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A SINGLE-BULLET DUEL WITH UNCERTAIN INFORMATION AVAILABLE TO THE DUELISTS

By

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Abstract

The purpose of this paper is to solve the single-bullet duel in which either silent or noisy is uncertain and the accuracy functions are arbitrary. It will be found the model in this paper is an extension of the single-bullet duels: silent, noisy, and silent-noisy duels.

1. Introduction

In this paper we genelarize the single-bullet duel in the work of Dresher [1] and Karlin [3] to include the possibility that either silent or noisy is uncertain, under the arbitrary accuracy functions.

Two duelists, player I and II, starting at time $t=0$ (at a distance 2 apart), walk toward each other at a constant (unit) speed with no opportunity for retreat, they will reach other at time $t=1$. Both are each allowed to fire only once. The accuracies of firing are described by the *accuracy function* $A_1(x)$=*the probability of I's hitting his opponent if he fires at time* x (*at a distance* $2(1-x)$). Similarly, $A_2(y)$ is defined for the player II. These functions are continuous and monotonically increasing on $[0, 1]$ with $A_1(0)=A_2(0)=0$ and $A_1(1)=A_2(1)=1$. Let the payoff be $+1$ to the surviving duelist and 0 to each duelist if both survives or neither survives. Each selects a time (distance) to fire.

As in all games, we need to describe the information available to the players. If a duelist is informed about his opponent's firings as soon as they take place, we usually call the duel a *noisy duel.* If neither duelist ever learns when or whether his opponent has fired, we usually call the duel a *silent duel.* A duel with silent and noisy bullets is called a *silent-noisy duel.*

There are four interesting papers which genelarized the single bullet duel mentioned above. Fox and Kimerdorf [2] solved the noisy duel with arbitrary accuracy functions and arbitrary numbers of bullets for both players. Restrepo [4] solved the silent case under the same assumptions. Styszyński [5] solved the silent vs. noisy duel when the accuracy functions are arbitrary, the silent player has n bullets and

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the noisy player has one. Sweat [6] genelarized the single-bullet noisy duel to include the possibility that one of the duelists has imperfect ability to perceive his opponent. Furtheremore, Teraoka analized the duels with uncertain existence of the bullet for silent [7, 8] noisy [9], and silent-noisy [10] cases.

Thinking the duels over, a duelist does not know whether his opponent possesses silent bullets or noisy bullets, so that it is natural to suppose the model in which either silent or noisy is uncertain. In this paper we shall examine the simplest case where each has only one bullet.

Section 2 presents our model suggested from the simplest assumptions mentioned above, that is, a single-bullet duel in which the state (silent or noisy) variable is a random variable with bivariate Bernoulli Distribution. In Section 3 we shall prepare two lemmas which are useful to derive our main results, one is given by Karlin [3] and the other will be proved by the author. The optimal strategies and the value of the game will be shown in Section 4. In section 5 it will be found that our main result covers the results for silent, noisy, and silent-noisy cases of the single-bullet duel. Section 6 deals with a simple example when $A_1(t)=A_2(t)=t$.

2. The model

Both players are informed of the accuracy functions described in section 1. We also assume that the accuracy functions A_1 and A_2 possess continuous derivatives with respect to t, which are denoted by $A_2'(t)$ and $A_2'(t)$, and their first derivatives are strictly positive in $[0, 1)$.

Furthermore we assume that each duelist is received a gun with only one bullet by their umpire. However, both duelists do not know whether their bullets are silent or noisy. Let define θ_i as follows:

$$\theta_i=\begin{Bmatrix}1\\0\end{Bmatrix} \text{ if player i's bullet is } \begin{Bmatrix}\text{silent}\\\text{noisy}\end{Bmatrix},$$

and let (θ_1, θ_2) is a random variable with bivariate Bernoulli distribution given as Table 1. The inner part of this double-entry table shows the joint distribution of θ_1 and θ_2. Let r be the correlation coefficient of θ_1 and θ_2, then $r=d/\sqrt{p_1q_1p_2q_2}$. That is to say θ_i denote the number of silencers fitted to player i's gun. Thus we have the expected payoff to player I for each (θ_1, θ_2) and its probability as Table 2,

Table 1. Bivariate Bernoulli Distribution

$0<p_i=1-q_i<1,\quad 0\leqq|a|\leqq\sqrt{p_1q_1p_2q_2},\quad 0\leqq|a|<\min\ (p_1q_2,\ p_2q_1)$

θ_1 \ θ_2	1 (silent)	0 (noisy)	Marginal Distribution
1 (silent)	p_1p_2+d	p_1q_2-d	p_1
0 (noisy)	q_1p_2-d	q_1q_2+d	q_1
Marginal Distribution	p_2	q_2	

Table 2. Expected Payoff to I for (θ_1, θ_2) and Its Probability

(θ_1, θ_2)	Probability	Expected Payoff to I		
		$x<y$	$x=y$	$x>y$
(1, 1)	p_1p_2+d	$A_1(x)-A_2(y)+A_1(x)A_2(y)$	$A_1(x)-A_2(x)$	$A_1(x)-A_2(y)-A_1(x)A_2(y)$
(1, 0)	p_1q_2-d	$A_1(x)-A_2(y)+A_1(x)A_2(y)$	$A_1(x)-A_2(x)$	$1-2A_2(y)$
(0, 1)	q_1p_2-d	$2A_1(x)-1$	$A_1(x)-A_2(x)$	$A_1(x)-A_2(y)-A_1(x)A_2(y)$
(0, 0)	q_1q_2+d	$2A_1(x)-1$	$A_1(x)-A_2(x)$	$1-2A_2(y)$

since the payoff is $+1$ to the surviving duelist and 0 to each duelist if both survives or neither survives.

After all, we get payoff kernel $M(x, y)$ to player I as follows:

$$(1)\qquad M(x, y)=\begin{cases} p_1\{A_1(x)-A_2(y)+A_1(x)A_2(y)\}+(1-p_1)\{2A_1(x)-1\} & x<y \\ A_1(x)-A_2(x) & x=y \\ p_2\{A_1(x)-A_2(y)-A_1(x)A_2(y)\}+(1-p_2)\{1-2A_2(y)\} & x>y \end{cases}$$

We observe that $M(x, y)$ is independent of d, so that this game is unchanged even if $d=0$, that is, θ_1 and θ_2 are independent. Furthermore it is found that $M(x, y)$ consists of a linear combination of the expected payoff for silent duel and one for noisy duel whose positive proportions are p_1 and $1-p_1$ when $x<y$, p_2 and $1-p_2$ when $x>y$, and arbitrary numbers when $x=y$.

This game is a two-person zero-sum game defined on the unit square in which the pure strategy spaces, $0\leqq x\leqq 1$ for player I and $0\leqq y\leqq$I for player II, represent the possible times during which a certain action can be taken. Each selects a time (distance) to fire in [0, 1], given that he survives. The maximum weakness to each duelist is to play noisy duelist when he has fired first and failed to beat his opponent. Our payoff kernel $M(x, y)$ shows the above situations.

Throughout this paper, we suppose that I and II use mixed strategies (distribution functions), $F(x)$ and $G(y)$, respectively, and we shall employ notations on expectation of $M(x, y)$ defined on the unit square as follows:

$$M(F, G)=\int_0^1\int_0^1 M(x, y)dF(x)dG(y), \quad \text{and}$$

$$M(x, G)=\int_0^1 M(x, y)dG(y)\,;\; M(F, y)=\int_0^1 M(x, y)dF(x).$$

Furthermore we shall define two functions $h_i(\cdot)$ and $U_i(\cdot)$ by

$$(2)\qquad h_i(t)=\frac{A_j'(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}$$

$$\text{for}\quad t_0<t\leqq 1,\ \{i, j\}=\{1, 2\},$$

$$(3)\qquad U_i(z)=\exp\left[-\int_a^z \{(2-p_1-p_2)+(p_1+p_2)A_i(t)\}\,h_i(t)dt\right]$$

$$\text{for}\quad t_0<a\leqq z\leqq 1,\ i=1, 2,$$

where t_0 is a unique root of equation

$$(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)=0$$

in the interval $[0, 1]$, and a is some number which will be given in Section 4.

3. Preliminary lemmas

In this section we shall prepare two lemmas which are useful to Section 4. Lemma 1 deals with a property of the optimal strategies for certain classes of timing games, which is given by Karlin [3]. Lemma 2 presents a property of function $U_i(\cdot)$ which will be used to determie the mixed strategies in Section 4.

LEMMA 1. *Let $M(x, y)$ be kernels of the form*

$$M(x, y)=\begin{cases} K(x, y) & x<y \\ \Phi(x) & x=y \\ L(x, y) & x>y \end{cases}$$

which satisfy the following conditions:

(a) *The functions $K(x, y)$ and $L(x, y)$ are defined over the closed triangles $0\leqq x \leqq y\leqq 1$ and $0\leqq y\leqq x\leqq 1$, respectively. Furthermore, they possess continuous second partial derivatives defined in their respective closed triangles.*

(b) $K(1, 1)>\Phi(1)>L(1, 1)$ *and* $K(0, 0)\leqq\Phi(0)\leqq L(0, 0)$.

(c) $K_x(x, y)>0$ *and* $L_x(x, y)>0$ *for* $x<1$,

$K_y(x, y)<0$ *and* $L_y(x, y)<0$ *for* $y<1$

(in their respective domains of definition).

Then the optimal strategies $F^*(x)$, $G^*(y)$ of zero-sum game $M(x, y)$ exists uniquely and take the following form: densities $f^*(x)$, $g^*(y)$ over a common support $[a, 1]$ and a possible jump at 1 for one of the two players, and then

$$M(x, G^*)\leqq v^*\leqq M(F^*, y) \quad \textit{for} \quad 0\leqq x,\ y\leqq 1 \tag{4}$$

$$M(x, G^*)\equiv v^*\equiv M(F^*, y) \quad \textit{for} \quad a\leqq x,\ y\leqq 1 \tag{5}$$

PROOF. See Karlin's work, Chapter 5.

LEMMA 2. *Given any $z>t_0$,*

$$\int_l^z \{(2-p_1-p_2)+(p_1+p_2)A_i(t)\}\, h_i(t)dt \uparrow \infty \ \textit{as}\ l\downarrow t_0,\ i=1, 2,$$

so that

$$\exp[-\int_l^z \{(2-p_1-p_2)+(p_1+p_2)A_i(t)\}\, h_i(t)dt] \downarrow 0 \ \textit{as}\ l\uparrow t_0$$

and this exponential form is a non-increasing function with respect to z for any fixed $l\in(t_0, 1)$.

PROOF. We shall prove the case where $i=1$ and $j=2$. Since $t>t_0$

$$\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}$$

$$>\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}.$$

Now we get

$$\lim_{t\to t_0}\left(\frac{\dfrac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}}{\dfrac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t_0)+A_2(t)-1\}+(p_1+p_2)A_1(t_0)A_2(t)}}\right)$$

$$=\lim_{t\to t_0}\frac{(2-p_1-p_2)\{A_1(t_0)+A_2(t)-1\}+(p_1+p_2)A_1(t_0)A_2(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}$$

$$=\lim_{t\to t_0}\frac{(2-p_1-p_2)A_2'(t)+(p_1+p_2)A_1(t_0)A_2'(t)}{(2-p_1-p_2)\{A_1'(t)+A_2'(t)\}+(p_1+p_2)\{A_1'(t)A_2(t)+A_1(t)A_2'(t)\}}$$

$$=\frac{(2-p_1-p_2)A_2'(t_0)+(p_1+p_2)A_1(t_0)A_2'(t_0)}{(2-p_1-p_2)\{A_1'(t_0)+A_2'(t_0)\}+(p_1+p_2)\{A_1'(t_0)A_2(t_0)+A_1(t_0)A_2'(t_0)\}}$$

$$=\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t_0)}{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t_0)+\{(2-p_1-p_2)+(p_1+p_2)A_2(t_0)\}A_1'(t_0)}>0$$

Hence, there exists $s\in(t_0, z)$ such that for any $t\in(t_0, s)$

$$\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}h_1(t)$$

$$>\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t)}{(2-p_1-p_2)A_1(t)+A_2(t)-1+(p_1+p_2)A_1(t)A_2(t)}$$

$$\geqq\frac{1}{2}\,\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t_0)}{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t_0)+\{(2-p_1-p_2)+(p_1+p_2)A_2(t_0)\}A_1'(t_0)}$$

$$\times\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t_0)+A_2(t)-1\}+(p_1+p_2)A_1(t_0)A_2(t)}$$

Then

$$\int_{t_0}^{z}\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}h_1(t)dt$$

$$=\left(\int_{t_0}^{s}+\int_{s}^{z}\right)\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}h_1(t)dt$$

$$>\frac{1}{2}\,\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t_0)}{\{(2-p_1-p_2)+(p_1+p_2)A_1(t_0)\}A_2'(t_0)+\{(2-p_1-p_2)+(p_1+p_2)A_2(t_0)\}A_1'(t_0)}$$

$$\times\int_{t_0}^{s}\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t_0)+A_2(t)-1\}+(p_1+p_2)A_1(t_0)A_2(t)}dt$$

$$+\int_{s}^{z}\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}dt$$

$$\times\ln\left(\frac{(2-p_1-p_2)\{A_1(t_0)+A_2(s)-1\}+(p_1+p_2)A_1(t_0)A_2(s)}{(2-p_1-p_2)\{A_1(t_0)+A_2(t_0)-1\}+(p_1+p_2)A_1(t_0)A_2(t_0)}\right)=\infty.$$

This completes the proof of Lemma 2.

4. The solution

The kernel (1) clearly satisfies conditions in Lemma 1. Supposing that $F(x)$ consists of a density part $f(x)>0$ over an interval $[a, 1)$ and a mass part α at $x=1$, and $G(y)$ consists of a density part $g(y)>0$ over the same interval and a mass part β at $y=1$, we get

$$\begin{aligned}(6)\qquad M(F, y)=&\int_a^y[p_1\{A_1(x)-A_2(y)+A_1(x)A_2(y)\}+(1-p_1)\{2A_1(x)-1\}]f(x)dx\\&+\int_y^1[p_2\{A_1(x)-A_2(y)-A_1(x)A_2(y)\}+(1-p_2)\{1-2A_2(y)\}]f(x)dx\\&+\alpha[1-2A_2(y)],\ a\leqq y<1\end{aligned}$$

and

$$\begin{aligned}(7)\qquad M(x, G)=&\int_a^x[p_2\{A_1(x)-A_2(y)-A_1(x)A_2(y)\}+(1-p_2)\{1-2A_2(y)\}]g(y)dy\\&+\int_x^1[p_1\{A_1(x)-A_2(y)+A_1(x)A_2(y)\}+(1-p_1)\{2A_1(x)-1\}]g(y)dy\\&+\beta[2A_1(x)-1],\ a\leqq x<1.\end{aligned}$$

Since

$$(8)\qquad M(F, y)\equiv v,\ a\leqq y\leqq 1;\ M(x, G)\equiv v,\ a\leqq x\leqq 1\quad\text{and}\quad \alpha\beta=0.$$

We obtain two integral equations. Differentiation of the first equation in (8) with respect to y followed by the normalization condition $\int_a^1 f(x)dx+\alpha=1$ gives

$$\begin{aligned}&[(2-p_1-p_2)\{A_1(y)+A_2(y)-1\}+(p_1+p_2)A_1(y)A_2(y)]f(y)\\&=\left[\left(\int_a^y p_1-\int_y^1 p_2-\int_a^y 2\right)f(x)dx-\left(\int_a^y p_1-\int_y^1 p_2\right)A_1(x)f(x)dx+2\right]A_2'(y).\end{aligned}$$

Hence $M(F, y)$ is independent of y for $y\in[a, 1]$ if

$$(9)\qquad \frac{f(y)}{\left(\int_a^y 2-\int_a^y p_1+\int_y^1 p_2\right)f(x)dx+\left(\int_a^y p_1-\int_y^1 p_2\right)A_1(x)f(x)dx-2}=-\frac{A_2'(y)}{(2-p_1-p_2)\{A_1(y)+A_2(y)-1\}+(p_1+p_2)A_1(y)A_2(y)}$$

which yields

$$(10)\qquad \frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(y)\}f(y)}{\left(\int_a^y 2-\int_a^y p_1+\int_y^1 p_2\right)f(x)dx+\left(\int_a^y p_1-\int_y^1 p_2\right)A_1(x)f(x)dx-2}$$

$$=-\frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(y)\}A_2'(y)}{(2-p_1-p_2)\{A_1(y)+A_2(y)-1\}+(p_1+p_2)A_1(y)A_2(y)}.$$

Integrating (10), we obtain

$$\left(\int_a^y 2-\int_a^y p_1+\int_y^1 p_2\right)f(x)dx+\left(\int_a^y p_1-\int_y^1 p_2\right)A_1(x)f(x)dx-2$$

$$=k_i'\exp\left[-\int_a^y \frac{\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}A_2'(t)}{(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)}dt\right].$$

Differentiation with respect to y yields

$$\{(2-p_1-p_2)+(p_1+p_2)A_1(y)\}f(y)$$
$$=-k_1'\{(2-p_1-p_2)+(p_1+p_2)A_1(y)\}h_1(y)$$
$$\times\exp\left[-\int_a^y\{(2-p_1-p_2)+(p_1+p_2)A_2'(t)\}h_1(t)dt\right].$$

Hence we have

$$f(x)=k_1h_1(x)\exp\left[-\int_a^x\{(2-p_1-p_2)+(p_1+p_2)A_1(t)\}h_1(t)dt\right]\quad\text{for}\quad x\in[a,1),\tag{11}$$

where $k_1=-k_1'$.

A similar argument on the second equation in (8) gives

$$g(y)=k_2h_2(y)\exp\left[-\int_a^y\{(2-p_1-p_2)+(p_1+p_2)A_1'(t)\}h_1(t)dt\right]\quad\text{for}\quad y\in[a,1).\tag{12}$$

We have proved the following:

LEMMA 3. *The densities $f^*(x)$ and $g^*(y)$ over a common support $[a,1]$ of the optimal strategies for* (1) *take the following form*:

$$f^*(x)=\begin{Bmatrix}0\\ k_1h_1(x)U_1(x)\end{Bmatrix}\quad\textit{for}\quad x\in\begin{Bmatrix}[0,a)\\ [a,1)\end{Bmatrix};$$

$$g^*(y)=\begin{Bmatrix}0\\ k_2h_2(y)U_2(y)\end{Bmatrix}\quad\textit{for}\quad y\in\begin{Bmatrix}[0,a)\\ [a,1)\end{Bmatrix},$$

where k_1 and k_2 are some positive constants.

The next lemma is trivial but is useful to determine k_1, k_2, and the value of game v^*.

LEMMA 4. *For $t_0<l\leqq u\leqq 1$*

$$\int_l^u\{(2-p_1-p_2)+(p_1+p_2)A_1(x)\}f(x)dx=k_1\{U_1(l)-U_1(u)\};$$

$$\int_l^u\{(2-p_1-p_2)+(p_1+p_2)A_2(y)\}g(y)dy=k_2\{U_2(l)-U_2(u)\},$$

where $f(x)$ and $g(y)$ are given as (11) *and* (12), *respectively.*

LEMMA 5. *Two constants k_1 of $f^*(x)$ and k_2 of $g^*(y)$ stated in Lemma 3 take the following from:*

$$k_1=\frac{2(p_1+\alpha p_2)}{p_1+p_2U_1(1)}\ ;\quad k_2=\frac{2(p_2+\beta p_1)}{p_2+p_1U_2(1)}\qquad \textit{for}\quad a\in(t_0,\ 1) \tag{13}$$

PROOF. Inserting (11) in (9) to make $f^*(x)$ a solution of the original equation, we obtain

$$\frac{k_1h_1(y)U_1(y)}{k_1\{1-U_1(y)\}+p_2\int_a^1\{1-A_1(x)\}f(x)dx-2}=-h_1(y)\qquad \text{for}\quad y\in[a,\ 1),$$

since the denominater in (9) equals to

$$\int_a^y\{(2-p_1-p_2)+(p_1+p_2)A_1(x)\}f(x)dx+p_2\int_a^1\{1-A_1(x)\}f(x)dx-2$$

$$=k_1\{U_1(a)-U_1(y)\}+p_2\int_a^1\{1-A_1(x)\}f(x)dx-2$$

which is given by Lemma 4. Thus we get

$$k_1=2-p_2\int_a^1\{1-A_1(x)\}f(x)dx\qquad \text{for}\quad a\in(t_0,\ 1). \tag{14}$$

Furthermore Lemma 4 yields

$$\int_a^1\{1-A_1(x)\}f(x)dx=\frac{1}{p_1+p_2}\left[2\int_a^1f(x)dx-k_1\{1-U_1(1)\}\right] \tag{15}$$

substituting (15) in (14), we have the first equation in (13).

A similar argument on $g^*(y)$ shows the second equation in (13). This completes the proof of lemma 5.

LEMMA 6. *There exists $a\in(t_0,\ 1)$ uniquely such that*

$$\alpha=1-F(1-0)\geqq0\ ;\ \beta=1-G(1-0)\geqq0$$

which are mass parts at 1. Then α and β are given by

$$\alpha=\frac{p_1+p_2U_1(1)-2p_1\int_a^1h_1(x)U_1(x)dx}{p_1+p_2U_1(1)+2p_2\int_a^1h_1(x)U_1(x)dx}\geqq0\ ;$$

$$\beta=\frac{p_2+p_1U_2(1)-2p_2\int_a^1h_2(y)U_2(y)dy}{p_2+p_1U_2(1)+2p_1\int_a^1h_2(y)U_2(y)dy}\geqq0\ ,$$

for such an appropriate $a\in(t_0,\ 1)$.

PROOF. First we shall prove the existence of $a\in(t_0,\ 1)$ such that $1-F(1-0)\geqq0$. $F(1-0)$ is a strictly decreasing function with respect to a and $F(1-0)=0$ at $a=1-0$. Now we get the following inequality:

(16) $$F(1-0)=\int_a^1 f(x)dx>\frac{1}{2}\int_a^1 \{(2-p_1-p_2)+(p_1+p_2)A_1(x)\}f(x)dx$$

$$=\frac{k_1}{2}\{U_1(a)-U_1(1)\}=\frac{k_1}{2}\{1-U_1(1)\}, \qquad \text{for} \quad a\in(t_0,\ 1),$$

since Lemma 4 holds. Lemma 2 and Lemma 5 suggest that

(17) $$F(1-0)>\frac{k_1}{2}\{1-U_1(1)\}\downarrow 1+\alpha\cdot\frac{p_2}{p_1}\geqq 1 \quad \text{as} \quad a\downarrow t_0.$$

A similar argument on $G(1-0)$ gives

(18) $$G(1-0)>\frac{k_2}{2}\{1-U_2(1)\}\downarrow 1+\beta\cdot\frac{p_1}{p_2}\geqq 1 \quad \text{as} \quad a\downarrow t_1,$$

showing the proof of the first part in Lemma 6.

Here we shall determine α and β for an appropriate $a\in(t_0,\ 1)$. Since $F(1-0)=1-\alpha$, we obtain

$$\frac{2(p_1+\alpha p_2)}{p_1+p_2U_1(1)}\int_a^1 h_1(x)U_1(x)dx=1-\alpha$$

so that we get the statement of the second part in Lemma 6.

This completes the proof of Lemma 6.

LEMMA 7. *The value of the game* v^* *is given by*

$$v^*=\begin{cases} 2\{1-A_2(a)\}\left\{\dfrac{p_1}{p_1+p_2U_2(1)}\right\}-1 & \textit{if} \quad \alpha=0 \\ -2\{1-A_1(a)\}\left\{\dfrac{p_2}{p_1+p_2U_1(1)}\right\}+1 & \textit{if} \quad \beta=0. \end{cases}$$

PROOF. Equation (8) shows $\alpha\beta=0$ and

(19) $$\begin{aligned} v^*&=M(F^*,\ y)=M(F^*,\ a) && \text{for all} \quad y\in[a,\ 1] \\ &=M(x,\ G^*)=M(a,\ G^*) && \text{for all} \quad x\in[a,\ 1]. \end{aligned}$$

If $\alpha=0$, the first equation yields

(20) $$v^*=\int_a^1[p_2\{A_1(x)-A_2(a)-A_1(x)A_2(a)\}+(1-p_2)\{1-2A_2(a)\}]f^*(x)dx$$

$$=p_2\{1-A_2(a)\}\int_a^1 A_1(x)f^*(x)dx-p_2\{1-A_2(a)\}+1-2A_2(a)$$

since equation (6) holds. Furthermore Lemma 4 gives

$$2-p_1-p_2+(p_1+p_2)\int_a^1 A_1(x)f^*(x)dx=k_1\{1-U_1(1)\},$$

which shows

(21) $$\int_a^1 A_1(x)f^*(x)dx=\frac{2}{p_1+p_2}\left[\frac{p_1}{p_1+p_2U_1(1)}\{1-U_1(1)\}-1\right]+1.$$

Substituting (21) into (20), we have

$$
\begin{aligned}
v^* &= \frac{2\{1-A_2(a)\}}{p_1+p_2}\left[\frac{p_1p_2\{1-U_1(1)\}}{p_1+p_2U_1(1)}-p_2\right]+1-2A_2(a)\\
&= \frac{2\{1-A_2(a)\}}{p_1+p_2}\left[\frac{p_1p_2\{1-U_1(1)\}}{p_1+p_2U_1(1)}+p_1\right]-1\\
&= 2\{1-A_2(a)\}\left\{\frac{p_1}{p_1+p_2U_1(1)}\right\}-1 .
\end{aligned}
$$

The similar argument on the second equation in (19) gives

$$
v^* = -2\{1-A_1(a)\}\left\{\frac{p_2}{p_2+p_1U_2(1)}\right\}+1 .
$$

This completes the proof of Lemma 7.

According to Lemma 1 the optimal strategies for the payoff kernel (1) exist uniquely, and two integral equations in (8) have unique solutions under appropriate boundary conditions, respectively. Hence we shall conclude this section by asserting Theorem 1, since Lemma 6 suggest that

$$
F(1-0)=\int_a^1 f^*(x)dx=1\,;\ G(1-0)=\int_a^1 g^*(y)dy=1
$$

have unique roots, respectively.

THEOREM *Let a_1 and a_2 be the unique roots of equations*

$$
2p_1\int_a^1 h_1(x)U_1(x)dx=p_1+p_2U_1(1)\,;
$$

$$
2p_2\int_a^1 h_2(y)U_2(y)dy=p_2+p_1U_2(1)
$$

in the interval $(t_0, 1)$, respectively, where t_0 is the unique root of equation

$$
(2-p_1-p_2)\{A_1(t)+A_2(t)-1\}+(p_1+p_2)A_1(t)A_2(t)=0 .
$$

And let $a=\max(a_1, a_2)$.

Then optimal strategies $F^*(x)$ for I and $G^*(y)$ for II of zero-sum game (1) *are the following mixed strategies*:

$$
F^*(x)=\begin{cases} 0 & ,\quad 0\leqq x\leqq a\\ \int_a^x k_1h_1(t)U_1(t)dt+\alpha I_1(x), & a\leqq x\leqq 1\,; \end{cases}
$$

$$
G^*(y)=\begin{cases} 0 & ,\quad 0\leqq y< a\\ \int_a^y k_2h_2(t)U_2(t)dt+\beta I_1(y), & a\leqq y\leqq 1\,, \end{cases}
$$

where $I_1(z)$ is the unit-step function at $z=1$, coefficients k_1 and k_2 are given by

$$
k_1=\frac{2(p_1+\alpha p_2)}{p_1+p_2U_1(1)}\,;\quad k_2=\frac{2(p_2+\beta p_1)}{p_2+p_1U_2(1)}\,,
$$

and mass parts α and β are determined as

$$\alpha=\frac{p_1+p_2U_1(1)-2p_1\int_a^1 h_1(x)U_1(x)dx}{p_1+p_2U_1(1)+2p_2\int_a^1 h_1(x)U_1(x)dx};$$

$$\beta=\frac{p_2+p_1U_2(1)-2p_2\int_a^1 h_2(y)U_2(y)dy}{p_2+p_1U_2(1)+2p_1\int_a^1 h_2(y)U_2(y)dy},$$

which have the following properties:

$$\alpha=\left\{\begin{matrix} = \\ = \\ > \end{matrix}\right\}0 \quad and \quad \beta=\left\{\begin{matrix} > \\ = \\ = \end{matrix}\right\}0 \quad if \quad a=\left\{\begin{matrix} a_1>a_2 \\ a_1=a_2 \\ a_2>a_1 \end{matrix}\right\}$$

The value of the game v^* is

$$v_*=\begin{cases} 2\{1-A_2(a)\}\left\{\dfrac{p_1}{p_1+p_2U_1(1)}\right\}-1 & if \quad a=a_1 \\ -2\{1-A_2(a)\}\left\{\dfrac{p_2}{p_2+p_1U_2(1)}\right\}+1 & if \quad a=a_2 \end{cases}$$

5. The special cases

The purpose of this section is to assert that the classical results of the single-bullet duel in the works of Dresher [1] and Karlin [3] are the special cases when $p_1=p_2\to 1$, $p_1=p_2\to 0$, and $(p_1, p_2)\to(1, 0)$, which are called as silent duel, noisy duel, and silent-noisy duel, respectively.

(i) The Case Where $p_1=p_2\to 1$.

Letting $p_1=p_2\to 1$, we have

$$h_i(t)=A_j'(t)/\{2A_1(t)A_2(t)\} \qquad \text{for} \quad 0<t\leqq 1, \ \{i, j\}=\{1, 2\}$$

and

$$U_i(z)=A_j'(a)/A_j(z) \qquad \text{for} \quad 0<a\leqq z\leqq 1, \ \{i, j\}=\{1, 2\},$$

so that a_1 and a_2 are the unique roots of equations

$$1+\frac{1}{A_2(a)}=\int_a^1\frac{A_2'(t)}{A_1(t)\{A_2(t)\}^2}dt;\ 1+\frac{1}{A_1(a)}=\int_a^1\frac{A_1'(t)}{\{A_1(t)\}^2A_2(t)}dt$$

in the interval (0, 1), respectively. Thus we conclude the following.

$$a=\max(a_1, a_2),$$

$$F^*(x)=\begin{cases} 0, & 0\leqq x<a \\ \displaystyle\int_a^x\frac{k_1A_2(a)A_2'(t)}{2A_1(t)\{A_2(t)\}^2}dt+\alpha I_1(x), & a\leqq x\leqq 1; \end{cases}$$

$$G^*(y)=\begin{cases} 0, & 0\leqq y<a \\ \displaystyle\int_a^y\frac{k_2A_1(a)A_1'(t)}{2A_2(t)\{A_1(t)\}^2}dt+\beta I_1(y), & a\leqq y\leqq 1, \end{cases}$$

$$k_1=2(1+\alpha)/\{1+A_2(a)\}\ ;\quad k_2=2(1+\beta)/\{1+A_1(a)\},$$

$$\alpha=\frac{1+A_2(a)-A_2(a)\int_a^1[A_2'(t)/A_1(t)\{A_2(t)\}^2]dt}{A+A_2(a)+A_2(a)\int_a^1[A_2'(t)/A_1(t)\{A_2(t)\}^2]dt}\ ;$$

$$\beta=\frac{1+A_1(a)-A_1(a)\int_a^1[A_1'(t)/A_2(t)\{A_1(t)\}^2]dt}{a+A_1(a)+A_1(a)\int_a^1[A_1'(t)/A_2(t)\{A_1(t)\}^2]dt},$$

and

$$v^*=\begin{cases}\{1-3A_2(a)\}/\{1+A_2(a)\} & \text{if} \quad a=a_1\\ \{1-3A_1(a)\}/\{1+A_1(a)\} & \text{if} \quad a=a_2,\end{cases}$$

in the sence of limitting point.

It is found that the above conclusions are the soution of single-bullet silent duel [1, 3, 4, 7, 8].

(ii) The Case Where $p_1=p_2\to 0$.

When $p_1=p_2\to 0$, we obtain

$$h_i(t)=\frac{A_j'(t)}{2\{A_1(t)+A_2\{t)-1\}}\qquad \text{for}\quad t_0<t\leqq 1,\ \{i,\ j\}=\{1,\ 2\}$$

and

$$U_i(z)=\exp\left[-\int_a^z\frac{A_j'(t)}{A_1(t)+A_2(t)-1}dt\right]\qquad \text{for}\quad t_0<a\leqq z\leqq 1,\ \{i,\ j\}=\{1,\ 2\},$$

where t_0 is the unique root of equation $A_1(t)+A_2(t)=1$ in the interval [0, 1]. Thus we get

$$2\int_a^z h_i(t)U_i(t)dt=1-U_i(z),$$

$$\{p_1+p_2U_1(1)\}/p_1\longrightarrow 1+U_1(1)\ ;\ \{p_2+p_1U_2(1)\}/p_2\longrightarrow 1+U_2(1)$$

and

$$\alpha\longrightarrow 0;\quad \beta\longrightarrow 0.$$

Since lemma 2 holds, we have

$$\lim a_1=\lim a_2=t_0,$$

$$\lim F^*(x)=\begin{cases}0 & \text{if}\quad 0\leqq x<t_0\\ 1 & \text{if}\quad t_0\leqq x\leqq 1;\end{cases}$$

$$\lim G^*(y)=\begin{cases}0 & \text{if}\quad 0\leqq y<t_0\\ 1 & \text{if}\quad t_0\leqq y\leqq 1,\end{cases}$$

and

$$\lim v^*=A_1(t_0)-A_2(t_0).$$

The above conclusions are the solution of single-bullet noisy duel [1, 2, 3, 9].

(iii) The Case Where $(p_1,\ p_2)\to(1,\ 0)$

When $(p_1,\ p_2)\to(1,\ 0)$, we have

$$h_i(t)=A_j'(t)/\{A_1(t)A_2(t)+A_1(t)+A_2(t)-1\}$$

$$\text{for} \quad t_0<t\leqq 1, \ \{i, j\}=\{1, 2\}$$

and

$$U_i(z)=\exp\left[-\int_a^z \{1+A_i(t)\}\, h_i(t)dt\right] \qquad \text{for} \quad t_0<a\leqq z\leqq 1, \ \{i, j\}=\{1, 2\}$$

Where t_0 is the unique root of equation

$$A_1(t)A_2(t)+A_1(t)+A_2(t)-1=0$$

in the interval [0, 1], so that a is the unique root of

$$2\int_a^1 h_1(x)\exp\left[-\int_a^x \{1+A_1(t)\}\, h_1(t)dt\right]dx=1$$

in the interval $(t_0, 1]$, since Lemma 2 and Lemma 4 suggest that equation $U_2(1)=0$ has unique root $a_2=t_0$ in $[t_0, 1]$ in the sense of limitting point. Then

$$\alpha=0;$$

$$\beta=U_2(1)\Big/\left(U_2(1)+2\int_a^1 h_2(t)\exp\left[-\int_a^t \{1+A_1(s)\}\, h_1(x)ds\right]dt\right)>0$$

Hence we conclude the following:

$$F^*(x)=\begin{cases} 0, & 0\leqq x<a \\ 2\int_a^x h_1(t)\exp\left[-\int_a^t \{1+A_2(s)\}\, h_2(s)ds\right]dt, & a\leqq x\leqq 1; \end{cases}$$

$$G^*(y)=\begin{cases} 0, & 0\leqq y<a \\ \beta\left(2\int_a^y h_2(t)\exp\left[\int_t^1 \{1+A_2(s)\}\, h_2(s)ds\right]dt+I_1(y)\right), & a\leqq y\leqq 1 \end{cases}$$

and

$$v^*=1-2A_2(a)$$

in the sence of limitting point.

The above solution is one for the singel-bullet silent-noisy duel [5, 10].

6. A **simple example**

As a simple example of the result obtained in Section 5 we shall examine the case where $A_1(t)=A_2(t)=t$ and $p_1\geqq p_2$. Then we have

$$h_i(t)=1/\{(p_1+p_2)t^2+2(2-p_1-p_2)t-(2-p_1-p_2)\} \qquad \text{for} \quad t_0<t\leqq 1$$

and

$$U_i(z)=\left\{\frac{(p_1+p_2)a^2+2(2-p_1-p_2)a-(2-p_1-p_2)}{(p_1+p_2)z^2+2(2-p_1-p_2)z-(2-p_1-p_2)}\right\}^{1/2} \qquad \text{for} \quad t_0<a<z\leqq 1,$$

where $t_0=[-(2-p_1-p_2)+\{2(2-p_1-p_2)\}^{1/2}]/(p_1+p_2)$ and $i=1, 2$. Putting

$$R(z)=\{(p_1+p_2)z^2+2(2-p_1-p_2)z-(2-p^{\mathrm{I}}-p_2)\}^{1/2},$$

we obtain

$$\int_a^z h_i(t)U_i(t)dt = \frac{R(a)}{2(2-p_1-p_2)}$$
$$\times\left[\frac{(p_1+p_2)a+(2-p_1-p_2)}{R(a)} - \frac{(p_1+p_2)z+(2-p_1-p_2)}{R(z)}\right]$$

since

$$\int \{(p_1+p_2)t^2+2(2-p_1-p_2)t-(2-p_1-p_2)\}^{-3/2}dt$$
$$= -\frac{(p_1+p_2)t+(2-p_1-p_2)}{2(2-p_1-p_2)\{(p_1+p_2)t^2+2(2-p_1-p_2)t-(2-p_1-p_2)\}^{1/2}}.$$

Hence a_1 and a_2 are the unique roots of

$$\{\sqrt{2p_1}/(2-p_2)\}\, a_1 + \{(p_1+p_2)a_1^2+2(2-p_1-p_2)a_1-(2-p_1-p_2)\}^{1/2},$$
$$\{\sqrt{2p_2}/(2-p_1)\}\, a_2 = \{(p_1+p_2)a_2^2+2(2-p_1-p_2)a_2-(2-p_1-p_2)\}^{1/2},$$

respectively. Now a_1 and a_2 are the abscissa in $\{(a, b)\,|\, a\geqq 0,\ b\geqq 0\}$ of intersecting points of

hyperbola: $b^2=\{R(a)\}^2$ and straight line: $b=\{\sqrt{2p_1}/(2-p_2)\}\, a$;

hyperbola: $b^2=\{R(a)\}^2$ and straight line: $b+\{\sqrt{2p_2}/(2-p_1)\}\, a$,

respectively. Thus we get

$$a = \left\{\begin{array}{c} a_1 > a_2 \\ a_1 = a_2 \\ a_2 > a_1 \end{array}\right\} \quad \text{if} \quad p_1 \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} p_2,$$

which gives

$$a = a^{\mathrm{I}} = (2-p_2)(2-p_1-p_2)[2-p_2-\{2(2+p_1-p_2)\}^{1/2}]/\{2p_1^2-(p_1+p_2)(2-p_2)^2\}$$

and

$$\alpha = 0;$$
$$\beta = \{\sqrt{2}(2-p_1)R(a)-2p_2a\}/\{2(2-p_1-p_2)-\sqrt{2}\,p_1R(a)+2p_1a\}$$
$$= \{p_1(2-p_1)-p_2(2-p_2)\}\,a/\{(2-p_1-p_2)(2-p_1+p_1a)\}$$

since

$$R(a) = \{\sqrt{2}\,p_1/(2-p_1)\}\,a.$$

Then we have

$$k_1 = 2p_1/\{p_1+p_2R(a)/\sqrt{2}\} = 2(2-p_2)/\{2-(1-a)p_2\};$$
$$k_2 = 2(p_2+\beta p_1)/\{p_2+p_1R(a)/\sqrt{2}\} = 2(2-p_1)(p_2+\beta p_1)/\{p_2(2-p_1)+p_1^2a\}.$$

The above considerations lead us to the following optimal strategies $F^*(x)$ and $G^*(y)$:

$$F^*(x)=\begin{cases} 0, & 0\leqq x<a \\ \displaystyle\int_a^x \frac{2\sqrt{2}\,p_1 a}{2-(1-a)p_2}\{(p_1+p_2)t^2+2(2-p_1-p_2)t-(2-p_1-p_2)\}^{-3/2}dt, & a\leqq x\leqq 1; \end{cases}$$

$$G^*(y)=\begin{cases} 0, & 0\leqq y<a \\ \dfrac{2(2-p_1)(p_2+\beta p_1)\{2-(1-a)p_2\}}{(2-p_2)\{p_2(2-p_1)+p_1^2 a\}}F^*(y)+\beta I_1(y), & a\leqq y\leqq 1 \end{cases}$$

and the value of the game v^*:

$$v^*=2(2-p_2)(1-a)/\{2-(1-a)p_2\}-1.$$

Letting $(p_1, p_2)\to(1, 0)$; $(p_1, p_2)\to(1, 1)$, the above conclusions converge to the solutions of single-bullet silent-noisy duel and single-bullet silent duel in which $A_1(t)=A_2(t)=t$, respectively (see Karlin and Dresher).

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