INFORMATION THEORETICAL APPROACHES IN GAME THEORY

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INFORMATION THEORETICAL APPROACHES
IN GAME THEORY

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Abstract

We give an information theoretical definition of a two person game and show some sufficient conditions under which each player can obtain the total information on the other player’s parameter from the sequence of observations.

1. Introduction

Generally in the game theory the player’s purpose is to maximize his own reward. In the stochastic games a player also wants to maximize his reward and uses the past steps only to find the optimal strategy in the sense. But one of the most important problem to win a game is to estimate the other players’ strategies and in the usual games there are a lot of cases such that a player chooses a strategy with mature consideration for the other players’ ones. For example, in the game of chess or bridge etc., if a player can estimate the other players’ strategies from a sequence of observations on the other players’ actions in the past, the player can win the game easily. Then in this paper we study two person games from the point of view of the information theory.

Rényi [1], [2] and Korsh [3] show sufficient conditions to obtain the total information on a parameter \( \theta \) from the sequence of observations in several cases. The present paper at first extends the theory to the system with two parameter \( \theta_1 \) and \( \theta_2 \) which are defined as the strategies of two person game, and we show some sufficient conditions under which each player can obtain the total information on the other player’s parameter from a sequence of observations \( \{(X_t, Y_t)\}, t=1, 2, \ldots \), of a independent case in section 3 and Markovian case in section 4 with respective examples.

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2. Definitions and elementary theorem

The two person game is defined as follows; let \( \theta_1 \) and \( \theta_2 \) be parameters of player I and player II, respectively, their values be determined by some probability law, \( \{X_t\}, \{Y_t\}, t=1, 2, \ldots \), be sequences of observations of player I and player II, respectively, and the distribution of \( \xi_t \equiv ((X_t, Y_t), \ldots, (X_t, Y_t)) \) depend on \( \theta_1 \) and \( \theta_2 \). Then each player tries to estimate the other player's parameter from the sequence of observations \( \xi_t \).

Let \( S=(\Omega, A, P) \) be a probability space and, for each \( \omega \in \Omega, (\theta_1, \theta_2) \) be a random vector with following probability distributions.

\[
p_k \equiv P((\theta_1(\omega), \theta_2(\omega))=(u_k, v_1)) \quad \text{and} \quad \sum_{k=1}^{n} p_k = 1,
\]

and

\[
q_l \equiv P(\theta_1=\theta_l) = \sum_{k=1}^{m} p_{kl}.
\]

Let us \( \{(X_t, Y_t)\}, t=1, 2, \ldots \), be an infinite sequence of random vectors on \( S \) and suppose that the distribution of \( \xi_t \equiv ((X_t, Y_t), \ldots, (X_t, Y_t)) \) depends on a value of \( (\theta_1, \theta_2) \). We suppose further in the present and following section that all distributions of \( \{\xi_t\}, t=1, 2, \ldots \), are absolutely continuous and define as follows:

\[
\phi_{kl}(z^{(t)}) = P'(\xi_t \leq z^{(t)} | (\theta_1, \theta_2) = (u_k, v_1)),
\]

\[
\phi_{kl}(z^{(t)}) = P'(\xi_t \leq z^{(t)} | \theta_1 = u_k)
\]

\[
= \frac{1}{p_k} \frac{\sum_{k=1} p_{kl} \phi_{kl}(z^{(t)})}{p_{k}}, \quad \text{a.e.,}
\]

where \( z^{(t)} = ((x_t, y_t), \ldots, (x_t, y_t)) \in R^{2t} \) and \( R^{2t} \) is a 2t-dimensional vector space, and \( \phi_{kl}(z^{(t)}) \) is the density function of the conditional distribution \( P(\xi_t \leq z^{(t)} | (\theta_1, \theta_2) = (u_k, v_1)) \) and \( \phi_{kl}(z^{(t)}) \) is the density function of \( P(\xi_t \leq z^{(t)} | \theta_1 = u_k) \). In the present paper we give some conclusions only about the parameter \( \theta_1 \), because the results are similarly true for \( \theta_2 \).

Now we have the following:

Theorem 1. If, for each \( k, l \) and \( t \geq 1 \),

\[
\phi_{kl}(z^{(t)}) > 0 \quad \text{for all} \quad z^{(t)} \in R^{2t},
\]

then there exists a positive constant \( c \) such that

\[
E[H(\theta_1 | \xi_t)] \leq c \sum_{k=1}^{\infty} \sum_{l=1}^{n} \sum_{l=1}^{m} \int_{R^{2t}} \{p_{kl} \phi_{kl}(z^{(t)})^2 \}^{\frac{1}{k}} dz^{(t)},
\]

where \( E[H(\theta_1 | \xi_t)] \) is the expectation of \( H(\theta_1 | \xi_t) \).

Proof. As \( H(\theta_1 | \xi_t) \) is the conditional entropy of \( \theta_1 \) given \( \xi_t \), so
By the Bayes' theorem and the assumption, we have

\[
P(\theta_i = u_k | \xi_i) = \frac{P_k \phi_i^{\text{av}}(\xi_i)}{\sum_{i=1}^{m} P_i \phi_i^{\text{av}}(\xi_i)}
\]

then, from the lemma of Rényi [1], there exists a positive constant \(c\) such that

\[
H(\theta_i | \xi_i) \leq c \sum_{k=1}^{m} \sqrt{P(\theta_i = u_k | \xi_i)}
\]

for all \(h=1, \ldots, m\).

Thus

\[
E[H(\theta_i | \xi_i) | \theta_i = u_h] = \int_{R^{x(t)}} H(\theta_i | x(t)) \phi_i^{\text{av}}(z(t)) dz(t)
\]

\[
\leq c \sum_{k=1}^{m} \int_{R^{x(t)}} \left( \frac{P_k \phi_k^{\text{av}}(z(t))}{\sum_{i=1}^{m} P_i \phi_i^{\text{av}}(z(t))} \right)^{1/2} \phi_k^{\text{av}}(z(t)) dz(t)
\]

\[
= c \sum_{h=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{l'=1}^{m} \int_{R^{x(t)}} \left( P_{h1} \phi_{h1}^{\text{av}}(z(t)) p_{k1} \phi_{k1}^{\text{av}}(z(t)) \right)^{1/2} dz(t).
\]

Consequently

\[
E[H(\theta_i | \xi_i)] = \sum_{h=1}^{m} p_h E[H(\theta_i | \xi_i) | \theta_i = u_h]
\]

\[
= c \sum_{h=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{l'=1}^{m} \int_{R^{x(t)}} \left( P_{h1} \phi_{h1}^{\text{av}}(z(t)) p_{k1} \phi_{k1}^{\text{av}}(z(t)) \right)^{1/2} dz(t).
\]

3. Amount of information in independent case.

We define the amount of information contained in \(\xi_i\) concerning \(\theta_i\) as follows:

\[
I_i = H(\theta_i) - E[H(\theta_i | \xi_i)].
\]

In the case that the random vectors of observations are independent, under the assumption in section 2, we have the following result:

**Theorem 2.** If, under each condition that \((\theta_1, \theta_2) = (u_k, v_l)\), the random vectors \((X_1, Y_1), (X_2, Y_2), \ldots\) are independent and

\[
P'(\langle x_t, y_t \rangle) \leq f(x_t, y_t) |
\]

for all \((x_t, y_t) \in R^2, t=1, 2, \ldots\), and if for each \(h, k(\neq h), l\) and \(l'\),

\[ \prod_{t=1}^{\infty} \lambda_{kl}^{(t)} = 0, \]

then

\[ \lim_{t \to \infty} I_t = H(\theta_t), \]

where

\[ \lambda_{kl}^{(t)} = \iint (f_k^{(t)}(x, y)f_l^{(t)}(x, y))^{1/2} \, dx \, dy. \]

**Proof.** By the independency of \((X_t, Y_t)\)'s, for each \((k, l)\)

\[ \phi_{kl}^{(t)}(z^{(t)}) = \prod_{i=1}^{k} f_k^{(t)}(x_i, y_i), \]

Then by the theorem 1

\[ E[H(\theta_1 | \xi_1)] \]

\[ \leq c \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \prod_{t=1}^{n} f_k^{(t)}(x_t, y_t) \prod_{t=1}^{n} f_l^{(t)}(x_t, y_t) ^{1/2} d\xi(t) \]

\[ \leq c \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \iint \lambda_{kl}^{(t)}. \]

This proves the theorem.

In the above theorem, \( \lim_{t \to \infty} I_t = H(\theta_t) \) means that player II can obtain the total information on the parameter of player I.

**Example.** For each \((k, l)\), let \((X_t, Y_t), t=1, 2, \ldots, \) be a two-dimensional normal distribution such that

\[ f^{(t)}(x, y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left[ -\frac{1}{2(1-\rho^2)} \frac{(x-m_1(k, l))^2}{\sigma_1^2} \right. \]

\[ -2\rho \frac{(x-m_1(k, l))(y-m_2(k, l))}{\sigma_1 \sigma_2} \left. \frac{(y-m_2(k, l))^2}{\sigma_2^2} \right]. \]

Then

\[ \lambda_{kl}^{(t)} = \iint \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left[ -\frac{1}{2(1-\rho^2)} \frac{(x-\alpha_{hl})^2}{\sigma_1^2} \right. \]

\[ -2\rho \frac{(x-\alpha_{hl})(y-\beta_{hl})}{\sigma_1 \sigma_2} \left. \frac{(y-\beta_{hl})^2}{\sigma_2^2} \right] \, dx \, dy \]

\[ = \exp \left[ -\frac{1}{8(1-\rho^2)} \left( \frac{\alpha_{hl}}{\sigma_1} \right)^2 - \frac{2\rho}{\sigma_1 \sigma_2} a_{hl} b_{hl} \left( \frac{b_{hl}}{2\sigma_2} \right)^2 \right] \],

where

\[ \alpha_{hl} = \frac{m_1(h, l) + m_1(k, l')}{2}, \quad \beta_{hl} = \frac{m_2(h, l) + m_2(k, l')}{2}, \]
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1. When \( a_{hkl} \cdot b_{hkl} \geq 0 \), since \(|\rho_i| \leq 1\),

\[
\left( \frac{a_{hkl}}{\sigma_{st}} \right)^2 - \frac{2\rho_t}{\sigma_{st}} a_{hkl} b_{hkl} + \left( \frac{b_{hkl}}{\sigma_{st}} \right)^2 \geq \left( \frac{a_{hkl}}{\sigma_{st}} - \frac{b_{hkl}}{\sigma_{st}} \right)^2,
\]

and

\[
\lambda_{hkl}^{(i)} \leq \exp \left\{ \frac{1}{8(1-\rho_t)} \left( \frac{a_{hkl}}{\sigma_{st}} - \frac{b_{hkl}}{\sigma_{st}} \right)^2 \right\}.
\]

Thus, if \( \sum_{t=1}^{\infty} \frac{1}{\rho_t^2} \left( \frac{a_{hkl}}{\sigma_{st}} - \frac{b_{hkl}}{\sigma_{st}} \right)^2 = \infty \),

we have

\[
\prod_{t=1}^{\infty} \lambda_{hkl}^{(i)} = 0.
\]

2. When \( a_{hkl} \cdot b_{hkl} < 0 \), similarly in (1),

\[
\lambda_{hkl}^{(i)} \leq \exp \left\{ \frac{1}{8(1-\rho_t)} \left( \frac{a_{hkl}}{\sigma_{st}} + \frac{b_{hkl}}{\sigma_{st}} \right)^2 \right\}.
\]

Thus, if \( \sum_{t=1}^{\infty} \frac{1}{\rho_t^2} \left( \frac{a_{hkl}}{\sigma_{st}} + \frac{b_{hkl}}{\sigma_{st}} \right)^2 = \infty \),

we have

\[
\prod_{t=1}^{\infty} \lambda_{hkl}^{(i)} = 0.
\]

4. Amount of information in Markovian case.

In this section we assume that the \( \{(X_t, Y_t)\} \) process is a denumerable state Markov chain under each condition that \((\theta_1, \theta_2) = (u_k, v_l)\).

Then

\[
P(\xi_t | (\theta_1, \theta_2) = (u_k, v_l)) = P^{(u)}(X_t, Y_t | X_{t-1}, Y_{t-1}),
\]

where \( P^{(u)}(X_t, Y_t) \equiv P^{(\theta)}(X_t, Y_t | (\theta_1, \theta_2) = (u_k, v_l)) \), the initial distribution,

and

\[
P^{(u)}(X_{t+1}, Y_{t+1} | X_t, Y_t) = P^{(\theta)}(X_{t+1}, Y_{t+1} | (X_t, Y_t), (\theta_1, \theta_2) = (u_k, v_l)),
\]

the transition probability at stage \( t \geq 1 \).

Now we define the following value:

\[
\delta_{hkl}^{(i)} \equiv \sup_{(X_t, Y_t) \in \{(X_{t+1}, Y_{t+1})\}} \left\{ \sum_{t=1}^{\infty} \{P^{(u)}(X_{t+1}, Y_{t+1} | X_t, Y_t) \cdot P^{(u)}(X_{t+1}, Y_{t+1} | X_t, Y_t)\}^{1/2} \right\}.
\]

Then we have the following:

**Theorem 3.** If, for each \((u_k, v_l)\), \( P(\xi_t | (\theta_1, \theta_2) = (u_k, v_l)) > 0 \) for all \( \xi_t \) and if
\[ \prod_{t=1}^{\infty} \delta^{(t)}_{hkl} = 0 \quad \text{for all} \quad h, k(\neq h), l, l'. \]

then

\[ \lim_{t \to \infty} I_t = H(\theta_1). \]

**Proof.** By the assumption and Bayes' theorem,

\[ P(\theta_1 = u_k | \xi_t) = \frac{p_{l}P(\xi_t | \theta_1 = u_k)}{\sum_{i} p_{i}P(\xi_t | \theta_1 = u_i)} > 0. \]

Then from (2.2)

\[
E[H(\theta_1 | \xi_t) | \theta_1 = u_k] \\
= \sum_{\xi_t} H(\theta_1 | \xi_t) P(\xi_t | \theta_1 = u_k) \\
\leq c \sum_{\xi_t} \left\{ \sum_{k=1}^{N} \left( \frac{p_{k}P(\xi_t | \theta_1 = u_k)}{\sum_{i} p_{i}P(\xi_t | \theta_1 = u_i)} \right)^{1/2} \right\} P(\xi_t | \theta_1 = u_k) \\
\leq c \sum_{k=1}^{N} \sum_{\xi_t} \left( \frac{p_{k}}{p_{h}} P(\xi_t | \theta_1 = u_h) P(\xi_t | \theta_1 = u_k) \right)^{1/2}.
\]

Thus

\[ E[H(\theta_1 | \xi_t)] \\
= \sum_{h} p_{h} E[H(\theta_1 | \xi_t) | \theta_1 = u_h)] \\
\leq c \sum_{h} \sum_{k} \sum_{\xi_t} \left( p_{k}p_{h} P(\xi_t | \theta_1 = u_h) P(\xi_t | \theta_1 = u_k) \right)^{1/2} \\
\leq c \sum_{h} \sum_{k} \sum_{\xi_t} \sum_{i=1}^{l} \prod_{v=1}^{t-1} (P_{hi}^{-1}(X_{i}, Y_{i})P_{ki}^{-1}(X_{i}, Y_{i}|X_{i-1}, Y_{i-1}))^{1/2} \\
\leq c \sum_{h} \sum_{k} \sum_{\xi_t} \prod_{i=1}^{t-1} \delta^{(i)}_{hkl}. \]

This proves the theorem.

**Example.** Let, for each \((\theta_1, \theta_2) = (u_k, v), \{(X_t, Y_t)\}\) be a stationary Markov process with finite state \(\{a_1, \cdots, a_s\}\) and let \(Q^{kt}\) be a transition matrix such that

\[ Q^{kt} = (q_{ij}^{kt}), \]

where

\[ q_{ij}^{kt} = P^{kt}(X_{t+i}, Y_{t+i} = a_j | (X_t, Y_t) = a_i) \quad \text{for all} \quad t = 1, 2, \cdots, \]

then

\[ \delta_{hkl}^{(t)} = \sup_{i} \sum_{j} (q_{ij}^{kt} q_{ji}^{kt})^{1/2}. \]

Thus, if \(q_{ij}^{kt} > 0\) for all \(k, l, i, j\) and a row vector \(q^{kt} = (q_{1}^{kt}, \cdots, q_{s}^{kt})\) is not equal to \(q^{kt'}\) for all \(k, k \neq h, l, l'\), we have \(\delta_{hkl}^{(t)} < 1\) and by the theorem

\[ \lim_{t \to \infty} I_t = H(\theta_1). \]
5. Remarks.

The present paper gives the sufficient conditions for the players to obtain the total information on the parameters $\theta_1$ and $\theta_2$. But practically in the games, the player's main object is to obtain earlier the total information than the other player obtains or to obtain at each step more amount of information on the other player's parameter than the other player obtains. Furthermore, the system that the distribution of $(X_t, Y_t)$ depends only on the initial distribution of $(\theta_1, \theta_2)$ is not so flexible as a model of the game. Then we want to generalize it to the system in which the player can choose the distribution of $(X_t, Y_t)$ with taking account of the finite history of $X_1, Y_1, \cdots, X_{t-1}, Y_{t-1}$ and $(\theta_1, \theta_2)$ so that he can obtain more amount of information on the other player's parameter than the other player can obtain. These problems have to be studied future.

References

