AN OPTIMALITY EQUATION IN CONTROLLED JUMP PROCESSES WITH A CHOICE OF STOPPING RULES

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AN OPTIMALITY EQUATION IN CONTROLLED JUMP PROCESSES WITH A CHOICE OF STOPPING RULES

By

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1. Introduction

Controlled Markov jump processes associated with a choice of stopping rules on general state and action spaces have been first introduced by Ohtsubo [5]. In that paper the author has given the optimal return in the form of the limit of successive approximations and also given necessary and sufficient conditions for a control (a pair of policy and stopping rule) to be optimal in terms of an inequality and a functional equation involving an infinitesimal operator. The latter result, however, is valid only when an optimal control exists. In general, the functional equation arising in the optimization of controlled processes is the so-called optimality equation that should be satisfied by an optimal return over infinite future. Main purpose of the present paper is to make a study of the functional equation in connection with the optimal return in controlled jump processes with a choice of stopping rules.

Kakumanu [4] and Doshi [2] have shown that the optimal return in a certain type of continuous time Markov decision model satisfies the optimality equation. Shiryayev [6] and Furukawa [3] have studyed optimality equations in optimal stopping problems and in stopped decision problems, respectively.

In Section 2, we shall introduce notations and definitions to be used throughout this paper, and in Section 3 we shall set up several assumptions and prepare fundamental lemmas to be used in Section 4. In Section 4, we shall show that the optimal return satisfies a functional equation involving an infinitesimal operator, that is to say, an optimality equation.

2. Preliminaries

The *state space* S is a non-empty Borel subset of a complete separable metric space. Z is the cartesian product of S and R^+ where the *time space* R^+ is the set of all non-negative real numbers. The *action space* A is a non-empty Borel subset of a complete separable metric space. $\mathscr{B}(S)$, $\mathscr{B}(Z)$ and $\mathscr{B}(A)$ mean σ-fields of Borel subsets of S, Z and A, respectively. The *terminal reward function* g is a bounded

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function on Z such that for each $t \geqq 0$ $g(\cdot, t)$ is $\mathscr{B}(S)$-measurable, and for each $x \in S$ $g(x, \cdot)$ is right continuous on R^+.

A mapping π from Z to A is called an *admissible policy* if for each $t \geqq 0$ $\pi(\cdot, t)$ is $\mathscr{B}(S)/\mathscr{B}(A)$-measurable and for each $x \in S$ $\pi(x, \cdot)$ is piecewise constant, right continuous on R^+ and has only a finite number of discontinuities on any finite interval. Let Π denote the set of all admissible policies.

In order to determine the processes, we need to introduce a real-valued function λ on $Z \times A$ and a Markov kernel Q. We assume the following conditions:

(i) For each $t \geqq 0$, $\lambda(\cdot, t, \cdot)$ is $\mathscr{B}(S) \times \mathscr{B}(A)$-measurable.

(ii) For each $(x, a) \in S \times A$, $\lambda(x, \cdot, a)$ is right continuous on R^+.

(iii) There exists a natural number $M < \infty$ such that $0 < \lambda(z, a) < M$ for all $(z, a) \in Z \times A$.

(iv) For each $(z, a) \in Z \times A$, $Q(\cdot \mid z, a)$ is a probability measure on $\mathscr{B}(S)$.

(v) For each $(x, t, a) \in Z \times A$, $Q(\{x\} \mid x, t, a) = 0$.

(vi) For each $(t, B) \in R^+ \times \mathscr{B}(S)$, $Q(B \mid \cdot, t, \cdot)$ is $\mathscr{B}(S) \times \mathscr{B}(A)$-measurable.

(vii) For each $(x, a) \in S \times A$, $Q(B \mid x, \cdot, a)$ is piecewise constant, right continuous on R^+ and has only a finite number of discontinuities on any finite interval, independently of all $B \in \mathscr{B}(S)$.

The jump process $Z^\pi = (Z_t^\pi)_{t \geqq 0} = (X_t^\pi, t)_{t \geqq 0}$ (corresponding to $\pi \in \Pi$) is defined by the following probability laws:

$$P_{(x,s)}^\pi[\nu(x, s) \leqq t] = 1 - \exp\left\{-\int_s^{s+t} \lambda(x, s', \pi(x, s'))ds'\right\},$$

$$P_{(x,s)}^\pi[X_{s+t}^\pi \in B \mid \nu(x, s) = t] = Q(B \mid x, s+t, \pi(x, s+t))$$

for each $(x, s, t) \in Z \times R^+$ and any $B \in \mathscr{B}(S)$, where $\nu(x, s)$ is a holding time such that the x-component of the process starts in x at time s and remains there a random time $\nu(x, s)$. Then the process Z^π is a Markov jump process.

By Blumental and Getoor [1] we may regard the *sample space* Ω as the set of all functions ω: $R^+ \to Z$ such that (i) for each $t \geqq 0$ there exists a state $y \in S$ satisfying $\omega(t) = (y, t)$, (ii) for each $t \geqq 0$ a left-hand limit $\omega(t-0)$ does exist, and (iii) for each $t \geqq 0$ if $\omega(t) = (x, t) \in Z$ then there exists a $h > 0$ such that $\omega(t+h') = (x, t+h')$ for all $h' \in [0, h)$.

For each $0 \leqq s \leqq t$ let $\mathscr{F}_t^s$ denote the σ-field of subsets of Ω generated by the sets $\{\omega \in \Omega \mid \omega(s') \in B\}$, $B \in \mathscr{B}(Z)$, $s \leqq s' \leqq t$.

For each $\pi \in \Pi$ and each $s \geqq 0$ let $C_s(\pi)$ denote the set of all stopping times τ with respect to $\{\mathscr{F}_t^s\}_{t \geqq s}$ satisfying that $P_{(x,s)}^\pi(s \leqq \tau < \infty) = 1$ for all $x \in S$.

The *expected terminal return* φ_τ^π of our system corresponding to $\pi \in \Pi$ and $\tau \in C_s(\pi)$ is defined by

$$\varphi_\tau^\pi(x, s) = E_{(x,s)}^\pi[g(Z_\tau^\pi)] \qquad \text{for} \quad (x, s) \in Z,$$

where $E_{(x,s)}^\pi$ denotes the expectation operator with respect to $P_{(x,s)}^\pi$.

The *optimal return* u^* is defined by

$$u^*(x, s) = \sup_{\pi \in \Pi,\, \tau \in C_s(\pi)} \varphi_\tau^\pi(x, s) \qquad \text{for} \quad (x, s) \in Z.$$

DEFINITION 2.1. Let f be a bounded universally measurable function on Z.

(i) f is said to be *excessive*, if it holds that for each $(x, s)\in Z$ and any $\pi\in\Pi$

$$f(x, s)\geqq E^{\pi}_{(x,s)}[f(Z^{\pi}_{t})] \qquad \text{for all} \quad t\geqq s\,,$$

and

$$\liminf_{t\downarrow s} f(Z^{\pi}_{t})\geqq f(x, s) \qquad P^{\pi}_{(x,s)}\text{—a. s.}.$$

(ii) f is called a *majorant of* g, if $f(z)\geqq g(z)$ for all $z\in Z$.

(iii) An excessive majorant f of g is called the *smallest excessive majorant of* g, if for any excessive majorant h of g, $f(z)\leqq h(z)$ for all $z\in Z$.

Let F be the set of all bounded Borel measurable functions on Z. For each $\pi\in\Pi$ let $\tilde{F}^{\pi}$ be the set of all $f\in F$ such that

$$\lim_{t\downarrow 0} E^{\pi}_{(x,s)}[f(Z^{\pi}_{s+t})]=f(x, s) \qquad \text{for} \quad (x, s)\in Z\,.$$

The *weak infinitesimal operator* (corresponding to $\pi\in\Pi$) is defined by

$$\mathcal{A}^{\pi}f(x, s)=\lim_{t\downarrow 0}\frac{1}{t}\,\{E^{\pi}_{(x,s)}[f(Z^{\pi}_{s+t})]-f(x, s)\}\,, \quad (x, s)\in Z\,,$$

for all $f\in\tilde{F}^{\pi}$ such that the limit of the right-hand side exists in the weak sence and belongs to $\tilde{F}^{\pi}$. Let $\mathcal{D}(\mathcal{A}^{\pi})\subset\tilde{F}^{\pi}$ denote the set of such functions and let $\mathcal{D}(\mathcal{A})=\bigcap_{\pi\in\Pi}\mathcal{D}(\mathcal{A}^{\pi})$.

In the remainder of this paper, even in the case when a random variable $\hat{\tau}$ is defined depending on a policy π and a time s, we shall not use an explicit expression such as $\hat{\tau}(\pi, s)$ but use the simple notation $\hat{\tau}$(or, if we need to more precisely, we use the notation $\hat{\tau}(s)$).

3. Fundamental lemmas

We have proved in the previous paper [5] that the optimal return u^* is a universally measurable function on Z and the smallest excessive majorant of g, and that for each $x\in S$ $u^*(x, \cdot)$ is right continuous on R^+. In this section, by using these results, some fundamental lemmas to be used in the following section will be prepared.

Let

$$\Gamma^*=\{z\in Z\,|\,u^*(z)=g(z)\}\,.$$

For each integer $m\geqq 1$, each $s\geqq 0$ and any $\pi\in\Pi$, let

$$\Gamma^*_m=\left\{z\in Z\,|\,u^*(z)\leqq g(z)+\frac{1}{m}\right\},$$

$$\tau_m=\tau_m(s)=\inf\,\{t>s\,|\,Z^{\pi}_{t}\in\Gamma^*_m\}\,,$$

and

$$\sigma_m=\inf\{t\geqq s\,|\,Z^{\pi}_{t}\in\Gamma^*_m\}\,.$$

Then it is obvious that $\Gamma^*=\bigcap_{m=1}^{\infty}\Gamma^*_m$ and $\Gamma^*_m\supset\Gamma^*_n$ for each $n\geqq m$. For each $m\geqq 1$, let's define

$$u_m(\pi)(z)=E^{\pi}_{z}[u^*(Z^{\pi}_{\tau_m})] \qquad \text{for} \quad z\in Z \text{ and } \pi\in\Pi\,,$$

and

$$\tilde{u}_m(z)=\sup_{\pi\in\Pi} u_m(\pi)(z) \qquad \text{for} \quad z\in Z.$$

Throughout this paper we impose the following assumptions.

ASSUMPTION (A). (i) For each integer $m\geqq 1$, $\tilde{u}_m$ is universally measurable on Z.

(ii) Let m be any natural number and let p be any probability measure on $\mathcal{B}(S)$. Then for given $\varepsilon>0$ and $t\geqq 0$, there exists a policy $\pi_{\varepsilon,t}\in\Pi$ such that

$$p\{x\in S:\ u_m(\pi_{\varepsilon,t})(x,\ t)+\varepsilon\geqq\tilde{u}_m(x,\ t)\}=1.$$

ASSUMPTION (B). For any $f\in\mathcal{D}(\mathcal{A})$ and for each $\varepsilon>0$, there exists a policy $\pi_\varepsilon\in\Pi$ such that

$$\sup_{\pi\in\Pi}\mathcal{A}^\pi f(z)<\mathcal{A}^{\pi_\varepsilon}f(z)+\varepsilon \qquad \text{for all} \quad z\notin\Gamma^*.$$

LEMMA 3.1. *For each* $m\geqq 1$, $\tilde{u}_m$ *is excessive.*

PROOF. Let $m\geqq 1$ be fixed but arbitrary.

First we shall show that for each $z=(x,\ s)\in Z$ and any $\pi\in\Pi$,

$$\tilde{u}_m(z)\geqq E_z^\pi[\tilde{u}_m(Z_t^\pi)] \qquad \text{for all} \quad t\geqq s.$$

Let $z=(x,\ s)\in Z$, $\pi\in\Pi$ and $t\geqq s$ be arbitrary. From Assumption (A) (ii), it holds that for any $\varepsilon>0$ there exists a policy $\pi_{\varepsilon,t}\in\Pi$ such that

$$P_z^\pi[u_m(\pi_{\varepsilon,t})(Z_t^\pi)+\varepsilon\geqq\tilde{u}_m(Z_t^\pi)]=1.$$

Thus we have

$$E_z^\pi[\tilde{u}_m(Z_t^\pi)]\leqq E_z^\pi[u_m(\pi_{\varepsilon,t})(Z_t^\pi)]+\varepsilon. \tag{3.1}$$

We define a policy $\hat{\pi}$ by

$$\hat{\pi}(y,\ t')=\begin{cases}\pi(y,\ t') & \text{if} \quad t'<t,\\ \pi_{\varepsilon,t}(y,\ t') & \text{if} \quad t'\geqq t.\end{cases}$$

Then $\hat{\pi}\in\Pi$. Since u^* is excessive and $\tau_m(s)\leqq\tau_m(t)$,

$$\begin{aligned}E_z^\pi[u_m(\pi_{\varepsilon,t})(Z_t^\pi)]&=E_z^\pi[E_{Z_t^\pi}^{\pi_{\varepsilon,t}}[u^*(Z_{\tau_m(t)}^{\pi_{\varepsilon,t}})]]\\&=E_z^{\hat{\pi}}[u^*(Z_{\tau_m(t)}^{\hat{\pi}})]\\&\leqq E_z^{\hat{\pi}}[u^*(Z_{\tau_m(s)}^{\hat{\pi}})]\\&\leqq\tilde{u}_m(z).\end{aligned}$$

This formula combined with (3.1) implies that

$$E_z^\pi[\tilde{u}_m(Z_t^\pi)]\leqq\tilde{u}_m(z)+\varepsilon.$$

Letting $\varepsilon\downarrow 0$ we have

$$E_z^\pi[\tilde{u}_m(Z_t^\pi)]\leqq\tilde{u}_m(z). \tag{3.2}$$

Next we shall show that for each $(x,\ s)\in Z$ and any $\pi\in\Pi$,

$$\liminf_{t\downarrow 0}\tilde{u}_m(Z_{s+t}^\pi)\geqq\tilde{u}_m(x,\ s) \qquad P_{(x,s)}^\pi\text{—a. s.}.$$

It follows by the excessiveness of u^* that for each $(x,\ s)\in Z$, each $0\leqq t\leqq t_0$ and any

$\pi\in\Pi$,

$$u_m(\pi)(x,\ s+t)=E^{\pi}_{(x,s+t)}[u^*(Z^{\pi}_{\tau_m(s+t)})]$$
$$\geqq E^{\pi}_{(x,s+t)}[u^*(Z^{\pi}_{\tau_m(s+t_0)})].$$

From Lemma 4.2 in [5], we get

$$\liminf_{t\downarrow 0} u_m(\pi)(x,\ s+t)\geqq\liminf_{t\downarrow 0} E^{\pi}_{(x,s+t)}[u^*(Z^{\pi}_{\tau_m(s+t_0)})]$$
$$=E^{\pi}_{(x,s)}[u^*(Z^{\pi}_{\tau_m(s+t_0)})].$$

Since $\tau_m(s+t_0)\downarrow\tau_m(s)$ as $t_0\downarrow 0$, by the right continuity of $u^*(Z^{\pi}_t)$ and Fatou's Lemma we have

$$\liminf_{t\downarrow 0} u_m(\pi)(x,\ s+t)\geqq\liminf_{t_0\downarrow 0} E^{\pi}_{(x,s)}[u^*(Z^{\pi}_{\tau_m(s+t_0)})]$$
$$\geqq E^{\pi}_{(x,s)}[\liminf_{t_0\downarrow 0} u^*(Z^{\pi}_{\tau_m(s+t_0)})]$$
$$=E^{\pi}_{(x,s)}[u^*(Z^{\pi}_{\tau_m(s)})]$$
$$=u_m(\pi)(x,\ s).$$

Hence it follows from Lemma 4.3 in [5] that for any $\pi\in\Pi$ and each $(x,\ s)\in Z$,

$$\liminf_{t\downarrow 0}\sup_{\tilde{\pi}\in\Pi} u_m(\tilde{\pi})(Z^{\pi}_{s+t})\geqq\sup_{\tilde{\pi}\in\Pi} u_m(\tilde{\pi})(x,\ s)\quad P^{\pi}_{(x,s)}\text{—a. s.},$$

that is,

$$\liminf_{t\downarrow 0}\tilde{u}_m(Z^{\pi}_{s+t})\geqq\tilde{u}_m(x,\ s)\qquad P^{\pi}_{(x,s)}\text{—a. s..}\tag{3. 3}$$

Finally from (3.2) and (3.3) $\tilde{u}_m$ is excessive.

LEMMA 3.2. *For each* $m\geqq 1$, *the following hold*:

(i) $\tilde{u}_m$ *is a majorant of* g.

(ii) $u^*(z)=\sup_{\pi\in\Pi}E^{\pi}_z[u^*(Z^{\pi}_{\sigma_m})]=\tilde{u}_m(z)$ *for all* $z\in Z$.

PROOF. (i) Let

$$c=\sup_{z\in Z}[g(z)-\tilde{u}_m(z)].$$

We have two cases: $c\leqq 0$ and $c>0$. In the first case, obviously $\tilde{u}_m(z)\geqq g(z)$ for all $z\in Z$.

Now suppose $c>0$. According to Lemma 3.1, $\tilde{u}_m$ is excessive, hence so is $c+\tilde{u}_m$. Also it is obvious that $c+\tilde{u}_m$ is a majorant of g. Since u^* is the smallest excessive majorant of g, we get

$$c+\tilde{u}_m(z)\geqq u^*(z)\qquad\text{for}\quad z\in Z.\tag{3. 4}$$

We take $0<\alpha<\min\left(c,\ \dfrac{1}{m}\right)$. Then there is a point $z_0=(x_0,\ s_0)\in Z$ such that

$$g(z_0)-\tilde{u}_m(z_0)>c-\alpha.\tag{3. 5}$$

Combining (3.4) and (3.5) leads us to

$$0\leqq u^*(z_0)-g(z_0)\leqq c+\tilde{u}_m(z_0)-g(z_0)<\alpha<\frac{1}{m},$$

that is,

$$u^*(z_0)<g(z_0)+\frac{1}{m}.$$

Thus the point $z_0\in\hat{\Gamma}_m^*$, where $\hat{\Gamma}_m^*$ is the set of all $(x, s)\in\Gamma_m^*$ for which there exists a $h_0>0$ depending on (x, s) such that $(x, s+h)\in\Gamma_m^*$ for all $h\in[0, h_0)$. Hence it follows that for any $\pi\in\Pi$ $\tau_m(s_0)=s_0$ $P_{z_0}^{\pi}$-a. s., and

$$\tilde{u}_m(z_0)=\sup_{\pi\in\Pi}E_{z_0}^{\pi}[u^*(Z_{\tau_m}^{\pi})]=u^*(z_0)\geqq g(z_0),$$

which together with (3.5) yields the inequality $\alpha>c$, which contradicts the assertion $0<\alpha<\min\left(c, \frac{1}{m}\right)$. This completes the proof of (i).

(ii) By (i) and Lemma 3.1, $\tilde{u}_m$ is an excessive majorant of g. Since u^* is the smallest excessive majorant of g, we have

$$u^*(z)\leqq\tilde{u}_m(z) \qquad \text{for} \quad z\in Z. \tag{3.6}$$

On the other hand, it follows from the excessiveness of u^* and $\tau_m\geqq\sigma_m$ that for any $\pi\in\Pi$,

$$u^*(z)\geqq E_z^{\pi}[u^*(Z_{\sigma_m}^{\pi})]\geqq E_z^{\pi}[u^*(Z_{\tau_m}^{\pi})],$$

hence

$$u^*(z)\geqq\sup_{\pi\in\Pi}E_z^{\pi}[u^*(Z_{\sigma_m}^{\pi})]\geqq\tilde{u}_m(z) \qquad \text{for} \quad z\in Z. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we immediately obtain the desired equality

$$u^*(z)=\sup_{\pi\in\Pi}E_z^{\pi}[u^*(Z_{\sigma_m}^{\pi})]=\tilde{u}_m(z) \qquad \text{for} \quad z\in Z.$$

For each $x\in S$, let $\frac{d^+}{ds}f(x, \cdot)$ denote the right-hand derivative of $f(x, \cdot)$.

Then we have

LEMMA 3.3. *Suppose that:*

(i) $\frac{d^+}{ds}u^*$ *exists and is bounded.*

(ii) *For each* $x\in S$, $\frac{d^+}{ds}u^*(x, \cdot)$ *is right continuous on* R^+.

Then it holds that for each $(x, s)\notin\Gamma^*$, $\sup_{\pi\in\Pi}\mathcal{A}^{\pi}u^*(x, s)$ *is right continuous at s.*

PROOF. From Lemma 5.1 in [5] we have that $u^*\in\mathcal{D}(\mathcal{A})$ and that for any $\pi\in\Pi$ and each $(x, s)\in Z$,

$$\mathcal{A}^{\pi}u^*(x, s)=\frac{d^+}{ds}u^*(x, s)+\lambda(x, s, \pi(x, s))\left[\int_S u^*(y, s)Q(dy\,|\,x, s, \pi(x, s)) - u^*(x, s)\right]. \tag{3.8}$$

Hence by the given conditions and the properties of π, λ and Q, it follows that for any $\pi\in\Pi$ $\mathcal{A}^{\pi}u^*(x, \cdot)$ is right continuous on R^+ for fixed $x\in S$.

Let $(x, s)\notin\Gamma^*$ be arbitrary. From Assumption (B), it holds that for any $\varepsilon>0$ there exists a policy $\pi_{\varepsilon}\in\Pi$ such that

(3.9) $$0\leqq \sup_{\pi\in\Pi}\mathcal{A}^{\pi}u^*(z)-\mathcal{A}^{\pi_\varepsilon}u^*(z)<\varepsilon/3$$

for all $z\notin\Gamma^*$. Also by the right continuity of $\mathcal{A}^{\pi_\varepsilon}u^*(x, s)$ at s, there exists a $h_0>0$ such that

(3.10) $$|\mathcal{A}^{\pi_\varepsilon}u^*(x, s+h)-\mathcal{A}^{\pi_\varepsilon}u^*(x, s)|<\varepsilon/3$$

and $(x, s+h)\notin\Gamma^*$ for all $h\in[0, h_0)$. Combining (3.9) and (3.10) we obtain

$$|\sup_{\pi\in\Pi}\mathcal{A}^{\pi}u^*(x, s+h)-\sup_{\pi\in\Pi}\mathcal{A}^{\pi}u^*(x, s)|<\varepsilon \qquad \text{for} \quad h\in[0, h_0),$$

which completes the proof.

We fix h_0 so that $0<h_0<\infty$ and define a stopping time by the following:

$$\mu=\min[h_0, \nu(Z_s^{\pi})] \qquad \text{for} \quad \pi\in\Pi \quad \text{and} \quad s\geqq 0.$$

LEMMA 3.4. *For each* $(x, s)\in Z$, *we have* $\inf_{\pi\in\Pi}E^{\pi}_{(x,s)}[\mu]>0$.

PROOF. Let $(x, s)\in Z$ be arbitrary and let

$$G^{\pi}(x, s; h)=P^{\pi}_{(x,s)}[\nu(x, s)\leqq h], \qquad h\geqq 0.$$

Then for any $\pi\in\Pi$ we get

$$\begin{aligned} E^{\pi}_{(x,s)}[\mu]&=(1-G^{\pi}(x, s; h_0))\cdot h_0+\int_0^{h_0}w\, G^{\pi}(x, s; dw) \\ &\geqq \exp\left\{-\int_0^{h_0}\lambda(x, s+s', \pi(x, s+s'))ds'\right\}\cdot h_0 \\ &\geqq e^{-Mh_0}h_0, \end{aligned}$$

since $0<\lambda(z, a)<M$ for all $(z, a)\in Z\times A$. Hence we have

$$\inf_{\pi\in\Pi}E^{\pi}_{(x,s)}[\mu]\geqq e^{-Mh_0}\cdot h_0>0.$$

4. The main result

In this section we shall prove the main result of the paper, that is, the optimal return satisfies the optimality equation, by using the fundamental lemmas obtained in Section 3.

THEOREM 4.1. *Suppose that:*

(i) $\dfrac{d^+}{ds}u^*$ *exists and is bounded.*

(ii) *For each* $x\in S$, $\dfrac{d^+}{ds}u^*(x, \cdot)$ *is right continuous on* R^+.

Then u^* *satisfies the following equation*:

$$\begin{cases} \sup_{\pi\in\Pi}\mathcal{A}^{\pi}f(z)=0 & \text{for} \quad z\notin\Gamma^*, \\ f(z)=g(z) & \text{for} \quad z\in\Gamma^*. \end{cases}$$

PROOF. We shall first show that

$$\sup_{\pi\in\Pi}\mathcal{A}^{\pi}u^*(z)\geqq 0 \qquad \text{for} \quad z\notin\Gamma^*.$$

Suppose for some $(x_0, s_0) \notin \Gamma^*$,

$$\sup_{\pi\in\Pi} \mathcal{A}^\pi u^*(x_0, s_0) < 0 .$$

Then there exists a $\delta > 0$ satisfying

$$\sup_{\pi\in\Pi} \mathcal{A}^\pi u^*(x_0, s_0) + \delta < 0 .$$

Take $\varepsilon > 0$ satisfying

$$\sup_{\pi\in\Pi} \mathcal{A}^\pi u^*(x_0, s_0) + \varepsilon + \delta < 0 . \tag{4.1}$$

From Lemma 3.3 it follows that there exists $h_1 > 0$ such that

$$\sup_{\pi\in\Pi} \mathcal{A}^\pi u^*(x_0, s_0 + h) < \sup_{\pi\in\Pi} \mathcal{A}^\pi u^*(x_0, s_0) + \varepsilon \tag{4.2}$$

for all $h \in [0, h_1)$. From (4.1) and (4.2) we get

$$\sup_{\pi\in\Pi} \mathcal{A}^\pi u^*(x_0, s_0 + h) + \delta < 0 \qquad \text{for} \quad h \in [0, h_1) . \tag{4.3}$$

Since $\Gamma^* = \bigcap_{m=1}^{\infty} \Gamma^*_m$ and $\Gamma^*_m \supset \Gamma^*_n$, $n \geqq m$, it is obvious that there exist $h_2 > 0$ and $m_0 \geqq 1$ such that $(x_0, s_0 + h) \notin \Gamma^*_m$ for all $m \geqq m_0$ and all $h \in [0, h_2)$. Let $h_0 = \min(h_1, h_2)$, $\mu = \text{mim}\,[h_0, \nu(Z^\pi_{s_0})]$ and let $\xi = s_0 + \mu$. Then it holds from (4.3) that for any $\pi \in \Pi$,

$$\mathcal{A}^\pi u^*(Z^\pi_t) + \delta < 0 \qquad \text{for} \quad t \in [s_0, \xi) \quad P^\pi_{(x_0, s_0)}-\text{a. s.},$$

hence

$$E^\pi_{(x_0, s_0)}\left[\int_{s_0}^{\xi} (\mathcal{A}^\pi u^*(Z^\pi_t) + \delta) dt\right] \leqq 0 .$$

By Lemma 5.4 in [5], we get

$$u^*(x_0, s_0) \geqq E^\pi_{(x_0, s_0)}[u^*(Z^\pi_\xi)] + \delta E^\pi_{(x_0, s_0)}[\mu] . \tag{4.4}$$

Since $\xi \leqq \sigma_m < \infty$ for all $m \geqq m_0 (P^\pi_{(x_0, s_0)}-\text{a. s.})$ and u^* is excessive, we have

$$E^\pi_{(x_0, s_0)}[u^*(Z^\pi_\xi)] \geqq E^\pi_{(x_0, s_0)}[u^*(Z^\pi_{\sigma_m})]$$

for any $\pi \in \Pi$ and each $m \geqq m_0$. This formula combined with (4.4) implies that

$$\begin{aligned} u^*(x_0, s_0) &\geqq E^\pi_{(x_0, s_0)}[u^*(Z^\pi_{\sigma_m})] + \delta\, E^\pi_{(x_0, s_0)}[\mu] \\ &\geqq E^\pi_{(x_0, s_0)}[u^*(Z^\pi_{\sigma_m})] + \delta \inf_{\pi\in\Pi} E^\pi_{(x_0, s_0)}[\mu] \end{aligned}$$

for any $\pi \in \Pi$ and each $m \geqq m_0$. Taking the supremum for $\pi \in \Pi$, we get

$$u^*(x_0, s_0) \geqq \sup_{\pi\in\Pi} E^\pi_{(x_0, s_0)}[u^*(Z^\pi_{\sigma_m})] + \delta \inf_{\pi\in\Pi} E^\pi_{(x_0, s_0)}[\mu] . \tag{4.5}$$

Lemma 3.2 implies that

$$u^*(z) = \sup_{\pi\in\Pi} E^\pi_z[u^*(Z^\pi_{\sigma_m})] \tag{4.6}$$

for all $m \geqq 1$ and all $z \in Z$. From (4.5) and (4.6) we obtain

$$u^*(x_0, s_0) \geqq u^*(x_0, s_0) + \delta \inf_{\pi\in\Pi} E^\pi_{(x_0, s_0)}[\mu] .$$

Since $\inf_{\pi\in\Pi} E^{\pi}_{(x_0,s_0)}[\mu]>0$ by Lemma 3.4,

$$u^*(x_0, s_0)>u^*(x_0, s_0),$$

which is a contradiction. So we get

$$\sup_{\pi\in\Pi} \mathcal{A}^{\pi}u^*(z)\geqq 0 \qquad \text{for} \quad z\notin\Gamma^*. \tag{4.7}$$

Let $(x, s)\notin\Gamma^*$ and $\pi\in\Pi$ be arbitrary. We take sufficient small $h>0$ so that $(x, s+h)\notin\Gamma^*$, and define $\mu=\min[h, \nu(Z^{\pi}_s)]$ and $\xi=s+\mu$. Then by the excessiveness of u^* we have

$$\begin{aligned} u^*(x, s)&\geqq E^{\pi}_{(x,s)}[u^*(Z^{\pi}_{\xi})] \\ &\geqq(1-G^{\pi}(x, s; h))u^*(x, s+h) \\ &\quad+\int_0^h\int_S G^{\pi}(x, s; dw)Q(dy\,|\,x, s+w, \pi(x, s+w))u^*(y, s+w)dw. \end{aligned}$$

From the properties of π, λ and Q it follows that for positive h small enough,

$$\begin{aligned} &\frac{1}{h}\{u^*(x, s)-u^*(x, s+h)\} \\ &\geqq\frac{1}{h}\int_0^h\lambda^{\pi}(x, s+h)\exp\left(-\int_0^w\lambda^{\pi}(x, s+s')ds'\right) \\ &\quad\cdot\left[\int_S u^*(y, s+w)Q^{\pi}(dy\,|\,x, s)-u^*(x, s+h)\right]dw, \end{aligned}$$

where

$$\lambda^{\pi}(x, w)=\lambda(x, w, \pi(x, s)) \qquad \text{for} \quad w\in[s, s+h),$$

and

$$Q^{\pi}(\cdot\,|\,x, s)=Q(\cdot\,|\,x, s, \pi(x, s)).$$

Letting $h\downarrow 0$ we have

$$-\frac{d^+}{ds}u^*(x, s)\geqq\lambda^{\pi}(x, s)\left[\int_S u^*(y, s)Q^{\pi}(dy\,|\,x, s)-u^*(x, s)\right],$$

which yields from (3.8) that

$$\mathcal{A}^{\pi}u^*(x, s)\leqq 0.$$

Since the above inequality holds for any $\pi\in\Pi$ and every $(x, s)\notin\Gamma^*$, we have

$$\sup_{\pi\in\Pi} \mathcal{A}^{\pi}u^*(z)\leqq 0 \qquad \text{for} \quad z\notin\Gamma^*. \tag{4.8}$$

Combining (4.7) and (4.8) we obtain

$$\sup_{\pi\in\Pi} \mathcal{A}^{\pi}u^*(z)=0 \qquad \text{for} \quad z\notin\Gamma^*.$$

This completes the proof.

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