prod_{j=2}^{n}(1-a_j) \qquad \text{for} \quad n\geqq 2.$$

It is clear that $\gamma_n>0$ for all $n\geqq 1$ and $\beta_{mn}=\gamma_n\gamma_m^{-1}$ for all $n\geqq m\geqq 1$. The author [6] gave the inequalities:

(2.0.1) $$C_1m^a n^{-a}\leqq\beta_{mn}\leqq C_2m^a n^{-a} \qquad \text{for all} \quad n\geqq m\geqq 1.$$

DEFINITION 2.1. A bounded real-valued function g defined on R^p is said to be locally Lipschitz of order λ, $0<\lambda\leqq 1$, at x_0 (abbreviated as loc. Lip. λ at x_0) if there exist two positive constants L and η, depending on x_0 and λ such that $\|y\|<\eta$ implies $|g(x_0+y)-g(x_0)|\leqq L\|y\|^\lambda$.
The following lemma was given by the author [6].

LEMMA 2.2. *Let* $Z_n=K_n(x, X_n)-EK_n(x, X_n)$ *for all* $n\geqq 1$. *Assume that* $\{h_n\}$ *satisfies* (H1), (H2) *and* (H6). *In addition suppose the following conditions*:

(2.2.1) *For some* a *in* (A) *with* $\frac{2}{3}\leqq a\leqq 1$ *there exists a positive constant* β *such that*

$$n^{1-2a}h_n^p\sum_{m=1}^{n}m^{2(a-1)}h_m^{-p}\longrightarrow\beta \qquad as \quad n\to\infty,$$

(2.2.2) $\|f\|_\infty=\sup_{y\in R^p}f(y)<\infty$.

Then, for each $x\in C(f)$

$$(nh_n^p)^{1/2}\sum_{m=1}^{n}a_m\beta_{mn}Z_m\underset{\mathcal{L}}{\longrightarrow}N(0, Bf(x)) \qquad as \quad n\to\infty,$$

where $B=a^2\beta\int K^2(y)dy$, $N(0, \sigma^2)$ *stands for the normal random variable with mean* 0 *and variance* σ^2 *and* "$\underset{\mathcal{L}}{\longrightarrow}$" *means convergence in law.*

3. Stopping rules.

In this section we shall propose a class of stopping rules which yields a confidence interval for $f(x)$ of prescribed width $2d$ and prescribed coverage probability $1-\alpha$. The asymptotic normality of the stopping rules will also be shown. Our stopping rules are analogous to those of Carroll [2], but in the specifications of the stopping rules the recursive estimators of Carroll [2] and Isogai [6] are different.

Let any α $(0<\alpha<1)$ be given. Define $D=D(\alpha)$ by $D(\alpha)=\Phi^{-1}\left(1-\frac{\alpha}{2}\right)$, where Φ is the distribution function of the standard normal random variable. We shall, now, define stopping rules as follows:

$$N(d)=\text{the smallest integer } n\geqq 1 \text{ such that}$$

(3.0) $$(D^2B)^{-1}nh_n^p d^2 \geqq f_n(x)>0\,,$$

where B is the same as in Lemma 2.2 and $f_n(x)$ is defined by (F). Also, define $n(d)$ by

$$n(d)=\text{the smallest integer } n\geqq 1 \text{ such that}$$

$$(D^2B)^{-1}nh_n^p d^2 \geqq f(x)\,.$$

The following lemma is a modification of Lemma 1 in Chow and Robbins [3].

LEMMA 3.1. *Let y_n $(n=1, 2, \cdots)$ be any sequence of random variables on a probability space $(\Omega, \mathfrak{F}, P)$ satisfying the following: There exists a null set $\tilde{N}_1$ (i.e. $P\{\tilde{N}_1\}=0$) such that for each $\omega\in\tilde{N}_1^c$, where $\tilde{N}_1^c$ is the complement of the set $\tilde{N}_1$,*

$$y_n(\omega)\geqq 0 \quad \text{for all} \quad n\geqq 1 \quad \text{and} \quad \lim_{n\to\infty} y_n(\omega)=1\,,$$

and if $y_m(\omega)>0$ for some $m=m(\omega)$ then $y_n(\omega)>0$ for all $n\geqq m$. Let $\{g(n)\}$ be any sequence of constants such that

$$g(n)>0\,, \qquad \lim_{n\to\infty} g(n)=\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{g(n)}{g(n-1)}=1\,,$$

and for each $t>0$ and any fixed integer $n_1\geqq 1$ define

$$N_t=N(t, n_1)=\text{the smallest integer } n\geqq n_1 \text{ such that}$$

$$0<y_n\leqq\frac{g(n)}{t}\,.$$

Then, N_t is well-defined and non-decreasing as a function of t,

(3.1.1) $$\lim_{t\to\infty} N_t=\infty \quad a.s.$$

and

(3.1.2) $$\lim_{t\to\infty}\frac{g(N_t)}{t}=1 \quad a.s.$$

PROOF. It is easy to see that N_t is well-defined and non-decreasing as a function of t. Hence there exists $\lim_{t\to\infty} N_t$ on $\tilde{N}_1^c$. First we shall show (3.1.1). Suppose that there exists a set $\Omega^1\in\mathfrak{F}$ with $P\{\Omega^1\}>0$ such that

(3.1.3) $$M=\lim_{t\to\infty} N_t<\infty \qquad \text{on} \quad \Omega^1(\subset\tilde{N}_1^c)\,.$$

Let any $\omega\in\Omega^1$ be fixed. By definition of N_t and (3.1.3) we get

$$y_M=\lim_{t\to\infty} y_{N_t}\leqq\lim_{t\to\infty}\frac{g(N_t)}{t}=0\,,$$

where we omit ω. Thus we have

$$y_M=0. \tag{3.1.4}$$

Let any $t>0$ be fixed. By the monotonicity of N_t we get $M\geqq N_t$. Taking account of $y_{N_t}>0$ and assumption of y_n, we have $y_M>0$, which contradicts (3.1.4). Thus we have $P\{\Omega^1\}=0$, concluding (3.1.1). Finally we shall show (3.1.2). Let $\tilde{N}_2^c=\tilde{N}_1^c\cap\{\omega\in\Omega: \lim_{t\to\infty}N_t=\infty\}$. It follows from the above relation that $P\{\tilde{N}_2^c\}=1$. Let any $\omega\in\tilde{N}_2^c$ be fixed. For convenience we omit ω. As $\lim_{n\to\infty}y_n=1$, there exists a positive integer $n_2>n_1$ such that

$$y_n>0 \quad \text{for all} \quad n\geqq n_2. \tag{3.1.5}$$

On the other hand, as $\lim_{t\to\infty}N_t=\infty$, there exists a positive number t_0 such that

$$N_t-1\geqq n_2 \quad \text{for all} \quad t\geqq t_0,$$

which, together with (3.1.5), yields

$$y_{N_t-1}>0 \quad \text{for all} \quad t\geqq t_0. \tag{3.1.6}$$

Hence by (3.1.6) and definition of N_t we get

$$y_{N_t}\leqq\frac{g(N_t)}{t}<\frac{g(N_t)}{g(N_t-1)}y_{N_t-1} \quad \text{for all} \quad t\geqq t_0. \tag{3.1.7}$$

Since $\lim_{t\to\infty}\frac{g(N_t)}{g(N_t-1)}=1$, it follows from (3.1.7) and assumption of y_n that $\lim_{t\to\infty}\frac{g(N_t)}{t}=1$ on $\tilde{N}_2^c$, concluding (3.1.2). Thus the proof is complete.

THEOREM 3.2. *Suppose that* $\{h_n\}$ *satisfies* (H1), (H3), (H5) *and* (H7). *Then, for each point* $x\in C(f)$ *with* $f(x)>0$, $N(d)$ *is well-defined. Furthermore,*

$$\lim_{d\downarrow 0}N(d)=\infty \quad a.s. \tag{3.2.1}$$

and

$$\lim_{d\downarrow 0}\frac{N(d)h_{N(d)}^p d^2}{D^2Bf(x)}=1 \quad a.s. \tag{3.2.2}$$

PROOF. By (3.0) we get

$$0<\frac{f_n(x)}{f(x)}\leqq\frac{nh_n^p}{f(x)}/(D^2Bd^{-2}).$$

In Lemma 3.1 we set

$$y_n=\frac{f_n(x)}{f(x)}, \quad g(n)=\frac{nh_n^p}{f(x)} \quad \text{and} \quad t=D^2Bd^{-2} \quad \text{for all} \quad n\geqq 1.$$

By Theorem 3.1 of Isogai [6] there exists a null set $\tilde{N}_1\in\mathcal{B}$ such that $\lim_{n\to\infty}y_n=1$ on $\tilde{N}_1^c$. By Remark in Section 2 it follows that

$$y_n\geqq 0 \quad \text{on} \quad \Omega \quad \text{for all} \quad n\geqq 1.$$

All conditions of $\{g(n)\}$ in Lemma 3.1 are satisfied by (H3) and (H5). Suppose that for any fixed $\omega\in\tilde{N}_1^c$ omitted below there exists a positive integer m such that $y_m>0$. If $y_n>0$ for $n>m$ which is equivalent to $f_n(x)>0$, according to $K_{n+1}(x, X_{n+1})\geqq 0$ and $1-a_{n+1}>0$ we have $f_{n+1}(x)>0$ which is equivalent to $y_{n+1}>0$. Thus by induction we get $y_n>0$ for all $n\geqq m$. Since all conditions of Lemma 3.1 are satisfied, Lemma 3.1 yields (3.2.1) and (3.2.2). The proof is complete.

By the same argument for Theorem 3.2 it follows that under (H1), (H3), (H5) and $f(x)>0$ $n(d)$ is non-decreasing as a function of d, and that

(3.2.3) $$\lim_{d\downarrow 0} n(d)=\infty$$

and

(3.2.4) $$\lim_{d\downarrow 0}\frac{n(d)h_{n(d)}^p d^2}{D^2Bf(x)}=1\,.$$

Thus by (3.2.2) and (3.2.4) we obtain

Corollary 3.3. *Under all conditions of Theorem* 3.2 *we have*

(3.3.1) $$\lim_{d\downarrow 0}\frac{N(d)h_{N(d)}^p}{n(d)h_{n(d)}^p}=1 \quad a.s.$$

By Theorem 3.1 of Isogai [6], Theorem 3.2 and Theorem 1 of Richter [8] we have

Proposition 3.4. *Under all conditions of Theorem* 3.2 *we have*

$$\lim_{d\downarrow 0} f_{N(d)}(x)=f(x) \quad a.s.$$

Now, we shall verify Anscombe's condition.

Lemma 3.5. *Suppose that* $\{h_n\}$ *satisfies* (H1), (H2), (H4) *and* (2.2.1). *Let any* $x\in C(f)$ *with* $f(x)>0$ *be fixed and set* $S_n=f_n(x)-Ef_n(x)$. *Then, for any positive numbers* ε *and* η, *there exist a positive constant* $\rho=\rho(\varepsilon, \eta)<1$ *and a positive integer* $\nu=\nu(\varepsilon, \eta)$ *such that*

(3.5.1) $$P\Big\{\max_{|i-n|\leqq n\rho}|S_i-S_n|\geqq\varepsilon(nh_n^p)^{-1/2}\Big\}<\eta \qquad \textit{for all} \quad n\geqq\nu\,.$$

Proof. Let $\varepsilon_n=\varepsilon(nh_n^p)^{-1/2}$. In what follows we consider ρ with $0<\rho\leqq\frac{1}{2}$ and n satisfying $n\rho>1$ for fixed ρ and $n\geqq 2n_0$ which is the same as in (H2) and (H4). For any fixed ρ and n define two positive integers $m_1=m_1(\rho, n)$ and $m_2=m_2(\rho, n)$ by

(3.5.2) $$(1-\rho)n\leqq m_1<(1-\rho)n+1 \quad\text{and}\quad (1+\rho)n-1<m_2\leqq(1+\rho)n\,.$$

We note that $m_2>m_1\geqq n_0$ and $m_2>n>m_1$ for each ρ and n. Since S_n can be rewritten as $\sum_{m=1}^n a_m\beta_{mn}Z_m$ with Z_m being the same as in Lemma 2.2, we have

$$P\Big\{\max_{|i-n|\leqq n\rho}|S_i-S_n|\geqq\varepsilon_n\Big\}$$

$$\leqq P\Big\{\max_{m_1\leqq i\leqq n-1}|S_i-S_n|\geqq\varepsilon_n\Big\}+P\Big\{\max_{n+1\leqq i\leqq m_2}|S_i-S_n|\geqq\varepsilon_n\Big\}$$

$$(3.5.3)\qquad \leq P\Big\{\max_{m_1\leq i\leq n-1}\Big|\sum_{m=1}^{i} a_m(\beta_{mi}-\beta_{mn})Z_m\Big|\geq\frac{\varepsilon_n}{2}\Big\}$$

$$+P\Big\{\max_{m_1\leq i\leq n-1}\Big|\sum_{m=i+1}^{n} a_m\beta_{mn}Z_m\Big|\geq\frac{\varepsilon_n}{2}\Big\}$$

$$+P\Big\{\max_{n+1\leq i\leq m_2}\Big|\sum_{m=1}^{n} a_m(\beta_{mi}-\beta_{mn})Z_m\Big|\geq\frac{\varepsilon_n}{2}\Big\}$$

$$+P\Big\{\max_{n+1\leq i\leq m_2}\Big|\sum_{m=n+1}^{i} a_m\beta_{mi}Z_m\Big|\geq\frac{\varepsilon_n}{2}\Big\}$$

$$=I_1+I_2+I_3+I_4,\quad \text{say}.$$

Define

$$d_{mm_1}=\beta_{mm_1}\qquad \text{if}\quad m\leqq m_1$$
$$=\beta_{m_1m}^{-1}\qquad \text{if}\quad m>m_1.$$

Then it is easy to see that $\beta_{mi}-\beta_{mn}=(1-\beta_{in})\beta_{m_1i}d_{mm_1}$ for $m_1\leqq i\leqq n-1$ and $m\leqq i$, and that $(1-\beta_{in})\beta_{m_1i}$ is positive and non-increasing as a function of i for $m_1\leqq i\leqq n-1$. Hence by the Hájek-Rényi inequality (see Hájek and Rényi [5]) we get

$$I_1=P\Big\{\max_{m_1\leq i\leq n-1}(1-\beta_{in})\beta_{m_1i}\Big|\sum_{m=1}^{i} a_m d_{mm_1}Z_m\Big|\geq\frac{\varepsilon_n}{2}\Big\}$$

$$(3.5.4)\qquad \leqq(2/\varepsilon_n)^2(1-\beta_{m_1n})^2\beta_{m_1m_1}^2\sum_{m=1}^{m_1} a_m^2 d_{mm_1}^2 EZ_m^2$$

$$+(2/\varepsilon_n)^2\sum_{m=m_1+1}^{n-1}(1-\beta_{mn})^2\beta_{m_1m}^2 a_m^2 d_{mm_1}^2 EZ_m^2$$

$$=J_1+J_2,\quad \text{say}.$$

As $\mathrm{Var}(S_n)=\sum_{m=1}^{n} a_m^2\beta_{mn}^2 EZ_m^2$, by Theorem 3.2 of Isogai [6] we get

$$(3.5.5)\qquad nh_n^p\sum_{m=1}^{n} a_m^2 d_{mn}^2 EZ_m^2\sim Bf(x)\qquad \text{as}\quad n\to\infty,$$

where "$\phi_n\sim\psi_n$ as $n\to\infty$" means that $\phi_n/\psi_n\to 1$ as $n\to\infty$.
The relation (3.5.2) implies that

$$(3.5.6)\qquad m_1/n\sim 1-\rho\qquad \text{as}\quad n\to\infty.$$

It follows from (H2) and (H4) that

$$(3.5.7)\qquad 1\leqq nh_n^p/(m_1h_{m_1}^p)\leqq n/m_1.$$

Let any ρ_1 with $0<\rho_1\leqq\frac{1}{2}$ be fixed. From (3.5.6) and (3.5.7) we get

$$(3.5.8)\qquad nh_n^p/(m_1h_{m_1}^p)\leqq 4\quad \text{for } n \text{ sufficiently large}.$$

It follows from (3.5.5) that

(3.5.9) $$nh_n^p \sum_{m=1}^{n} a_m^2 d_{mn}^2 EZ_m^2 \leqq C_3 \quad \text{for all} \quad n \geqq 1 .$$

Combining (3.5.8) and (3.5.9) we have

(3.5.10) $$J_1 \leqq C_4(\varepsilon)(1-\beta_{m_1 n})^2 \quad \text{for } n \text{ sufficiently large} .$$

By making use of (3.5.6) and the fact that $\beta_{mn} \sim m^a n^{-a}$ as $m \to \infty$, we get $(1-\beta_{m_1 n})^2 \sim \{1-(1-\rho_1)^a\}^2$ as $n \to \infty$. Hence we have $(1-\beta_{m_1 n})^2 \leqq 2\{1-(1-\rho_1)^a\}^2$ for n sufficiently large, which, together with (3.5.10), implies that

(3.5.11) $$J_1 \leqq C_5(\varepsilon)\{1-(1-\rho_1)^a\}^2 \quad \text{for } n \text{ sufficiently large} .$$

Choose $0<\rho_1=\rho_1(\varepsilon, \eta)<\dfrac{1}{2}$ such that $C_5(\varepsilon)\{1-(1-\rho)^a\}^2<\dfrac{\eta}{8}$ for all $0<\rho \leqq \rho_1$. Thus, for any fixed ρ_2 with $0<\rho_2 \leqq \rho_1$, it follows from (3.5.11) that

(3.5.12) $$J_1<\eta/8 \quad \text{for } n \text{ sufficiently large} .$$

In the proof of Theorem 3.2 of Isogai [6] it has been shown that

(3.5.13) $$h_n^p EZ_n^2 \leqq C_6 \quad \text{for all} \quad n \geqq 1 .$$

In view of (H2) and (3.5.13) we have

$$J_2 \leqq C_7(\varepsilon) nh_n^p \sum_{m=m_1+1}^{n-1} m^{-1} m^{-1} h_m^{-p}$$

(3.5.14) $$\leqq C_7(\varepsilon)(n/m_1) \sum_{m=m_1+1}^{n-1} m^{-1}$$

$$\leqq C_7(\varepsilon)(n/m_1) \log (n/m_1) .$$

Since by (3.5.6) $(n/m_1)\log(n/m_1) \sim (1-\rho_2)^{-1} \log(1-\rho_2)^{-1}$ as $n \to \infty$, it follows that

(3.5.15) $$(n/m_1) \log (n/m_1) \leqq 2(1-\rho_2)^{-1} \log (1-\rho_2)^{-1}$$

for n sufficiently large. Choose $0<\rho_2=\rho_2(\varepsilon, \eta) \leqq \rho_1$ such that

$$C_7(\varepsilon)(1-\rho)^{-1} \log (1-\rho)^{-1}<\eta/16 \quad \text{for all} \quad 0<\rho \leqq \rho_2 .$$

Thus, for any fixed ρ_3 with $0<\rho_3 \leqq \rho_2$, it follows from (3.5.14) and (3.5.15) that

(3.5.16) $$J_2<\eta/8 \quad \text{for } n \text{ sufficiently large} .$$

Combining (3.5.4), (3.5.12) and (3.5.16) we have the following:

For any fixed ρ_3 such that $0<\rho_3 \leqq \rho_2$

(3.5.17) $$I_1<\eta/4 \quad \text{for } n \text{ sufficiently large.}$$

Now, let any ρ_3 with $0<\rho_3 \leqq \rho_2$ be fixed. By Kolmogorov's inequality we get

(3.5.18) $$I_2 \leqq (2/\varepsilon)^2 \left\{ nh_n^p \sum_{m=1}^{n} a_m^2 \beta_{mn}^2 EZ_m^2 - nh_n^p \sum_{m=1}^{m_1} a_m^2 \beta_{mn}^2 EZ_m^2 \right\}$$

$$\leqq (2/\varepsilon)^2 T(n) \{1-\beta_{m_1 n}^2 (nh_n^p/m_1 h_{m_1}^p)(T(m_1)/T(n))\} ,$$

where $T(n)=nh_n^p \sum_{m=1}^{n} a_m^2 \beta_{mn}^2 EZ_m^2$. It follows from (3.5.9) that

(3.5.19) $$T(n) \leqq C_3 \quad \text{for all} \quad n \geqq 1\,.$$

Hence, in view of (3.5.7), (3.5.18) and (3.5.19) we have

(3.5.20). $$I_2 \leqq C_8(\varepsilon)\{1-\beta_{m_1 n}^2(T(m_1)/T(n))\} \qquad \text{for all } n.$$

As it is easy to see that $1-\beta_{m_1 n}^2(T(m_1)/T(n)) \leqq 2\{1-(1-\rho_3)^{2\alpha}\}$ for n sufficiently large, by (3.5.20) and the discussion similar to (3.5.17) we can choose $0<\rho_3=\rho_3(\varepsilon, \eta) \leqq \rho_2$ such that for any fixed ρ_4 with $0<\rho_4 \leqq \rho_3$

(3.5.21) $$I_2 < \eta/4 \quad \text{for } n \text{ sufficiently large}\,.$$

Let any ρ_4 with $0<\rho_4 \leqq \rho_3$ be fixed. By making use of (3.5.9) and Chebychev's inequality we have

(3.5.22)
$$\begin{aligned} I_3 &= P\left\{(\gamma_n-\gamma_{m_2})\left|\sum_{m=1}^{n} a_m \gamma_m^{-1} Z_m\right| \geqq \frac{\varepsilon_n}{2}\right\} \\ &\leqq (2/\varepsilon)^2 (1-\beta_{n m_2})^2 n h_n^p \sum_{m=1}^{n} a_m^2 \beta_{mn}^2 EZ_m^2 \\ &\leqq C_9(\varepsilon)(1-\beta_{n m_2})^2\,. \end{aligned}$$

As

(3.5.23) $$m_2/n \sim 1+\rho_4 \qquad \text{as} \quad n \to \infty\,,$$

we get $(1-\beta_{n m_2})^2 \leqq 2\{1-(1+\rho_4)^{-\alpha}\}^2$ for n sufficiently large. Thus, using (3.5.22) we can choose $0<\rho_4=\rho_4(\varepsilon, \eta) \leqq \rho_3$ such that for any fixed ρ with $0<\rho \leqq \rho_4$

(3.5.24) $$I_3 < \eta/4 \quad \text{for } n \text{ sufficiently large}\,.$$

Finally, let any ρ with $0<\rho \leqq \rho_4$ be fixed. Using (H4), (3.5.13) and the Hájek-Rényi inequality, we have

(3.5.25)
$$\begin{aligned} I_4 &= P\left\{\max_{n+1 \leq i \leq m_2} \gamma_i \left|\sum_{m=n+1}^{i} a_m \gamma_m^{-1} Z_m\right| \geqq \frac{\varepsilon_n}{2}\right\} \\ &\leqq (2/\varepsilon)^2 n h_n^p \sum_{m=n+1}^{m_2} a_m^2 EZ_m^2 \\ &\leqq C_{10}(\varepsilon) \sum_{m=n+1}^{m_2} m^{-1} \\ &\leqq C_{10}(\varepsilon) \log(m_2/n)\,. \end{aligned}$$

As by (3.5.23) $\log(m_2/n) \sim \log(1+\rho)$ as $n \to \infty$, we get $\log(m_2/n) \leqq 2\log(1+\rho)$ for n sufficiently large. Thus, using (3.5.25) we can choose $0<\rho=\rho(\varepsilon, \eta) \leqq \rho_4$ such that for ρ

(3.5.26) $$I_4 < \eta/4 \quad \text{for } n \text{ sufficiently large}\,.$$

For ρ chosen above, choose a positive integer ν such that for all $n \geqq \nu$ (3.5.17), (3.5.21), (3.5.24) and (3.5.26) hold. Thus, combining (3.5.3), (3.5.17), (3.5.21), (3.5.24) and (3.5.26)

we obtain (3.5.1). This completes the proof.
The following theorem is concerned with the asymptotic normality of $f_{N(d)}(x)$.

THEOREM 3.6. *Assume* (2.2.1). *Suppose that* f *is loc. Lip.* λ *at* x *and* $f(x)>0$, *and that* $\{h_n\}$ *satisfies* (H1)~(H7) *except for* (H3). *Furthermore suppose the following conditions*:

$$\sum_{m=1}^{n} m^{a-1}h_m^{\lambda}=O(n^a h_n^{\lambda}) \quad \text{as} \quad n\to\infty \tag{3.6.1}$$

with a *being the same as in* (2.2.1),

$$\lim_{n\to\infty} n h_n^{2\lambda+p}=0\,, \tag{3.6.2}$$

$$\lim_{d\downarrow 0} N(d)/n(d)=1 \quad \text{in probability.} \tag{3.6.3}$$

Then we have

$$(N(d)h_{N(d)}^{p})^{1/2}(f_{N(d)}(x)-f(x)) \xrightarrow[\mathcal{L}]{} N(0,\ Bf(x)) \quad \text{as} \quad d\downarrow 0\,, \tag{3.6.4}$$

where B *is the same as in Lemma* 2.2.

PROOF. Since, by (F) in Section 2,

$$\begin{aligned}(nh_n^p)^{1/2}(f_n(x)-f(x)) &= (nh_n^p)^{1/2}\sum_{m=1}^{n} a_m\beta_{mn}Z_m+(nh_n^p)^{1/2}\sum_{m=1}^{n} a_m\beta_{mn}\delta_m \\ &\quad +(nh_n^p)^{1/2}\beta_{0n}(f_0(x)-f(x))\,,\end{aligned}$$

where $\delta_m=EK_m(x,\ X_m)-f(x)$ and Z_m is the same as in Lemma 2.2, we have (3.6.4) if it holds that as $d\downarrow 0$

$$(N(d)h_{N(d)}^{p})^{1/2}\sum_{m=1}^{N(d)} a_m\beta_{mN(d)}Z_m \xrightarrow[\mathcal{L}]{} N(0,\ Bf(x))\,, \tag{3.6.5}$$

$$(N(d)h_{N(d)}^{p})^{1/2}\sum_{m=1}^{N(d)} a_m\beta_{mN(d)}\delta_m \longrightarrow 0 \quad \text{in probability} \tag{3.6.6}$$

and

$$(N(d)h_{N(d)}^{p})^{1/2}\beta_{0N(d)}(f_0(x)-f(x)) \longrightarrow 0 \quad \text{in probability.} \tag{3.6.7}$$

First we shall show (3.6.6). For convenience N denotes $N(d)(\omega)$ for each $\omega\in\Omega$ and d. Let any $d_0>0$ be fixed. Setting $T=\{\omega\in\Omega: N(d)(\omega)<\infty$ for all $0<d<d_0$ and $\lim_{d\downarrow 0} N(d)(\omega)=\infty\}$, it follows from Theorem 3.2 that $P\{T\}=1$. Let any $\omega\in T$ and d with $0<d<d_0$ be fixed. Using (3.4.4) in the proof of Theorem 3.4 of Isogai [6], we have

$$(Nh_N^p)^{1/2}\left|\sum_{m=1}^{N} a_m\beta_{mN}\delta_m\right| \leqq C_3(Nh_N^{2\lambda+p})^{1/2}\,, \tag{3.6.8}$$

where C_3 is independent of d and ω. By (3.6.2) and (3.6.8) we get (3.6.6). Second we shall show (3.6.7). Let any $\omega\in T$ and any d $(0<d<d_0)$ be fixed. As $|f_0(x)-f(x)| \leqq \|K\|_\infty+\|f\|_\infty<\infty$, it follows from (2.0.1) that

(3.6.9) $$(Nh_N^p)^{1/2}\beta_{0N}|f_0(x)-f(x)|\leqq C_4N^{1/2-a}h_N^{p/2},$$

where C_4 is independent of d and ω. Combining (H1), $a>\frac{1}{2}$ and (3.6.9) we obtain (3.6.7). Finally we shall show (3.6.5). By Lemma 2.2, Lemma 3.5 and (3.6.3) we can use Theorem 1 of Anscombe [1], which implies that

(3.6.10) $$(n(d)h_{n(d)}^p)^{1/2}\sum_{m=1}^{N(d)}a_m\beta_{mN(d)}Z_m \xrightarrow[\mathcal{L}]{} N(0,\ Bf(x)) \quad \text{as} \quad d\downarrow 0.$$

Combining (3.6.10) and Corollary 3.3, we have (3.6.5). This completes the proof.
The following is one of main results.

THEOREM 3.7. *Under all conditions of Theorem* 3.6, *we have*

$$\lim_{d\downarrow 0} P\{|f_{N(d)}(x)-f(x)|\leqq d\}=1-\alpha.$$

PROOF. Since

$$Dd^{-1}(f_{N(d)}(x)-f(x))$$
$$=(D^2Bf(x)/(N(d)h_{N(d)}^p d^2))^{1/2}(N(d)h_{N(d)}^p/(Bf(x)))^{1/2}(f_{N(d)}(x)-f(x)),$$

by Theorems 3.2 and 3.6 we obtain

$$Dd^{-1}(f_{N(d)}(x)-f(x)) \xrightarrow[\mathcal{L}]{} N(0,\ 1) \quad \text{as} \quad d\downarrow 0,$$

which yields that

$$\lim_{d\downarrow 0} P\{|f_{N(d)}(x)-f(x)|\leqq d\}$$
$$=P\{|N(0,\ 1)|\leqq D\}=1-\alpha.$$

Thus the proof is complete.

COROLLARY 3.8. *Suppose that f is loc. Lip. λ at x and $f(x)>0$. Let $\{h_n\}$ be given by*

(3.8.1) $$h_n=n^{-r/p} \quad with \quad \frac{2}{3}\leqq a\leqq 1 \quad and \quad \frac{p}{2\lambda+p}<r<\min\left(\frac{ap}{\lambda},\ 1\right).$$

Then we have

(3.8.2) $$\lim_{d\downarrow 0} P\{|f_{N(d)}(x)-f(x)|\leqq d\}=1-\alpha.$$

PROOF. It is easily verified that the sequence $\{h_n\}$ given by (3.8.1) satisfies all conditions on $\{h_n\}$ of Theorem 3.6 with $\beta=(2a+r-1)^{-1}$. On the other hand, by Corollary 3.3 we have $\lim_{d\downarrow 0} N(d)/n(d)=1$ with probability one, which implies (3.6.3). Thus by Theorem 3.7 we have (3.8.2).

In the remainder of this section we shall deal with the asymptotic normality of the stopping rules $N(d)$.

LEMMA 3.9. *Let $\{\xi_n\}$ be a sequence of real numbers such that $\xi_n=1+o(1)$ where $o(1)\to 0$ as $n\to\infty$. Then, for any constant b,*

$$\xi_n^b-1=(\xi_n-1)\{b+o(1)\}.$$

PROOF. By the Taylor expansion we obtain the result of lemma.
The following theorem presents the asymptotic normality.

THEOREM 3.10. *Let* $\{h_n\}$ *satisfy all conditions of Theorem* 3.6 *replacing* (H5) *by the following condition*:

(3.10.1) $$(n-1)h_{n-1}^p/(nh_n^p)=1+o((nh_n^p)^{-3/2}) \quad as \quad n\to\infty,$$

where $o(\phi_n)$ *means that* $o(\phi_n)/\phi_n\to 0$ *as* $n\to\infty$. *Then, under all conditions of Theorem* 3.6 *we have*

(3.10.2) $$(DB)^{-1}d(N(d)h_{N(d)}^p-D^2Bd^{-2}f(x)) \xrightarrow[\mathcal{L}]{} N(0,1) \quad as \quad d\downarrow 0.$$

PROOF. By (H3) and (3.10.1) we get (H5). From (3.2.3) and (3.6.3) it follows that

(3.10.3) $$\lim_{d\downarrow 0}(N(d)-1)/n(d)=1 \quad \text{in probability}.$$

In the same manner as in the proof of (3.6.4) we get

(3.10.4) $$((N(d)-1)h_{N(d)-1}^p)^{1/2}(f_{N(d)-1}(x)-f(x)) \xrightarrow[\mathcal{L}]{} N(0, Bf(x))$$

as $d\downarrow 0$. By (3.2.1) and the argument similar to Proposition 3.4 there exists a null set $\tilde{N}_1$ such that on $\tilde{N}_1^c$ $N(d)<\infty$ for all $d>0$, $\lim_{d\downarrow 0} N(d)=\infty$ and

(3.10.5) $$\lim_{d\downarrow 0} f_{N(d)-1}(x)=f(x)>0.$$

For any $d>0$ we set $A_d=\{\omega\in\Omega: f_{N(d)-1}(x)=0\}\cap\tilde{N}_1^c$. By (3.10.5) we get

(3.10.6) $$\lim_{d\downarrow 0} P\{A_d\}=0.$$

Suppose that $N(d)<\infty$ and $f_{N(d)-1}(x)>0$. Then, by definition of $N(d)$ we have

(3.10.7) $$\begin{aligned} x_d &\equiv (N(d)h_{N(d)}^p)^{1/2}(d^2N(d)h_{N(d)}^p-D^2Bf(x)) \\ &\geqq D^2B(N(d)h_{N(d)}^p)^{1/2}(f_{N(d)}(x)-f(x))\equiv y_d \end{aligned}$$

and

(3.10.8) $$\begin{aligned} w_d &\equiv ((N(d)-1)h_{N(d)-1}^p)^{1/2}(d^2(N(d)-1)h_{N(d)-1}^p-D^2Bf(x)) \\ &< D^2B((N(d)-1)h_{N(d)-1}^p)^{1/2}(f_{N(d)-1}(x)-f(x))\equiv v_d. \end{aligned}$$

Setting $B_d=\{\omega\in\Omega: f_{N(d)-1}(x)>0\}\cap\tilde{N}_1^c$, the relations (3.10.7) and (3.10.8) hold on B_d. By (H3) and (3.10.1) we get that for each $\omega\in\tilde{N}_1^c$ omitted below

(3.10.9) $$(N(d)-1)h_{N(d)-1}^p/(N(d)h_{N(d)}^p)=1+o(1),$$

where $o(1)\to 0$ as $d\downarrow 0$. Let any $\omega\in\tilde{N}_1^c$ be fixed. In view of (3.10.1), (3.10.9) and Lemma 3.9 we have that as $d\downarrow 0$

(3.10.10) $$(N(d)h_{N(d)}^p)^{3/2}-((N(d)-1)h_{N(d)-1}^p)^{3/2}=o(1)$$

and

(3.10.11) $$(N(d)h_{N(d)}^p)^{1/2}-((N(d)-1)h_{N(d)-1}^p)^{1/2}=o(1).$$

Hence by (3.10.10) and (3.10.11) we obtain

(3.10.12) $$\lim_{d\downarrow 0}|x_d-w_d|=0 \quad \text{a. s.}$$

Let F denote the distridution function of $N(0,\ D^4B^3f(x))$ and let any real number y be fixed. It follows from (3.10.7) that

(3.10.13) $$\begin{aligned}&P\{x_d\leqq y\}\\&=P(\{x_d\leqq y\}\cap B_d)+P(\{x_d\leqq y\}\cap A_d)\\&\leqq P\{y_d\leqq y\}+P\{A_d\}\ ,\end{aligned}$$

where $\{x_d\leqq y\}$ denotes ω-set such that $x_d\leqq y$. Since by (3.6.4) $y_d\xrightarrow[\mathcal{L}]{}N(0,\ D^4B^3f(x))$ as $d\downarrow 0$, by (3.10.6) and (3.10.13) we get

(3.10.14) $$\limsup_{d\downarrow 0} P\{x_d\leqq y\}\leqq F(y)\,.$$

Let any $\varepsilon>0$ be fixed. It follows (3.10.8) that for any $d>0$

$$\begin{aligned}&P\{v_d\leqq y-\varepsilon\}\\&\leqq P\{w_d<y-\varepsilon\}+P\{A_d\}\\&\leqq P\{x_d\leqq y\}+P\{|x_d-w_d|>\varepsilon\}+P\{A_d\}\ ,\end{aligned}$$

which yields that

(3.10.15) $$P\{v_d\leqq y-\varepsilon\}-P\{|x_d-w_d|>\varepsilon\}-P\{A_d\}\leqq P\{x_d\leqq y\}\ .$$

Since by (3.10.4) $v_d\xrightarrow[\mathcal{L}]{}N(0,\ D^4B^3f(x))$ as $d\downarrow 0$, by (3.10.6), (3.10.12) and (3.10.15) we have $F(y-\varepsilon)\leqq\liminf_{d\downarrow 0} P\{x_d\leqq y\}$, which yields that as $\varepsilon\to 0$

(3.10.16) $$F(y)\leqq\liminf_{d\downarrow 0} P\{x_d\leqq y\}\ .$$

Thus combining (3.10.14) and (3.10.16) we obtain

(3.10.17) $$x_d\xrightarrow[\mathcal{L}]{}N(0,\ D^4B^3f(x)) \qquad \text{as} \quad d\downarrow 0\,.$$

Since

$$\begin{aligned}&(DB)^{-1}d(N(d)h^p_{N(d)}-D^2Bd^{-2}f(x))\\&=(D^2Bf(x)/(N(d)h^p_{N(d)}d^2))^{1/2}x_d/(D^4B^3f(x))^{1/2}\,,\end{aligned}$$

taking account of (3.2.2) and (3.10.17) we obtain (3.10.2) This completes the proof. The next lemma is a modification of Theorem (i) of Rao [7] (page 319).

LEMMA 3.11. *Let h be a mapping from* $(0,\ \infty)$ *into* $(0,\ \infty)$ *with* $\lim_{d\to 0} h(d)=\infty$. *Let* $\{T(d)\}\,(d\in(0,\ \infty))$ *be a family of random variables. Suppose that*

(3.11.1) $$h(d)(T(d)-\theta)\xrightarrow[\mathcal{L}]{}N(0,\ \sigma^2) \qquad \textit{as} \quad d\to 0\,,$$

where θ is a real number and σ^2 is a positive number. Let g be a real-valued measurable function on R^1 such that there exists the first derivative $g'(\theta)$ of g at θ with $g'(\theta)$

$\neq 0$. *Then*

(3.11.2) $$h(d)(g(T(d))-g(\theta)) \xrightarrow[\mathcal{L}]{} N(0, \sigma^2(g'(\theta))^2) \quad as \quad d \to 0 .$$

Proof. By the Taylor expansion we get

(3.11.3) $$g(T(d))-g(\theta)=(T(d)-\theta)(g'(\theta)+\varepsilon(d)),$$

where $\varepsilon(d)\to 0$ as $T(d)-\theta\to 0$. It follows (3.11.1) that

(3.11.4) $$\lim_{d\to 0} \varepsilon(d)=0 \quad \text{in probability}.$$

Combining (3.11.1), (3.11.3) and (3.11.4) we have (3.11.2), concluding the lemma.

Corollary 3.12. *Suppose that* f *is loc. Lip.* λ *at* x *and* $f(x)>0$. *Let* $\{h_n\}$ *be given by* (3.8.1). *Then we have*

(3.12.1) $$(1-r)Df(x)(d\mu(d))^{-1}(N(d)-\mu(d)) \xrightarrow[\mathcal{L}]{} N(0, 1) \quad as \quad d \downarrow 0,$$

where $\mu(d)=(D^2Bf(x)/d^2)^{1/(1-r)}$.

Proof. Since $r>\frac{p}{2\lambda+p}\geqq\frac{1}{3}$ for $p\geqq 1$ and $0<\lambda\leqq 1$, we get (3.10.1). For convenience N denotes $N(d)$. As stated in Corollary 3.8 all conditions on $\{h_n\}$ of Theorem 3.6 are satisfied. Thus, using (3.3.1) and Theorem 3.10 we have

$$(DB)^{-1}d(N^{1-r}-D^2Bd^{-2}f(x)) \xrightarrow[\mathcal{L}]{} N(0, 1) \quad \text{as} \quad d \downarrow 0,$$

which is equivalent to

(3.12.2) $$(dDB)^{-1}(d^2N^{1-r}-D^2Bf(x)) \xrightarrow[\mathcal{L}]{} N(0, 1) \quad \text{as} \quad d \downarrow 0.$$

By making use of (3.12.2) and Lemma 3.11 we obtain (3.12.1), which concludes the corollary.

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