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https://doi.org/10.5109/13131

出版情報：統計数理研究. 18 (3/4), pp.61-68, 1979-03. Research Association of Statistical Sciences

TESTS OF HOMOGENEITY FOR ORDERED ALTERNATIVES IN THE NORMAL POPULATIONS

By

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(Received March 27, 1978)

Abstract

Four new statistics to test the homogeneity of means and of variances against the ordered alternatives are considered and their power properties are investigated with the aid of computer simulations. These statistics are extensions of a partial ordering on the sample space induced by the likelihood ratio. It is found that they are more powerful than the likelihood ratio tests.

1. Introduction.

Let X_{ij} for $j=1, \cdots, n_i$ and $i=1, \cdots, k$ be independent, normally distributed random variables with $E(X_{ij})=\theta_i$ and $var(X_{ij})=\sigma_i^2$. We shall consider the following two problems. The one is to test the null hypothesis H_1; $\theta_1= \cdots =\theta_k$ against the ordered alternative K_1; $\theta_1\geqq \cdots \geqq\theta_k$ where at least one inequality is strict, under the constraint that $\sigma_1^2= \cdots =\sigma_k^2=\sigma^2$ and σ^2 is unknown. The other is to test the null hypothesis H_2; $\sigma_1^2= \cdots =\sigma_k^2$ against the ordered alternative K_2; $\sigma_1^2\geqq \cdots \geqq\sigma_k^2$ where at least one inequality is strict. If σ^2 is assumed to be known, the testing problem (H_1, K_1) is included in Shirahata [15].

The method to approach to the problems is as follows. Suppose X be a random variable with a density function $f(\boldsymbol{x}; \theta)$ where $\boldsymbol{x}\in R^p$ and $\boldsymbol{\theta}\in R^q$ for some p and q and consider the problem of testing H_0; $\boldsymbol{\theta}\in\theta_{H_0}$ against the alternative K_0; $\boldsymbol{\theta}\in\theta_{K_0}$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two elements of the sample space of X and define

$$\boldsymbol{x}\geqq_*\boldsymbol{y}$$

if

$$f(\boldsymbol{x}; \boldsymbol{\theta})/f(\boldsymbol{x}; \boldsymbol{\xi})\geqq f(\boldsymbol{y}; \boldsymbol{\theta})/f(\boldsymbol{y}; \boldsymbol{\xi}) \qquad {}^\forall\boldsymbol{\theta}\in\theta_{K_0}, {}^\forall\boldsymbol{\xi}\in\theta_{H_0}.$$

The relation $\geqq_*$ induces a partial ordering on the sample space and a test statistic must be a generalization of $\geqq_*$. Let $A(\boldsymbol{x})=\{\boldsymbol{y}; \boldsymbol{y}\geqq_*\boldsymbol{x}\}$ and $B(\boldsymbol{x})=\{\boldsymbol{y}; \boldsymbol{x}\geqq_*\boldsymbol{y}\}$. We

AMS 1970 Subject classifications. Primary 62F05.

Key words and phrases. Normal populations, ordered alternative of means, ordered alternative of variances, partial ordering on sample space, test of homogeneity.

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assume that the relation $\geqq_*$ is not trivial, i. e. $A(\boldsymbol{x})\neq\phi$ or $B(\boldsymbol{x})\neq\phi$ for some $\boldsymbol{x}$. The test statistics we shall consider are

$$t_1=P\{A(\boldsymbol{X})\}/P\{A(\boldsymbol{X})\cup B(\boldsymbol{X})\},$$

$$t_2=P\{A(\boldsymbol{X})\},$$

$$t_3=P\{A(\boldsymbol{X})\}-P\{B(\boldsymbol{X})\}$$

and

$$t_4=-P\{B(\boldsymbol{X})\}$$

where the probability P is calculated under some adequate member of θ_{H_0}. In many problems, it will be clear how to choose the element. Making use of each t_i as a test statistic, the null hypothesis H_0 is rejected if t_i is too small.

In Section 2 the problem (H_1, K_1) is considered. This problem was studied by Bartholomew [2, 3, 4, 5] first and some generalizations were made by Chacko [6], Kudô [10], Nüesch [12], Shorack [17] and Perlman [13] and some nonparametric considerations were given by Jonckheere [10], Puri [14] and Tryon and Hettmansperger [18]. The book of Barlow et. al. gives a good summary. However, the parametric works of them are all based on the likelihood ratio test. Shirahata [15] studied bivariate one-sided problem which includes the problem (H_1, K_1) for $k=3$ and known variance σ^2 from a new stand of view. It was shown that t_1 is better than the Bartholomew's $\bar{x}_3^2$ test by computer simulations. Hence it is expected that each t_i is better than the likelihood ratio test for the unknown but common variance case. The statistic t_1 is also used to analyze a counted data in Shirahata [16]. A comparative study was given by Hirotsu [8].

In Section 3 the problem (H_2, K_2) is considered. This problem may occur when we want to compare the accuracies of e. g. several instruments to measure. For the test of homogeneity of variances there are works of Fujino [7] and Vincent [19].

The explicit forms of $\geqq_*$ are given for the both problems. However, the distributions of t_i's are very complicated and to derive them are impossible analytically up to now even in the known variance case. The distributions vary according as $\boldsymbol{n}'=(n_1, \cdots, n_k)$ varies and hence to determine cutoff points by simulations are also impractical. Therefore, in Section 4 we shall consider some artificial data, estimate their significances by simulations and compare the significances to the significances given by their competitors. It will be found that t_i's are good tests, especially t_3 and t_4 are very good. Hence we recommend to use t_3 or t_4 for the real data in both (H_1, K_1) and (H_2, K_2) by estimating the significance of the data empirically and then comparing it with the significance level.

2. The problem of testing H_1 against K_1

Put $\boldsymbol{\delta}'=(\delta_1, \cdots, \delta_{k-1})$ where $\delta_i=\theta_i-\theta_{i+1}$. Then H_1 and K_1 can be rewritten as H_1 ; $\boldsymbol{\delta}=\boldsymbol{0}$ and K_1 ; $\boldsymbol{\delta}\in D$ where $D=\{\boldsymbol{\delta}\,;\,\delta_i\geqq 0$ for $i=1, \cdots, k-1$ and $\max_i \delta_i>0\}$. Thus, it is natural to consider tests based on

$$\boldsymbol{T}'=(T_1, \cdots, T_{k-1})$$

where

$$T_i = a_i(\bar{X}_i - \bar{X}_{i+1})/S\,,$$

$$\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i\,,$$

$$a_i = \{m/(n_i^{-1} + n_{i+1}^{-1})\}^{1/2}\,, \qquad m = \sum_{i=1}^{k} n_i - k$$

and

$$S^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2\,.$$

The statistic $\boldsymbol{T}$ is a multivariate central t-distribution under H_1 and is a noncentral t-distribution with nonnegative noncentrality parameters under K_1. From Johnson and Kotz [9] the density function of $\boldsymbol{T}$ is

$$f(\boldsymbol{t}\,;\,\boldsymbol{\delta}) = \exp(-\boldsymbol{\delta}' R^{-1}\boldsymbol{\delta}/2)(m\pi)^{-(k-1)/2}|R|^{-1/2}\Gamma^{-1}\Big(\frac{m}{2}\Big)$$
$$\times\Big(1 + \frac{\boldsymbol{t}'R^{-1}\boldsymbol{t}}{m}\Big)^{-(m+k-1)/2} h(\boldsymbol{t}\,;\,\boldsymbol{\delta})$$

where R is a $(k-1)\times(k-1)$ matrix (r_{ij}) with $r_{ii}=1$, $r_{i,i+1}=r_{i+1,i}=\rho_i$ for $\rho_i = -a_i a_{i+1}/n_{i+1}m$ and $r_{ij}=0$ otherwise and where

$$h(\boldsymbol{t}\,;\,\boldsymbol{\delta}) = \sum_{j=0}^{\infty} \frac{1}{j!}\Gamma\Big(\frac{m+k+j}{2}\Big)\Big\{\Big(\frac{2}{m+\boldsymbol{t}'R^{-1}\boldsymbol{t}}\Big)^{1/2}\boldsymbol{\delta}'R^{-1}\boldsymbol{t}\Big\}^j\,.$$

Hence $f(\boldsymbol{t}\,;\,\boldsymbol{\delta})/f(\boldsymbol{t}\,;\,\boldsymbol{0})$ is an increasing function of $h(\boldsymbol{t}\,;\,\boldsymbol{\delta})$. Now the function

$$\sum_{j=0}^{\infty} \frac{1}{j!}\Gamma\Big(i + \frac{j}{2}\Big)x^j$$

is always positive for each $i>0$, $x \geqq 0$ and it is also found that it is an increasing function of x. Hence, considering the transformation

$$\bar{\boldsymbol{T}} = R^{-1}\boldsymbol{T}/(m + \boldsymbol{T}'R^{-1}\boldsymbol{T})^{1/2} \tag{2.1}$$

it follows that $\boldsymbol{t} \geqq_* \boldsymbol{s}$ if and only if $\boldsymbol{\delta}'\bar{\boldsymbol{t}} \geqq \boldsymbol{\delta}'\bar{\boldsymbol{s}}$ for each $\boldsymbol{\delta} \in D$. Therefore $\boldsymbol{t} \geqq_* \boldsymbol{s}$ if and only if

$$\bar{t}_i \geqq \bar{s}_i \qquad \text{for} \quad i=1, \cdots, k-1\,. \tag{2.2}$$

Now let us derive the density function of $\bar{\boldsymbol{T}}$ under H_1. It is easily shown that $0 \leqq \bar{\boldsymbol{T}}'R\bar{\boldsymbol{T}} \leqq 1$, $\boldsymbol{T}'R^{-1}\boldsymbol{T} = m\bar{\boldsymbol{T}}'R\bar{\boldsymbol{T}}/(1-\bar{\boldsymbol{T}}'R\bar{\boldsymbol{T}})$. It is also shown that

$$|\partial\boldsymbol{t}/\partial\bar{\boldsymbol{t}}| = \Big\{\frac{m}{(1-\bar{\boldsymbol{t}}'R\bar{\boldsymbol{t}})^3}\Big\}^{(k-1)/2}|R|\times|(1-\bar{\boldsymbol{t}}'R\bar{\boldsymbol{t}})I_{k-1} + R\bar{\boldsymbol{t}}\bar{\boldsymbol{t}}'| \tag{2.3}$$

where I_{k-1} is the $k-1$ dimensional unit matrix. Using a relation $|M+\boldsymbol{x}\boldsymbol{y}'| = |M|(1+\boldsymbol{y}'M^{-1}\boldsymbol{x})$, (2.3) is

$$|\partial\boldsymbol{t}/\partial\bar{\boldsymbol{t}}| = m^{(k-1)/2}|R|(1-\bar{\boldsymbol{t}}'R\bar{\boldsymbol{t}})^{-(k+1)/2}\,.$$

Therefore the density function $f(\bar{\boldsymbol{t}})$ of $\bar{\boldsymbol{T}}$ under H_1 is given by

$$(2.4)\qquad f(\bar{t})=\begin{cases}\dfrac{\Gamma\left(\dfrac{m+k-1}{2}\right)}{\pi^{(k-1)/2}\Gamma\left(\dfrac{m}{2}\right)}|R|^{1/2}(1-\bar{t}'R\bar{t})^{(m/2)-1} & \text{for } \bar{t}'R\bar{t}\leqq 1\\ 0 & \text{otherwise.}\end{cases}$$

From (2.1), (2.2) and (2.4), $P\{A(\boldsymbol{T})\}$ and $P\{B(\boldsymbol{T})\}$ are calculated and hence $\boldsymbol{t}_i$'s can be obtained. Note that if we consider further transformation $\bar{\bar{\boldsymbol{T}}}=R^{1/2}\bar{\boldsymbol{T}}$, then the density function $\bar{f}(\bar{\bar{t}})$ of $\bar{\bar{\boldsymbol{T}}}$ is

$$(2.5)\qquad \bar{f}(\bar{\bar{t}})=\begin{cases}\dfrac{\Gamma\left(\dfrac{m+k-1}{2}\right)}{\pi^{(k-1)/2}\Gamma\left(\dfrac{m}{2}\right)}(1-\bar{\bar{t}}'\bar{\bar{t}})^{(m/2)-1} & \text{for } \bar{\bar{t}}'\bar{\bar{t}}\leqq 1\\ 0 & \text{otherwise.}\end{cases}$$

The formula (2.5) is very simple. It may be convenient to calculate t_i's from (2.5).

3. The problem of testing H_2 against K_2

Without loss of generality it can be assumed that each θ_i is unknown. Then it is natural to consider tests based on

$$\boldsymbol{F}'=(F_1, \cdots, F_{k-1})$$

where

$$F_i=(n_{i+1}-1)S_i^2/(n_i-1)S_{i+1}^2$$

and

$$S_i^2=\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2 .$$

The statistic F_i is a F-distribution with degrees of freedom $(n_i-1,\ n_{i+1}-1)$ under H_2. Under K_2 it is distributed as $\sigma_i^2/\sigma_{i+1}^2$ multiplied by the above F-distribution and hence it is stochastically larger than the distribution under H_2.

After a simple calculation the natural log of the density function of $\boldsymbol{F}$ is given by

$$\log h(\boldsymbol{f}\,;\ \sigma_1, \cdots, \sigma_k)=c-\sum_{i=1}^{k}(n_i-1)\log\sigma_i+\sum_{i=1}^{k-1}\left(-1+\sum_{j=1}^{i}\frac{n_j-1}{2}\log f_i\right)$$
$$-\frac{m}{2}\log\left(\frac{n_k-1}{\sigma_k^2}+\sum_{i=1}^{k-1}\frac{n_i-1}{\sigma_i^2}\prod_{j=i}^{k-1}f_j\right)$$

where c is a constant depending on $\boldsymbol{n}'$ and k. Thus,

$$\begin{aligned}&\log L(\boldsymbol{f}\,;\ \sigma, \sigma_1, \cdots, \sigma_k)\\ &=\log h(\boldsymbol{f}\,;\ \sigma_1, \cdots, \sigma_k)-\log h(\boldsymbol{f}\,;\ \sigma, \cdots, \sigma)\\ &=-\sum_{i=1}^{k}(n_i-1)\log\sigma_i+\frac{m}{2}\log\left\{n_k-1+\sum_{i=1}^{k-1}(n_i-1)\prod_{j=i}^{k-1}f_j\right\}\\ &\quad-\frac{m}{2}\log\left(\frac{n_k-1}{\sigma_k^2}+\sum_{i=1}^{k-1}\frac{n_i-1}{\sigma_i^2}\prod_{j=i}^{k-1}f_j\right).\end{aligned}$$

Hence, considering $\bar{F}'=(\bar{F}_1, \cdots, \bar{F}_{k-1})$ where

$$\bar{F}_i=\frac{n_i-1}{n_k-1}\prod_{j=i}^{k-1}F_j, \tag{3.1}$$

the necessary and sufficient condition of

$$L(f\,;\ \sigma, \sigma_1, \cdots, \sigma_k)\geqq L(g\,;\ \sigma, \sigma_1, \cdots, \sigma_k)$$

for each $\sigma_1\geqq\cdots\geqq\sigma_k>0$ is

$$\sum_{i=1}^{k-1}(\bar{f}_i-\bar{g}_i)\geqq\sum_{i=1}^{k-1}\frac{\sigma_k^2}{\sigma_i^2}\{\bar{f}_i-\bar{g}_i+\sum_{j=1}^{k-1}(\bar{f}_i\bar{g}_j-\bar{f}_j\bar{g}_i)\} \tag{3.2}$$

$$=\sum_{i=1}^{k-1}\frac{\sigma_k^2}{\sigma_i^2}a_i \text{ (say)}.$$

Since the inequality (3.2) is

$$\sum_{i=1}^{k-1}(\lambda_i-\lambda_{i+1})\sum_{j=1}^{i}a_i\leqq 0$$

where $\lambda_i=\sigma_k^2/\sigma_i^2$, (3.2) holds if and only if $\sum_{j=1}^{i}a_j\geqq 0$ for $i=1, \cdots, k-1$ and hence $f\geqq_* g$ if and only if

$$\sum_{j=1}^{i}\bar{f}_j/(1+\sum_{j=1}^{k-1}\bar{f}_j)\geqq\sum_{j=1}^{i}\bar{g}_j/(1+\sum_{j=1}^{k-1}\bar{g}_j)\qquad\text{for}\quad i=1, \cdots, k-1. \tag{3.3}$$

Put $\bar{\bar{F}}'=(\bar{\bar{F}}_1, \cdots, \bar{\bar{F}}_{k-1})$ where

$$\bar{\bar{F}}_i=\sum_{j=1}^{i}\bar{F}_j/(1+\sum_{j=1}^{k-1}\bar{F}_j) \tag{3.4}$$

$$=\sum_{j=1}^{i}S_j^2/\sum_{j=1}^{k}S_j^2.$$

Then the density function of $\bar{\bar{F}}$ is

$$g(w_1, \cdots, w_{k-1})=\frac{\Gamma\left(\frac{m}{2}\right)}{\prod_{i=1}^{k}\Gamma\left(\frac{n_i-1}{2}\right)}\prod_{i=1}^{k}(w_i-w_{i-1})^{(n_i-3)/2} \tag{3.5}$$

for $w_0=0<w_1<\cdots w_{k-1}<1=w_k$ and $g(w_1, \cdots, w_{k-1})=0$ otherwise.

From (3.1), (3.3), (3.4) and (3.5), $P\{A(F)\}$ and $P\{B(F)\}$ can be calculated.

4. Some numerical results

From the results in Sections 2 and 3, we can calculate four statistics t_1, t_2, t_3 and t_4. But, the distributions of them depend on $n'=(n_1, \cdots, n_k)$ and further it is impossible to obtain the distributions even for a fixed n. However, to perform a significance test, it is enough to know the significance of the given data. The power of the t_i test can be guessed by estimating the significances of several artificial data.

Hence we estimate the significances of some artificial data by simulations and then compare with the significances given by the competitors. It will be found that our statistics are powerful.

We consider only the case $k=3$ and $\boldsymbol{n}'=(7, 5, 3)$, $(5, 5, 5)$ and $(3, 5, 7)$. The artificial data to be investigated are $(\bar{X}_1-\bar{X}_2, \bar{X}_2-\bar{X}_3, S^2)=(0.8, 0, 12)$, $(0.4, 0.4, 12)$ and $(0, 0.8, 12)$ for the problem (H_1, K_1). They correspond to the cases $\theta_1>\theta_2=\theta_3$, $\theta_1-\theta_2=\theta_2-\theta_3>0$ and $\theta_1=\theta_2>\theta_3$, respectively. For the problem (H_2, K_2) we consider

$$\left(\frac{S_1^2}{n_1-1}, \frac{S_2^2}{n_2-1}, \frac{S_3^2}{n_3-1}\right)=(3, 3, 1), (3, \sqrt{3}, 1)$$

and $(3, 1, 1)$. They correspond to $\sigma_1=\sigma_2>\sigma_3$, $\sigma_1/\sigma_2=\sigma_2/\sigma_3>1$ and $\sigma_1>\sigma_2=\sigma_3$ respectively. These data are analyzed except the cases $\boldsymbol{n}'=(3, 5, 7)$ and $\boldsymbol{n}'=(5, 5, 5)$ with $(\bar{X}_1-\bar{X}_2, \bar{X}_2-\bar{X}_3, S^2)=(0, 0.8, 12)$ for the problem (H_1, K_1). There exceptions are due to the fact that for the problem (H_1, K_1) we can show as in Shirahata [15] that the power for the alternative (δ_1, δ_2) with (n_1, n_2, n_3) is equal to the power for the alternative (δ_2, δ_1) with (n_3, n_2, n_1).

For five cases in the problem (H_1, K_1) we simulated 100 times in $\{A(\boldsymbol{X})\cup B(\boldsymbol{X})\}^c$

Table 1. Empirical significances of artificial data in the problem (H_1, K_1).

$\boldsymbol{n}'$	data $(\bar{X}_1-\bar{X}_2, \bar{X}_2-\bar{X}_3, S^2)$	t_1	t_2	t_3	t_4	$\bar{F}_3$	$\bar{E}^2$	t
(7, 5, 3)	(0.8, 0, 12)	0.095	0.105	0.091	0.091	0.205	0.124	0.097
	(0.4, 0.4, 12)	0.068	0.093	0.052	0.050	0.294	0.205	0.127
	(0, 0.8, 12)	0.151	0.164	0.135	0.140	0.282	0.194	0.165
(5, 5, 5)	(0.8, 0, 12)	0,116	0.130	0.112	0.116	0.222	0.139	0.115
	(0.4, 0.4, 12)	0.136	0.150	0.098	0.092	0.272	0.184	0.115

Table 2. Empirical significances of artificial data in the problem (H_2, K_2).

$\boldsymbol{n}'$	data $\left(\frac{S_1^2}{n_1-1}, \frac{S_2^2}{n_2-1}, \frac{S_3^2}{n_3-1}\right)$	t_1	t_2	t_3	t_4	B_a	LR
(7, 5, 3)	(3, 3, 1)	0.297	0.339	0.274	0.253	0.716	0.380
	(3, $\sqrt{3}$, 1)	0.258	0.280	0.198	0.179	0.664	0.360
	(3, 1, 1)	0.185	0.210	0.161	0.163	0.437	0.199
(5, 5, 5)	(3, 3, 1)	0.186	0.208	0.173	0.163	0.542	0·192
	(3, $\sqrt{3}$, 1)	0.160	0.191	0.122	0.112	0.573	0.249
	(3, 1, 1)	0.138	0.154	0.108	0.107	0.454	0.174
(3, 5, 7)	(3, 3, 1)	0.135	0.140	0.116	0.117	0.479	0.134
	(3, $\sqrt{3}$, 1)	0.141	0.173	0.104	0.095	0.636	0.213
	(3, 1, 1)	0.172	0.188	0.129	0.118	0.588	0.169

and for nine cases in the problem (H_2, K_2), we simulated 150 times in $\{A(\boldsymbol{X})\cup B(\boldsymbol{X})\}^c$. Table 1 and 2 gives

$$P(A)+\{1-P(A)-P(B)\}r_2/r_1 \tag{4.1}$$

where we denote by $P(A)$ and $P(B)$ the probabilities of the sets $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ of our artificial data. Here $r_1=100$ in the problem (H_1, K_1), $r_1=150$ in the problem (H_2, K_2) and r_2 is the number of times of random numbers more significant than the artificial data. The formula (4.1) gives an estimate of the significance of the data. Table 1 also includes theoretical significances of Bartholomew's $\bar{F}_3$, $\bar{E}^2$ and usual t-statistic

$$t=-m^{1/2}\sum_{i=1}^{k} in_i(\bar{X}_i-\bar{X})/S\left\{\sum_{i=1}^{k} n_i\left(i-\sum_{j=1}^{k} j\frac{n_j}{n}\right)^2\right\}^{1/2}$$

where $\bar{X}=n^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}$ and $n=\sum_{i=1}^{k} n_i$. The statistic t has t-distribution with degree of freedom m. Table 2 includes the empirical significances given by Bartlett's test

$$B_a=m\left\{\log(m^{-1}\sum_{i=1}^{k} S_i^2)-m^{-1}\sum_{i=1}^{k}(n_i-1)\log\frac{S_i^2}{n_i-1}\right\}$$

and the likelihood ratio test LR based on (S_1^2, S_2^2, S_3^2) where

$$LR=\begin{cases} m\log\dfrac{S_1^2+S_2^2+S_3^2}{m}-\sum_{i=1}^{3}(n_i-1)\log\dfrac{S_i^2}{n_i-1}, & \\ \qquad\qquad \text{for } \dfrac{S_1^2}{n_1-1}\geqq\dfrac{S_2^2}{n_2-1}\geqq\dfrac{S_3^2}{n_3-1}, & \\ m\log\dfrac{S_1^2+S_2^2+S_3^2}{m}-(n_1-1)\log\dfrac{S_1^2}{n_1-1}-(n_2+n_3-2)\log\dfrac{S_2^2+S_3^2}{n_2+n_3-2}, & \\ \qquad\qquad \text{for } \dfrac{S_1^2}{n_1-1}\geqq\dfrac{S_3^2}{n_3-1}\geqq\dfrac{S_2^2}{n_2-1} & \\ m\log\dfrac{S_1^2+S_2^2+S_3^2}{m}-(n_1+n_2-2)\log\dfrac{S_1^2+S_2^2}{n_1+n_2-2}-(n_3-1)\log\dfrac{S_3^2}{n_3-1}, & \\ \qquad\qquad \text{for } \dfrac{S_2^2}{n_2-1}\geqq\dfrac{S_1^2}{n_1-1}\geqq\dfrac{S_3^2}{n_3-1}, & \\ 0 \qquad\qquad \text{otherwise.} & \end{cases}$$

For B_a and LR, the calculations were done 1000 times. In general the smaller significance a test gives, the more powerful the test is. Therefore, from the Tables we can conclude that tests based on t_3 and t_4 are very powerful in the both problems.

REMARK 4.1. It will be worth mentioned that the significance given by t_i is larger than $P(A)$ and is smaller than $1-P(B)$. Therefore if $1-P(B)$ is smaller than the significance level or $P(A)$ is larger than it, we need not simulate. Any test which is an extension of $\geqq_*$ has this property. But, because Bartholomew's $\bar{F}_k$ and $\bar{E}^2$ are not based on $\boldsymbol{T}$ they do not necessarily have this property in our formulation.

REMARK 4.2. It is very difficult to calculate t_i's for large n_i's and k. Considering, however, their power properties, they are useful at least for $k=3$.

References

[1] BARLOW, R.E., BARTHOLOMEW, D.J., BREMNER, J.M. and BRUNK, H.D.: Statistical Inference under Order Restrictions. (1972). John Wiley and Sons, New York.

[2] BARTHOLOMEW, D.J.: *A test of homogeneity for ordered alternatives.* Biometrika, **46** (1959), 36-48.

[3] BARTHOLOMEW, D.J.: *A test of homogeneity for ordered alteranatives* II. Biometrika, **46** (1959), 328-335.

[4] BARTHOLOMEW, D.J.: *A test of homogeneity of means under restricted alternatives.* (with discussion). J.R. Statist. Soc. B, **23** (1961), 239-281.

[5] BARTHOLOMEW, D.J.: *Ordered tests in the analysis of variance.* Biometrika, **48** (1961), 325-332.

[6] CHACKO, V.J.: *Testing homogeneity against ordered alternatives.* Ann. Math. Statist., **34** (1963), 945-956.

[7] FUJINO, K.: *A likelihood ratio test for homogeneity of variances against ordered alternatives.* Jap. J. Appl. Statist., **6** (1977), 3-12. (in Japanese)

[8] HIROTSU, C.: The Analysis of Variance. (1976), Kyoiku Shuppan, Tokyo. (in Japanese)

[9] JOHNSON, N.L. and KOTZ, S.: Distributions in Statistics: Continuous Multivariate Distributions. (1972), John Wiley and Sons, New York.

[10] JONCKHEERE, A.R.: *A distribution-free k-sample test against ordered alternatives.* Biometrika **41** (1954), 133-145.

[11] KUDÔ, A.: *A multivariate analogue of the one-sided test.* Biometrika, **50** (1963), 403-418.

[12] NÜESCH, P.E.: *On the problem of testing locations in multivariate populations for restricted alternatives.* Ann. Math. Statist., **37** (1966), 113-119.

[13] PERLMAN, M.D.: *One-sided testing problems in multivariate analysis.* Ann. Math. Statist., **40** (1969), 549-567.

[14] PURI, M.L.: *Some distribution-free k-sample rank tests of homogeneity against ordered alternatives.* Comm. Pure Appl. Math., **18** (1965), 51-63.

[15] SHIRAHATA, S.: *An approach to the one-sided test in the bivariate normal population.* Biometrika, **65** (1978), 61-67.

[16] SHIRAHATA, S.: *Exact test on percentages in a two-way table.* Biometrics, to appear.

[17] SHORACK, G.R.: *Testing against ordered alternatives in model I analysis of variance; normal theory and nonparametric.* Ann. Math. Statist., **38** (1967), 1740-1753.

[18] TRYON, P.V. and HETTMANSPERGER, T.P.: *A class of non-parametric tests for homogeneity against ordered alternatives.* Ann. Statist., **1** (1973), 1061-1070.

[19] VINCENT, S.E.: *A test of homogeneity for ordered variances.* J.R. Statist. Soc. B, **23** (1961), 195-206.