WEAK PARETO OPTIMALITY OF MULTIOBJECTIVE PROBLEM IN A BANACH SPACE

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By

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I. Introduction.

In the previous paper ([5]), we studied the ordinary multiobjective convex program on a locally convex linear topological space in the case that the objective functions and the constraint functions were continuous and convex, but not always Gâteaux differentiable. In the case, we showed that the generalized Kuhn-Tucker conditions given by a subdifferential formula were necessary and sufficient for weak Pareto optimum.

In this paper, we consider the ordinary multiobjective program on a Banach space in the case that objective functions and constraint functions are locally Lipschitzian but not always convex, and derive Kuhn-Tucker forms given by Clarke's generalized gradients ([1]) as necessary conditions for weak Pareto optimum. Theorem 2.1 is a generalization of Theorem 1.1 of Schechter ([6]) which is concerned to ordinary program with a scalar-valued objective function.

In this paper, X and X^* are a real Banach space and its continuous dual, whose origins are denoted by θ and θ^*, respectively. By $\emptyset$ we denote the empty set.

2. Necessary condition for weak Pareto optimality.

Let $f_1, f_2, \cdots, f_m, g_1, g_2, \cdots, g_n$ be real-valued functions on a real Banach space X and Λ be a subset of X. We denote $\Omega_j=\{x\in X: g_j(x)\leqq 0\}$, $j=1, 2, \cdots, n$ and $\Omega=\bigcap_{j=1}^{n} \Omega_j\cap\Lambda$.

A vector $x_0\in\Omega$ is called to be *weak Pareto optimal* if there is no $x\in\Omega$ such that $f_i(x)<f_i(x_0)$ for every $i=1, 2, \cdots, m$.

The *ordinary multiobjective program* is the problem to find vectors $x_0\in\Omega$ of weak Pareto optimum and written by (Ω, f), where $f=(f_1, f_2, \cdots, f_m)$.

A real-valued function f on X is said to be *locally Lipschitzian* if any point in X admits a neighborhood U such that, for some constant K, for all y and z in U, we have

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$$|f(y)-f(z)| \leqq K\|y-z\|,$$

where $\|x\|$ denotes the norm of x.

For a locally Lipschitz function f and for x_0, x in X, $f^\circ(x_0; x)$ defined as follows is said to be the *generalized directional derivative* in the direction x at x_0;

$$f^\circ(x_0; x)=\limsup_{\substack{h\to\theta \\ t\downarrow 0}} \frac{f(x_0+h+tx)-f(x_0+h)}{t}.$$

We denote by $\partial^* f(x_0)$ the subdifferential of the convex continuous function $f^\circ(x_0; \cdot)$ at θ, that is,

$$\partial^* f(x_0)=\{\phi\in X^* : f^\circ(x_0; x)\geqq\phi(x) \text{ for every } x\in X\}.$$

We call it *Clarke gradient* of f at x_0 ([1]).

By the *normal cone* of convex set Λ at the point x_0 in Λ we mean the set

$$\{\phi\in X^* : \phi(z-x_0)\leqq 0 \text{ for every } z\in\Lambda\}$$

and denote it by $N(x_0, \Lambda)$ ([4]).

The main theorem is the following:

THEOREM 2.1. *Assume that all objective functions f_i, $i=1, 2, \cdots, m$ and all constraint functions g_j, $j=1, 2, \cdots, n$, are locally Lipschitzian and that the constraint set Λ is a convex subset of X. Then, if x_0 is weak Pareto optimal for the ordinary multiobjective program (Ω, f), there exist some real numbers $\lambda_i\geqq 0$ $(i=1, 2, \cdots, m)$, $\mu_j\geqq 0$ $(j=1, 2, \cdots, n)$ with $(\lambda_1, \cdots, \lambda_m, \mu_1, \cdots, \mu_n)\neq(0, \cdots, 0)$ and some vectors $y_i\in\partial^* f_i(x_0)$ $(i=1, 2, \cdots, m)$, $z_j\in\partial^* g_j(x_0)$ $(j=1, 2, \cdots, n)$, $w\in N(x_0, \Lambda)$ such that*

$$\sum_{i=1}^{m} \lambda_i y_i+\sum_{j=1}^{n} \mu_j z_j+w=\theta^*$$

and $\mu_j g_j(x_0)=0$ $(j=1, 2, \cdots, n)$.

3. Proof of Theorem 2.1.

In this section, we recall a few definitions and state lemmas which we use to prove Theorem 2.1.

DEFINITION 3.1. Let $f: X\to R^1$ and $x_0\in X$. A vector $\bar{x}\in X$ is called a *direction of decrease* of f at x_0 if there exist a neighborhood V of the vector $\bar{x}$ and a real $\varepsilon>0$ such that $x\in V$ and $0<t<\varepsilon$ imply $f(x_0+tx)<f(x_0)$. The set of all such vectors $\bar{x}$ is denoted by $DC(x_0, f)$.

REMARK 3.1. The set $DC(x_0, f)$ is an open cone with vertex at the origin θ in X, or else it is void.

DEFINITION 3.2. Let Ω be a subset of X and let $x_0\in X$. A vector $\bar{x}\in X$ is called an *admissible direction* with respect to Ω at x_0 if there exist a neighborhood V of the vector $\bar{x}$ and a real $\varepsilon>0$ such that $x\in V$ and $0<t<\varepsilon$ imply $x_0+tx\in\Omega$. The set of all such vectors $\bar{x}$ is denoted by $AC(x_0, \Omega)$.

REMARK 3.2. The set $AC(x_0, \Omega)$ is an open cone with vertex at the origin θ in X.

DEFINITION 3.3. Let Λ be a subset of X and let $x_0\in X$. A vector $\bar{x}\in X$ is called a *tangent direction* to Λ at x_0 if there exist a real $\varepsilon>0$ and a mapping $r:[0, \varepsilon]\to X$ such that $x_0+t\bar{x}+r(t)\in\Lambda$ for all t in $[0, \varepsilon]$ and $r(t)/t\to\theta$ as $t\downarrow 0$ where $[0, \varepsilon]$ denotes the closed interval $0\leqq t\leqq\varepsilon$. The set of all such vectors $\bar{x}$ is denoted by $TC(x_0, \Lambda)$.

REMARK 3.3. The set $TC(x_0, \Lambda)$ is a cone with vertex at the origin θ in X.

DEFINITION 3.4. The *negative polar cone* of a cone K with vertex at the origin θ in X is the set

$$K^{\circ}=\{\phi\in X^*: \phi(x)\leqq 0,\ x\in K\}\ .$$

LEMMA 3.1. ([4]). *Let $g: X\to R^1$ be any function and $\Omega=\{x\in X: g(x)\leqq 0\}$. Then, if $g(x_0)=0$, $DC(x_0, g)\subset AC(x_0, \Omega)$.*

LEMMA 3.2. ([6]). *Let $f: X\to R^1$ be a locally Lipschitz function and $x_0\in X$. If $f^{\circ}(x_0; \bar{x})<0$, then $\bar{x}\in DC(x_0, f)$.*

LEMMA 3.3. ([6]). *Let $f: X\to R^1$ be a locally Lipschitz function and $x_0\in X$. Define $C=\{x\in X: f^{\circ}(x_0; x)<0\}$ and assume $\theta^*\notin\partial^*f(x_0)$, then $C^{\circ}=\{\lambda y: \lambda\geqq 0,\ y\in\partial^*f(x_0)\}$.*

LEMMA 3.4. (Dubovitskii-Milyutin, [2]-[4]). *Let K_0 be a convex cone with vertex at the origin θ of X and $K_1, \cdots, K_n$ open convex cones with vertex at the origin θ of X. Then $\bigcap_{i=0}^{n} K_i=\emptyset$ if and only if there exist some $y_i^*\in K_i^{\circ}$, $i=0, 1, \cdots, n$, not all identically θ^*, such that*

$$y_0^*+y_1^*+\cdots+y_n^*=\theta^*\ .$$

LEMMA 3.5. ([4]). *For a subset Λ of X and $x_0\in X$, define $K_0=TC(x_0, \Lambda)$. Then, if Λ is a convex subset of X, $K_0^{\circ}=N(x_0, \Lambda)$.*

LEMMA 3.6. *Uuder the assumption on Theorem 2.1, define $K_0=TC(x_0, \Lambda)$,*

$$K_j=\{x\in X: g_j^{\circ}(x_0; x)<0\}\ ,\quad j=1, 2, \cdots, n$$

and

$$K_{n+i}=\{x\in X: f_i^{\circ}(x_0; x)<0\}\ ,\quad i=1, 2, \cdots, m\ ,$$

then K_0 is a convex cone at θ and K_i are open convex cones at θ for $i=1, 2, \cdots, n+m$. If x_0 is weak Pareto optimal for the ordinary multiobjective program (Ω, f), then there exist some $y_i^*\in K_i^{\circ}$ for i in $\{0, n+1, \cdots, n+m\}\cup J$, not all identically θ^*, such that

$$y_0^*+\sum_{j\in J} y_j^*+\sum_{i=1}^{m} y_{n+i}^*=\theta^*$$

where $J=\{j\in N: g_j(x_0)=0\}$ and $N=\{1, 2, \cdots, n\}$.

PROOF of LEMMA 3.6. By Lemma 3.4, it is sufficient to prove that

$$K_0\cap\Big(\bigcap_{j\in J} K_j\Big)\cap\Big(\bigcap_{i=1}^{m} K_{n+i}\Big)=\emptyset$$

holds. Suppose the contrary that there exists some

$$\bar{x}\in K_0\cap\Big(\bigcap_{j\in J} K_j\Big)\cap\Big(\bigcap_{i=1}^{m} K_{n+i}\Big)\ .$$

By Lemma 3.1 and Lemma 3.2,

$$\bar{x}\in DC(x_0, f_i)\qquad \text{for each}\quad i=1, 2, \cdots, m\ ,$$

and

$$\bar{x} \in DC(x_0, g_j) \subset AC(x_0, \Omega_j) \qquad \text{for each} \quad j \in J.$$

Then there exist a sufficiently small $t_1 > 0$ and some vector x_1 in a neighborhood of $\bar{x}$ such that

$$f_i(x_0 + t_1 x_1) < f_i(x_0) \qquad \text{for every} \quad i = 1, 2, \cdots, m,$$

$$x_0 + t_1 x_1 \in \Omega_j \qquad \text{for every} \quad j \in J,$$

$$x_0 + t_1 x_1 \in \Lambda$$

and

$$x_0 + t_1 x_1 \in \Omega_j \qquad \text{for every} \quad j \in N - J.$$

The last follows from $N - J = \{j \in N : g_j(x_0) < 0\}$ and from that x_0 is an interior point of a subset Ω_j in X for each $j \in N - J$. This contradicts the weak Pareto optimality of x_0. Hence we have the conclusion of Lemma 3.6.

PROOF of THEOREM 2.1. Let x_0 be weak Pareto optimal. In the case that $\theta^* \notin \partial^* f_i(x_0)$ for every $i = 1, 2, \cdots, m$ and that $\theta^* \notin \partial^* g_j(x_0)$ for every j in J, by Lemma 3.6 there exist some $y_i^* \in K_i^\circ$ for i in $\{0, n+1, \cdots, n+m\} \cup J$, not all identically θ^*, such that

$$y_0^* + \sum_{j \in J} y_j^* + \sum_{i=1}^{m} y_{n+i}^* = \theta^*.$$

Furthermore, by Lemma 3.3 and Lemma 3.5 we have

$$K_{n+i}^\circ = \{\lambda y : \lambda \geqq 0, \ y \in \partial^* f_i(x_0)\} \qquad \text{for} \quad i = 1, 2, \cdots, m,$$

$$K_j^\circ \ = \{\mu z : \mu \geqq 0, \ z \in \partial^* g_j(x_0)\} \qquad \text{for} \quad j \text{ in } J$$

and $K_0^\circ = N(x_0, \Lambda)$. For j in $N - J$, put $\mu_j = 0$, then we have the conclusion of Theorem 2.1 except the condition $(\lambda_1, \cdots, \lambda_m, \mu_1, \cdots, \mu_n) \neq (0, \cdots, 0)$. Supposed the contrary that $(\lambda_1, \cdots, \lambda_m, \mu_1, \cdots. \mu_n) = (0, \cdots, 0)$, then $y_i^* = \theta^*$ for $i \in \{0, n+1, \cdots, n+m\} \cup J$ which is the contradiction.

In the case that $\theta^* \in \partial^* f_{i_0}(x_0)$ for some i_0, sufficiently let λ_{i_0} be any positive number, $\lambda_i = 0$ except for $i = i_0$, $\mu_j = 0$ for $j = 1, 2, \cdots, n$, $y_{i_0} = \theta^*$, y_i be any vector in $\partial^* f_i(x_0)$ except for $i = i_0$, z_j be any vector in $\partial^* g_j(x_0)$ for $j = 1, 2, \cdots, n$ and $w = \theta^*$.

In the other case that $\theta^* \in \partial^* g_{j_0}(x_0)$ for some j_0 in J, sufficiently let $\lambda_i = 0$ for $i = 1, 2, \cdots, m$, μ_{j_0} be any positive number, $\mu_j = 0$ except for $j = j_0$, y_i be any vector in $\partial^* f_i(x_0)$ for $i = 1, 2, \cdots, m$, $z_{j_0} = \theta^*$, z_j be any vector in $\partial^* g_j(x_0)$ except for $j = j_0$ and $w = \theta^*$.

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