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ON FINITE PARAMETRIC LINEAR MODELS OF DYADIC STATIONARY PROCESSES

By

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Summary

On replacing the usual arithmetic addition by the dyadic addition and trigonometric functions by Walsh functions, daydic stationary processes defined in the sense that their covariance functions are invariant under the shift by the dyadic addition have almost the same structures as those of ordinary stationary processes.

However, there are some differences between finite parametric linear models of dyadic stationary processes and those of ordinary stationary processes.

It is shown in theorem 2 that a dyadic autoregressive process of finite order analogously defined to the ordinary autoregressive models is always inverted into a dyadic moving average process of finite order, and vice versa. As is well known, these results are not true for the ordinary stationary processes (see, e.g., Box and Jenkins [1]).

1. Preliminaries

Let x and y be two non-negative real numbers and have dyadic expansions

$$x=\sum_{k=-\infty}^{\infty} x_k\cdot 2^k, \quad \text{with} \quad x_k=0 \text{ or } 1,$$

$$y=\sum_{k=-\infty}^{\infty} y_k\cdot 2^k, \quad \text{with} \quad y_k=0 \text{ or } 1.$$

Then, the dyadic addition $\oplus$ is defined by

$$x\oplus y=\sum_{k=-\infty}^{\infty}(x_k\oplus y_k)\cdot 2^k, \tag{1.1}$$

where $(x_k\oplus y_k)$ denotes addition mod. 2 of $\{0, 1\}$, that is, $0\oplus 0=1\oplus 1=0$ and $0\oplus 1=1\oplus 0=1$.

We denote by $\{W(n, x), 0\leqq x\leqq 1\}$, $n=0, 1, 2, \cdots$, the system of Walsh functions.

The following properties of Walsh functions are well-known:

(i) For any non-negative integers n and m,

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$$W(n, x)W(m, x)=W(n \oplus m, x), \qquad \text{for a.e.} \quad x \in I,$$

where by I we denote the unit interval [0, 1].

(ii) For each non-negative integer n and for each $y \in I$,

$$W(n, x)W(n, y)=W(n, x \oplus y), \qquad \text{for a.e.} \quad x \in I.$$

(For these matters, see, e.g., Gulamhusein and Fallside [3] and Harmuth [4]).

2. Finite parametric linear models

We call a random process $\{X(n), n=0, 1, 2, \cdots,\}$ with finite second moments to be dyadic stationary if its mean value function is constant and its covariance function

$$R(n, m)=\text{Cov.}(X(n), X(m)),$$

is invariant under the shift by the dyadic addition, that is, for any $n=0, 1, 2, \cdots,$

$$E[X(n)]=\mu, \text{ say},$$

and for non-negative integers n, m and k,

$$R(n, m)=R(n \oplus k, m \oplus k)$$
$$=R(n \oplus m, 0).$$

For simplicity, we only treat the case where mean values of dyadic stationary processes are zero.

As is shown in Nagai [6], every covariance function $R(n, m)$ of dyadic stationary process has the spectral representation

$$R(n, m)=\int_0^1 W(n \oplus m, x)dG(x),$$

where $G(x)$, $0 \leqq x \leqq 1$, is the dyadic spectral distribution which is a monotone increasing function with $G(0)=0$ and $G(1)=R(0, 0)$.

It is also shown in Nagai [6] that a dyadic stationary process $\{X(n), n=0, 1, 2, \cdots,\}$ with its dyadic spectral distribution $G(x)$ has the following representation

$$X(n)=\int_0^1 W(n, x)dZ(x),$$

where $\{Z(x), 0 \leqq x \leqq 1\}$ is a random process with orthogonal increments such that

$$E[dZ(x)]=0,$$

and

$$E[(dZ(x))^2]=dG(x).$$

We call a random process $\{u(n), n=0, 1, 2, \cdots,\}$ a white noise process with variances σ^2, if its mean values are zero and its covariances are such that

$$E[u(n)u(m)]=\sigma^2, \qquad \text{for} \quad n=m,$$
$$=0, \qquad \text{for} \quad n \neq m.$$

Since a white noise process $\{u(n),\ n=0, 1, 2, \cdots,\}$ with variances σ^2 is dyadic stationary, it has the spectral representation

(2.1) $$u(n)=\int_0^1 W(n,\ x)dU(x)\,,$$

where $\{U(x),\ 0\leqq x\leqq 1\}$ is a random process with orthogonal increments such that $E[dU(x)]=0$ and

(2.2) $$E[(dU(x))^2]=\sigma^2\cdot dx\,.$$

We define a linear dyadic process $\{X(n),\ n=0, 1, 2, \cdots,\}$ after Morettin [5] by

(2.3) $$X(n)=\sum_{k=0}^{\infty} a(k)u(n\oplus k)\,,$$

where $\{u(n),\ n=0, 1, 2, \cdots,\}$ is a white noise process and $a(k),\ k=0, 1, 2, \cdots$, are real numbers such that $\sum_{k=0}^{\infty} a(k)^2<\infty$. Linear dyadic processes are dyadic stationary. In particular, if $a(q)\neq 0$ and $a(q+1)=a(q+2)=\cdots=0$, with an integer q, we call the linear dyadic process given by (2.3) a dyadic moving average process of order q (DMA(q)-process).

We also define a dyadic autoregressive process of order p (DAR(p)-process) by a dyadic stationary process $\{X(n),\ n=0, 1, 2, \cdots,\}$ which satisfies the following equation:

(2.4) $$\sum_{k=0}^{p} b(k)X(n\oplus k)=u(n)\,,$$

where $\{u(n),\ n=0, 1, 2, \cdots,\}$ is a white noise process and $b(k),\ k=0, 1, \cdots, p$ are real numbers with $b(0)=1$ and $b(p)\neq 0$.

Concerning the existence condition of DAR-process and their spectral representations, we have the following theorem.

THEOREM 1. *Let us put*

(2.5) $$B(x)=\sum_{k=0}^{p} b(k)W(k,\ x)\,,\qquad 0\leqq x\leqq 1\,,$$

where $b(k),\ k=0, 1, 2, \cdots, p$ *are real numbers given in* (2.4).

Then, if $B(x)\neq 0$, *a.e.* x, *there exists a unique dyadic stationary process* $\{X(n),\ n=0, 1, 2, \cdots,\}$ *satisfying* (2.4).

The process $\{X(n),\ n=0, 1, 2, \cdots,\}$ *has the spectral representation*

(2.6) $$X(n)=\int_0^1 [W(n,\ x)/B(x)]dU(x)\,,$$

where $\{U(x),\ 0\leqq x\leqq 1\}$ *is such the process given by the spectral representation* (2.1) *of the white noise process* $\{u(n),\ n=0, 1, \cdots,\}$ *in* (2.4).

PROOF. Since by assumption $B(x)\neq 0$, a.e. $x\in I$ and the values of $W(n,\ x)$ are only $+1$ or -1, we have the following inequality:

$$|B(x)|\geqq \min\{|1\pm b(l)\pm\cdots\pm b(p)|\}>0\,.$$

Hence, $1/B(x)$ is square integrable and so is $W(n, x)/B(x)$.

Let us put

$$X(n)=\int_0^1 [W(n, x)/B(x)]dU(x). \tag{2.7}$$

Then, from (2.2) and the square integrability of $1/B(x)$, we see that the process $\{X(n), n=0, 1, 2\cdots,\}$ is well-defined (see e. g., Doob [2]). Its mean values are zero and its covariance function is given by

$$R(n, m)=E[X(n)X(m)]$$

$$=\sigma^2\cdot\int_0^1 [W(n\oplus m, x)/(B(x))^2]dx.$$

This depends only on $n\oplus m$. Thus, it is seen that the process $\{X(n), n=0, 1, 2, \cdots,\}$ defined by (2.7) is dyadic stationary.

We show next that the process satisfies the equation (2.4). Indeed, substituting $X(n)$ given by (2.7) into (2.4), we have

$$\sum_{k=0}^{p} b(k)X(n\oplus k)=\sum_{k=0}^{p} b(k)\int_0^1 [W(n\oplus k, x)/B(x)]dU(x)$$

$$=\int_0^1 \left[W(n, x)\left(\sum_{k=0}^{p} b(k)W(k, x)\right)\Big/B(x)\right]dU(x)$$

$$=\int_0^1 W(n, x)dU(x)$$

$$=u(n).$$

Thus, we see that the process defined by (2.7) is a dyadic stationary solution of the equation (2.4). In order to prove uniqueness of the dyadic stationary solution of (2.4), let us suppose that there is another dyadic stationary process $\{\tilde{X}(n), n=0, 1, 2, \cdots,\}$ satisfying (2.4). Let the spectral representation of $\tilde{X}(n)$ be given by

$$\tilde{X}(n)=\int_0^1 W(n, x)d\tilde{Z}(x), \tag{2.8}$$

where $\{\tilde{Z}(x), 0\leqq x\leqq 1\}$ is random process with orthogonal increments. Then, it follows from (2.8) that

$$\sum_{k=0}^{p} b(k)X(n\oplus k)=\int_0^1 W(n, x)B(x)d\tilde{Z}(x) \tag{2.9}$$

$$=u(n)$$

$$=\int_0^1 W(n, x)dU(x).$$

From this relation (2.9), the dyadic spectral distributions corresponding to the processes $\{U(x), 0\leqq x\leqq 1\}$ and $\{\tilde{Z}(x), 0\leqq x\leqq 1\}$ are absolutely continuous. Hence, for every subinterval J of I, we have

$$U(J)=\int_0^1 I_J(x)dU(x)$$

$$=\underset{M\to\infty}{\text{l.i.m}}\sum_{k=0}^{M} c_J(k)\cdot\int_0^1 W(k,\ x)dU(x)$$

$$=\underset{M\to\infty}{\text{l.i.m}}\sum_{k=0}^{M} c_J(k)\cdot\int_0^1 W(k,\ x)B(x)d\tilde{Z}(x)$$

$$=\int_0^1 I_J(x)B(x)d\tilde{Z}(x),$$

where $I_J(x)$ is the indicator function of J whose Walsh-Fourier expansion is given by

$$I_J(x)=\sum_{k=0}^{\infty} c_J(k)W(k,\ x), \qquad \text{a.e.} \quad x,$$

with

$$c_J(k)=\int_J W(k,\ x)dx.$$

Thus, we have

$$d\tilde{Z}(x)=[1/B(x)]dU(x),$$

and hence for every n,

$$\tilde{X}(n)=\int_0^1 W(n,\ x)d\tilde{Z}(x)$$

$$=\int_0^1 [W(n,\ x)/B(x)]dU(x)$$

$$=X(n).$$

This completes the proof of theorem 1. Q. E. D.

In order to prove the equivalency between DMA-processes and DAR-processes, we prepare some notations and a lemma. Proof of lemma will be given in the next section.

For a positive integer m, we put $d(m)=2^m$ and we denote by $K(m)$ the set of all non-negative integers less than or equal to $d(m)-1$.

Let us call a function $\varphi(x)$, $0\leqq x\leqq 1$, to be of order m if it is written as

$$\varphi(x)=\sum_{k=0}^{p}\alpha(k)W(k,\ x), \qquad \text{a.e.} \quad x\in I,$$

with an integer $p\in K(m)$. In particular, we call a function $\varphi(x)$ to be strictly of order m if it is of order m but not of order $m-1$.

LEMMA. Let a function $\varphi(x)$ be strictly of order m and not zero a.e. x. Then, we can find a function $h(x)$ of order m strictly and satisfying that

$$h(x)\cdot\varphi(x)=c\neq 0, \qquad \text{a.e.} \quad x, \tag{2.10}$$

where c is a non-zero constant.

By means of this lemma, we can establish the equivalency between DMA-processes and DAR-processes.

THEOREM 2. (i) *Let* $\{X(n),\ n=0, 1, 2, \cdots, \}$ *be a* DMA(p)*-process such that* $d(m-1) \leqq p \leqq d(m)-1$. *Then, the process is a* DMA(q)*-precess such that* $d(m-1) \leqq q \leqq d(m)-1$.

(ii) *Conversely, let* $\{X(n),\ n=0, 1, 2, \cdots,\}$ *be a* DMA(q)*-process such that* $d(m-1) \leqq q \leqq d(m)-1$ *and be defined by*

$$X(n)=\sum_{k=0}^{q} a(k)u(n\oplus k)\,, \tag{2.11}$$

with a white noise process $\{u(n),\ n=0, 1, 2, \cdots,\}$. *Then, the process is a* DAR(p)*-process such that* $d(m-1) \leqq p \leqq d(m)-1$ *if*

$$\sum_{k=0}^{q} a(k)W(k,\ x) \neq 0 \qquad a.e. \quad x\,.$$

PROOF. (i) Suppose that the DAR(p)-process $\{X(n),\ n=0, 1, 2, \cdots,\}$ satisfies the equation (2.4) with $d(m-1) \leqq p \leqq d(m)-1$. Then, the function $B(x)$, $0 \leqq x \leqq 1$, defined by (2.5) in theorem 1 is strictly of order m and must not be zero a.e. x. (Otherwise, there exists no dyadic stationary process satisfying (2.4).)

From lemma, we can find a function $h(x)$ strictly of order m such that

$$h(x)B(x)=\hat{c} \neq 0\,, \qquad \text{a.e.} \quad x\,,$$

where $\hat{c}$ is a non-zero constant. The function $h(x)$ may be written as

$$\begin{aligned} h(x) &= \hat{c}/B(x) \\ &= \sum_{k=0}^{d(m)-1} a(k)W(k,\ x)\,, \qquad \text{a.e.} \quad x\,, \end{aligned} \tag{2.12}$$

and it is not zero a.e. x.

From theorem 1, the DAR(p)-process has the spectral representation (2.6), that is,

$$X(n)=\int_0^1 [W(n,\ x)/B(x)]dU(x)\,,$$

with the random process $\{U(x),\ 0 \leqq x \leqq 1\}$ defined by (2.1). Therefore, substituting (2.12) into (2.6), we have

$$\begin{aligned} X(n) &= \int_0^1 W(n,\ x)\left[\sum_{k=0}^{d(m)-1} a(k)W(k,\ x)\right]dU(x)/\hat{c} \\ &= \sum_{k=0}^{d(m)-1} a(k)\cdot\int_0^1 W(n\oplus k,\ x)dU(x)/\hat{c} \\ &= \sum_{k=0}^{d(m)-1} a(k)u(n\oplus k)/\hat{c}\,. \end{aligned}$$

Thus, it is seen that the DAR(p)-process turns out to be a DMA(q)-process such that $d(m-1) \leqq q \leqq d(m)-1$.

(ii) Since the white noise process $\{u(n),\ n=0, 1, 2, \cdots,\}$ has the spectral representation (2.1), the spectral representation of the DMA(q)process (2.11) is given by

(2.13) $$X(n)=\int_0^1 W(n, x)A(x)dU(x),$$

where we put

$$A(x)=\sum_{k=0}^{q} a(k)W(k, x).$$

The function $A(x)$ is strictly of order m. If we assume that $A(x)\neq 0$, a.e. x, then from lemma we can find a function $B(x)$, $0\leqq x\leqq 1$, strictly of order m such that

$$B(x)=c_1/A(x)$$

$$=\sum_{k=0}^{d(m)-1} b(k)W(k, x), \quad \text{a.e. } x,$$

where c_1 is a non-zero constant. Thus, we have

(2.14) $$A(x)=c_1/B(x), \quad \text{a.e. } x.$$

Substituting (2.14) into (2.13), we see that $X(n)$ has the spectral representation

(2.15) $$X(n)=\int_0^1 [c_1 W(n, x)/B(x)]dU(x).$$

It follows from theorem 1 that the dyadic stationary process $\{X(n), n=0, 1, 2, \cdots, \}$ having the spectral representation (2.15) satisfies the following equation:

$$\sum_{k=0}^{d(m)-1} b(k)X(n\oplus k)=c_1 u(n).$$

Thus, the DMA(q)-process has turned out to be a DAR(p)-process such that $d(m-1)\leqq p\leqq d(m)-1$. Q.E.D.

3. Proof of lemma

For $m=1$, a function $\varphi(x)$ of order 1 may be written as

$$\varphi(x)=a(0)+a(1)W(1, x), \quad \text{a.e. } x,$$

with some constants $a(0)$ and $a(1)$. Let us put

$$h(x)=a(0)-a(1)W(1, x).$$

Then, $h(x)$ is of order 1. If $\varphi(x)\neq 0$, a.e. x, clearly $h(x)\neq 0$ a.e. x and the product of $\varphi(x)$ and $h(x)$ is a non-zero constant a.e. x, that is,

$$\varphi(x)h(x)=a(0)^2-a(1)^2\neq 0, \quad \text{a.e. } x.$$

Thus, lemma holds for $m=1$.

Suppose that for some positive integer m, our lemma holds.

Let us consider a function $\varphi(x)$ being strictly of order $m+1$ and such that

(3.1) $$\varphi(x)\neq 0, \quad \text{a.e. } x.$$

Then, it is written as

$$\varphi(x)=\sum_{k\in K(m+1)}\alpha(k)W(k, x)$$

$$=\sum_{k\in K(m)}\alpha(k)W(k, x)+\sum_{k=d(m)}^{d(m+1)-1}\alpha(k)W(k, x)$$

$$=\sum_{k\in K(m)}\alpha(k)W(k, x)+\sum_{k\in K(m)}\alpha(d(m)+k)W(d(m)+k, x), \quad \text{a.e. } x.$$

By noting that for every $n\in K(m)$, $d(m)+n=d(m)\oplus n$ and therefore $W(d(m)+n, x)=W(d(m), x)W(n, x)$, we have

$$\varphi(x)=g(x)+W(d(m), x)f(x), \quad \text{a.e. } x,$$

where we put

$$g(x)=\sum_{k\in K(m)}\alpha(k)W(k, x),$$

and

$$f(x)=\sum_{k\in K(m)}\alpha(d(m)+k)W(k, x).$$

Both $g(x)$ and $f(x)$ are of order m and $f(x)$ is not identically zero.

Let us put

(3.2) $$h_1(x)=g(x)-W(d(m), x(f(x).$$

Then, $h_1(x)$ is strictly of order $m+1$ and not zero a.e. x since $W(d(m), x)=+1$ or -1. The product of $\varphi(x)$ and $h_1(x)$ is therefore not zero a.e. x and

(3.3) $$\varphi(x)h_1(x)=g(x)^2-W(d(m, x)^2f(x)^2$$
$$=g(x)^2-f(x)^2.$$

Since $K(m)$ is closed under the dyadic addition $\oplus$, we see that $g(x)^2$, $f(x)^2$ and therefore $g(x)^2-f(x)^2$ are of order m. Thus, we see that the product $\varphi(x)h_1(x)$ is of order m and not zero a.e. x.

From the assumption for induction, we can find a function $h_2(x)$ of order m such that

(3.4) $$h_2(x)\neq 0, \quad \text{a.e. } x, \text{ and}$$

(3.5) $$[\varphi(x)h_1(x)]\cdot h_2(x)=c_2\neq 0, \quad \text{a.e. } x,$$

where c_2 is a non-zero constant. Let us put $h(x)=h_1(x)h_2(x)$. Then, $h(x)$ is strictly of order $m+1$, since $h_1(x)$ is strictly of order $m+1$ and $h_2(x)$ of order m. From (3.1) and (3.4) together with the fact that the product $\varphi(x)h_1(x)\neq 0$, a.e. x, it follows that $h(x)=h_1(x)h_2(x)\neq 0$, a.e. x. It is clear from (3.5) that $h(x)$ satisfies (2.10). This completes the proof of lemma. Q.E.D.

References

[1] BOX, G.E.P. and JENKINS, G.M., Time Series Analysis, Forecasting and Control, Holden-Day, 1970.

[2] DOOB, J.L., Stochastic Processes, Wiley, New York, 1953.
[3] GULAMHUSEIN, M.N. and FALLSIDE, F., *Short-time spectral and autocorrelation analysis in the Walsh domain*, IEEE Trans. Information Theory, IT-19 (1973), 615-623.
[4] HARMUTH, H.F., Transmission of Information by Orthogonal Functions, Springer-V., Berlin, 1972.
[5] MORETTIN, P.A., *Walsh function analysis of a certain class of time series*, Stochastic Processes and Their Applications, 2 (1974), 183-193.
[6] NAGAI, T., *Dyadic stationary processes and their spectral representations*, Bull. Math. Statist., 17 (1977), 65-73.