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INFORMATION THEORETICAL APPROACHES IN GAME THEORY

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Abstract

We give an information theoretical definition of a two person game and show some sufficient conditions under which each player can obtain the total information on the other player's parameter from the sequence of observations.

1. Introduction

Generally in the game theory the player's purpose is to maximize his own reward. In the stochastic games a player also wants to maximize his reward and uses the past steps only to find the optimal strategy in the sense. But one of the most important problem to win a game is to estimate the other players' strategies and in the usual games there are a lot of cases such that a player chooses a strategy with mature consideration for the other players' ones. For example, in the game of chess or bridge etc., if a player can estimate the other players' strategies from a sequence of observations on the other players' actions in the past, the player can win the game easily. Then in this paper we study two person games from the point of view of the information theory.

Rényi [1], [2] and Korsh [3] show sufficient conditions to obtain the total information on a parameter θ from the sequence of observations in several cases. The present paper at first extends the theory to the system with two paramenter θ_1 and θ_2 which are defined as the strategies of two person game, and we show some sufficient conditions under which each player can obtain the total information on the other player's parameter from a sequence of obsevations $\{(X_t, Y_t)\}$, $t=1, 2, \cdots$, of a independent case in section 3 and Markovian case in section 4 with respective examples.

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2. Definitions and elementary theorem

The two person game is defined as follows; let θ_1 and θ_2 be parametes of player I and player II, respectively, their values be determined by some probability law, $\{X_t\}$, $\{Y_t\}$, $t=1, 2, \cdots$, be sequences of observations of player I and player II, respectively, and the distribution of $\xi_t\equiv((X_1, Y_1), \cdots, (X_t, Y_t))$ depend on θ_1 and θ_2. Then each player tries to estimate the other player's parameter from the sequence of observations ξ_t.

Let $S=(\Omega, \mathcal{A}, P)$ be a probability space and, for each $\omega\in\Omega$, (θ_1, θ_2) be a random vector with following probability distributions.

$$p_{kl}\equiv P((\theta_1(\omega), \theta_2(\omega))=(u_k, v_l)), \quad \sum_{k=1}^{m}\sum_{l=1}^{n} p_{kl}=1,$$

and

$$p_k\equiv P(\theta_1=u_k)=\sum_{l=1}^{n} p_{kl}$$

$$q_l\equiv P(\theta_2=v_l)=\sum_{k=1}^{m} p_{kl}$$

Let us $\{(X_t, Y_t)\}$, $t=1, 2, \cdots$, be an infinite sequence of random vectors on S and suppose that the distribution of $\xi_t\equiv((X_1, Y_1), \cdots, (X_t, Y_t))$ depends on a value of (θ_1, θ_2). We suppose further in the present and following section that all distributions of $\{\xi_t\}$, $t=1, 2, \cdots$, are absolutely continuous and define as follows:

$$\phi_{kl}^{(t)}(z^{(t)})=P'(\xi_t\leqq z^{(t)}\,|\,(\theta_1, \theta_2)=(u_k, v_l)),$$

$$\phi_k^{(t)}(z^{(t)})=P'(\xi_t\leqq z^{(t)}\,|\,\theta_1=u_k)$$

$$=\frac{\sum_{l=1}^{n} p_{kl}\phi_{kl}^{(t)}(z^{(t)})}{p_k} \qquad \text{a. e.,}$$

where $z^{(t)}=((x_1, y_1), \cdots, (x_t, y_t))\in R^{2t}$ and R^{2t} is a $2t$-dimensional vector space, and $\phi_{kl}^{(t)}(z^{(t)})$ is the density function of the conditional distribution $P(\xi_t\leqq z^{(t)}|(\theta_1, \theta_2)=(u_k, v_l))$ and $\phi_k^{(t)}(z^{(t)})$ is the density function of $P(\xi_t\leqq z^{(t)}\,|\,\theta_1=u_k)$. In the present paper we give some conclusions only about the parameter θ_1, because the results are similarly true for θ_2.

Now we have the following:

THEOREM 1. *If, for each k, l and $t\geqq 1$,*

$$\phi_{kl}^{(t)}(z^{(t)})>0 \qquad \textit{for oll} \quad z^{(t)}\in R^{2t},$$

then there exists a positive constant c such that

$$E[H(\theta_1|\xi_t)]$$

$$\leqq c\sum_{h=1}^{m}\sum_{\substack{k=1\\k\neq h}}^{m}\sum_{l,l'=1}^{n}\int_{R^{2t}}\{p_{hl}\phi_{hl}^{(t)}(z^{(t)})p_{kl'}\phi_{kl'}^{(t)}(z^{(t)})\}^{1/2}dz^{(t)},$$

where $E[H(\theta_1|\xi_t)]$ is the expectation of $H(\theta_1|\xi_t)$.

PROOF. As $H(\theta_1|\xi_t)$ is the conditional entropy of θ_1 given ξ_t, so

$$H(\theta_1|\xi_t)=-\sum_{k=1}^{m}P(\theta_1=u_k|\xi_t)\log P(\theta_1=u_k|\xi_t)\,.$$

By the Bayes' theorem and the assumption, we have

$$P(\theta_1=u_k|\xi_t)=\frac{p_k\phi_k^{(t)}(\xi_t)}{\sum_{i=1}^{m}p_i\phi_i^{(t)}(\xi_t)} \tag{2.1}$$
$$>0\,,$$

then, from the lemma of Rényi [1], there exists a positive constant c such that

$$H(\theta_1|\xi_t)\leqq c\sum_{\substack{k=1\\k\neq h}}^{m}\sqrt{P(\theta_1=u_k|\xi_t)}\qquad\text{for all}\quad h=1,\cdots,m\,. \tag{2.2}$$

Thus

$$\begin{aligned}
&E[H(\theta_1|\xi_t)|\theta_1=u_h]\\
&=\int_{R^{2t}}H(\theta_1|z^{(t)})\phi_h^{(t)}(z^{(t)})dz^{(t)}\\
&\leqq c\sum_{\substack{k=1\\k\neq h}}^{m}\int_{R^{2t}}\left(\frac{p_k\phi_k^{(t)}(z^{(t)})}{\sum_{i=1}^{m}p_i\phi_i^{(t)}(z^{(t)})}\right)^{1/2}\phi_h^{(t)}(z^{(t)})dz^{(t)}\\
&\leqq c\sum_{\substack{k=1\\k\neq h}}^{m}\sqrt{\frac{p_k}{p_h}}\int_{R^{2t}}(\phi_h^{(t)}(z^{(t)})\phi_k^{(t)}(z^{(t)}))^{1/2}dz^{(t)}\\
&\leqq c\sum_{\substack{k=1\\k\neq h}}^{m}\sum_{l,l'=1}^{n}\frac{1}{p_h}\int_{R^{2t}}(p_{hl}\phi_{hl}^{(t)}(z^{(t)})p_{kl'}\phi_{kl'}^{(t)}(z^{(t)}))^{1/2}dz^{(t)}\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
&E[H(\theta_1|\xi_t)]=\sum_{h=1}^{m}p_hE[H(\theta_1|\xi_t)|\theta_1=u_h]\\
&\leqq c\sum_{h=1}^{m}\sum_{\substack{k=1\\k\neq h}}^{m}\sum_{l,l'=1}^{n}\int_{R^{lt}}(p_{hl}\phi_{hl}^{(t)}(z^{(t)})p_{kl'}\phi_{kl'}^{(t)}(z^{(t)}))^{1/2}dz^{(t)}\,.
\end{aligned}$$

3. Amount of information in independent case.

We define the amount of information contained in ξ_t concerning θ_1 as follows:

$$I_t^1\equiv H(\theta_1)-E[H(\theta_1|\xi_t)]\,.$$

In the case that the random vectors of observations are independent, under the assumption in section 2, we have the following result:

THEOREM 2. *If, under each condition that* $(\theta_1,\theta_2)=(u_k,v_l)$, *the random vectors* $(X_1,Y_1),(X_2,Y_2),\cdots$ *are independent and*

$$P'((X_t,Y_t)\leqq(x_t,y_t)|(\theta_1,\theta_2)=(u_k,v_l))$$
$$=f_{kl}^{(t)}(x_t,y_t)>0$$

for all $(x_t,y_t)\in R^2$, $t=1,2,\cdots$, *and if for each* h, $k(\neq h)$, l *and* l',

$$\prod_{t=1}^{\infty} \lambda_{hkll'}^{(t)}=0\,,$$

then

$$\lim_{t\to\infty} I_t^1=H(\theta_1)\,,$$

where

$$\lambda_{hkll'}^{(t)}\equiv\iint(f_{hl}^{(t)}(x,\ y)f_{kl'}^{(t)}(x,\ y))^{1/2}dxdy\,.$$

PROOF. By the independency of $(X_t,\ Y_t)$'s, for each $(k,\ l)$

$$\phi_{kl}^{(t)}(z^{(t)})=\prod_{\nu=1}^{t} f_{kl}^{(\nu)}(x_\nu,\ y_\nu)\,,$$

Then by the theorem 1

$$\begin{aligned}
&E[H(\theta_1|\xi_t)]\\
&\leqq c\sum_{h=1}^{m}\sum_{\substack{k=1\\k\neq h}}^{m}\sum_{l,l'=1}^{n}\int_{R^{2t}}\{p_{hl}p_{kl'}\prod_{\nu=1}^{t} f_{hl}^{(\nu)}(x_\nu,\ y_\nu)\prod_{\nu=1}^{t} f_{kl'}^{(\nu)}(x_\nu,\ y_\nu)\}^{1/2}dz^{(t)}\\
&\leqq c\sum_{h=1}^{m}\sum_{\substack{k=1\\k\neq h}}^{m}\sum_{l,l'=1}^{n}\prod_{\nu=1}^{t}\lambda_{hkll'}^{(\nu)}\,.
\end{aligned}$$

This proves the theorem.

In the above theorem, $\lim I_t^1=H(\theta_1)$ means that player II can obtain the total information on the parameter of player I.

EXAMPLE. For each $(k,\ l)$, let $(X_t,\ Y_t)$, $t=1,\ 2,\ \cdots$, be a two-dimensional normal distribution such that

$$\begin{aligned}
f_{kl}^{(t)}(x,\ y)=&\frac{1}{2\pi\sigma_{1t}\sigma_{2t}\sqrt{1-\rho_t^2}}\exp\Big[-\frac{1}{2(1-\rho_t^2)}\Big\{\frac{(x-m_1(k,\ l))^2}{\sigma_{1t}{}^2}\\
&-2\rho_t\frac{(x-m_1(k,\ l))(y-m_2(k,\ l))}{\sigma_{1t}\sigma_{2t}}+\frac{(y-m_2(k,\ l))^2}{\sigma_{2t}{}^2}\Big\}\Big]\,.
\end{aligned}$$

Then

$$\begin{aligned}
\lambda_{hkll'}^{(t)}=&\iint\frac{1}{2\pi\sigma_{1t}\sigma_{2t}\sqrt{1-\rho_t^2}}\exp\Big[-\frac{1}{2(1-\rho_t^2)}\frac{(x-\alpha_{hkll'})^2}{\sigma_{1t}{}^2}\\
&-\frac{2\rho_t}{\sigma_{1t}\sigma_{2t}}(x-\alpha_{hkll'})(y-\beta_{hkll'})+\frac{(y-\beta_{hkll'})^2}{\sigma_{2t}{}^2}\\
&+\Big(\frac{a_{hkll'}}{2\sigma_{1t}}\Big)^2-\frac{\rho_t}{2\sigma_{1t}\sigma_{2t}}a_{hkll'}b_{hkll'}+\Big(\frac{b_{hkll'}}{2\sigma_{2t}}\Big)^2\Big\}\Big]dxdy\\
=&\exp\Big[-\frac{1}{8(1-\rho_t^2)}\Big\{\Big(\frac{a_{hkll'}}{\sigma_{1t}}\Big)^2-\frac{2\rho_t}{\sigma_{1t}\sigma_{2t}}a_{hkll'}b_{hkll'}\\
&+\Big(\frac{b_{hkll'}}{\sigma_{2t}}\Big)^2\Big\}\Big],
\end{aligned}$$

where $\alpha_{hkll'}\equiv\dfrac{m_1(h,\ l)+m_1(k,\ l')}{2}$, $\quad\beta_{hkll'}\equiv\dfrac{m_2(h,\ l)+m_2(k,\ l')}{2}$,

$$a_{hkll'} \equiv m_1(h, l) - m_1(k, l'), \qquad b_{hkll'} \equiv m_2(h, l) - m_2(k, l').$$

(1) When $a_{hkll'} \cdot b_{hkll'} \geqq 0$, since $|\rho_t| \leqq 1$,

$$\left(\frac{a_{hkll'}}{\sigma_{1t}}\right)^2 - \frac{2\rho_t}{\sigma_{1t}\sigma_{2t}} a_{hkll'} b_{hkll'} + \left(\frac{b_{hkll'}}{\sigma_{1t}}\right)^2$$
$$\geqq \left(\frac{a_{hkll'}}{\sigma_{1t}} - \frac{b_{hkll'}}{\sigma_{2t}}\right)^2,$$

and
$$\lambda_{hkll'}^{(t)} \leqq \exp\left\{-\frac{1}{8(1-\rho_t^2)}\left(\frac{a_{hkll'}}{\sigma_{1t}} - \frac{b_{hkll'}}{\sigma_{2t}}\right)^2\right\}.$$

Thus, if $\sum_{t=1}^{\infty} \frac{1}{1-\rho_t^2}\left(\frac{a_{hkll'}}{\sigma_{1t}} - \frac{b_{hkll'}}{\sigma_{2t}}\right)^2 = \infty$,
we have

$$\prod_{t=1}^{\infty} \lambda_{hkll'}^{(t)} = 0.$$

(2) When $a_{hkll'} \cdot b_{hkll'} < 0$, similarly in (1),

$$\lambda_{hkll'}^{(t)} \leqq \exp\left\{-\frac{1}{8(1-\rho_t^2)}\left(\frac{a_{hkll'}}{\sigma_{1t}} + \frac{b_{hkll'}}{\sigma_{2t}}\right)^2\right\}.$$

Thus, if $\sum_{t=1}^{\infty} \frac{1}{1-\rho_t^2}\left(\frac{a_{hkll'}}{\sigma_{1t}} + \frac{b_{hkll'}}{\sigma_{2t}}\right)^2 = \infty$,
we have

$$\prod_{t=1}^{\infty} \lambda_{hkll'}^{(t)} = 0.$$

4. Amount of information in Markovian case.

In this section we assume that the $\{(X_t, Y_t)\}$ process is a denumerable state Markov chain under each condition that $(\theta_1, \theta_2) = (u_k, v_l)$.

Then

$$P(\xi_t | (\theta_1, \theta_2) = (u_k, v_l))$$
$$= P_{kl}^{(0)}(X_1, Y_1) P_{kl}^{(1)}(X_2, Y_2 | X_1, Y_1) \cdots P_{kl}^{(t-1)}(X_t, Y_t | X_{t-1}, Y_{t-1}),$$

where $P_{kl}^{(0)}(X_1, Y_1) \equiv P^{(0)}(X_1, Y_1 | (\theta_1, \theta_2) = (u_k, v_l))$, the initial distribution,

and $\qquad P_{kl}^{(t)}(X_{t+1}, Y_{t+1} | X_t, Y_t) \equiv P^{(t)}(X_{t+1}, Y_{t+1} | (X_t, Y_t), (\theta_1, \theta_2) = (u_k, v_l))$,

the transition probability at stage $t \geqq 1$.

Now we define the following value:

$$\delta_{hkll'}^{(t)} \equiv \sup_{(X_t, Y_t)} \sum_{(X_{t+1}, Y_{t+1})} \{P_{hl}^{(t)}(X_{t+1}, Y_{t+1} | X_t, Y_t)$$
$$\cdot P_{kl'}^{(t)}(X_{t+1}, Y_{t+1} | X_t, Y_t)\}^{1/2}.$$

Then we have the following:

THEOREM 3. *If, for each* (u_k, v_l), $P(\xi_t | (\theta_1, \theta_2) = (u_k, v_l)) > 0$ *for all* ξ_t *and if*

$$\prod_{t=1}^{\infty} \delta_{hkll'}^{(t)}=0 \qquad \text{for all} \quad h,\ k(\neq h),\ l,\ l'.$$

then

$$\lim_{t\to\infty} I_t^1=H(\theta_1).$$

PROOF. By the assumption and Bayes' theorem,

$$P(\theta_1=u_k\,|\,\xi_t)=\frac{p_k P(\xi_t\,|\,\theta_1=u_k)}{\sum_i p_i P(\xi_t\,|\,\theta_1=u_i)}>0\,.$$

Then from (2.2)

$$\begin{aligned}
&E[H(\theta_1\,|\,\xi_t)\,|\,\theta_1=u_h]\\
&=\sum_{\xi_t} H(\theta_1\,|\,\xi_t)P(\xi_t\,|\,\theta_1=u_h)\\
&\leqq \sum_{\xi_t}\left\{c\sum_{k\neq h}\left\{\frac{p_k P(\xi_t\,|\,\theta_1=u_k)}{\sum_i p_i P(\xi_t\,|\,\theta_1=u_i)}\right\}^{1/2}\right\}P(\xi_t\,|\,\theta_1=u_h)\\
&\leqq c\sum_{k\neq h}\sum_{\xi_t}\left\{\frac{p_k}{p_h}P(\xi_t\,|\,\theta_1=u_h)\,P(\xi_t\,|\,\theta_1=u_k)\right\}^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&E[H(\theta_1\,|\,\xi_t)]\\
&=\sum_h p_h E[H(\theta_1\,|\,\xi_t)\,|\,\theta_1=u_h)]\\
&\leqq c\sum_h\sum_{k\neq h}\sum_{\xi_t}\{p_h p_k P(\xi_t\,|\,\theta_1=u_h)P(\xi_t\,|\,\theta_1=u_k)\}^{1/2}\\
&\leqq c\sum_h\sum_{k\neq h}\sum_{l,l'}\sum_{\xi_t}\prod_{\nu=2}^{t}\{P_{hl}^{(\nu-1)}(X_\nu,\ Y_\nu\,|\,X_{\nu-1},\ Y_{\nu-1})P_{kl'}^{(\nu-1)}(X_\nu,\ Y_\nu\,|\,X_{\nu-1},\ Y_{\nu-1})\}^{1/2}\\
&\leqq c\sum_h\sum_{k\neq h}\sum_{l,l'}\prod_{\nu=1}^{t-1}\delta_{hkll'}^{(\nu)}\,.
\end{aligned}$$

This proves the theorem.

EXAMPLE. Let, for each $(\theta_1,\ \theta_2)=(u_k,\ v_l)$, $\{(X_t,\ Y_t)\}$ be a stationary Markov process with finite state $\{a_1,\ \cdots,\ a_s\}$ and let Q^{kl} be a transition matrix such that

$$Q^{kl}=(q_{ij}^{kl})\,,$$

where $\qquad q_{ij}^{kl}\equiv P_{kl}^{(t)}((X_{t+1},\ Y_{t+1})=a_j\,|\,(X_t,\ Y_t)=a_i) \qquad$ for all $\quad t=1,\ 2,\ \cdots,$

then $$\delta_{hkll'}\equiv\delta_{hkll'}^{(t)}=\sup_i\sum_j\{q_{ij}^{hl}q_{ij}^{kl'}\}^{1/2}\,.$$

Thus, if $q_{ij}^{kl}>0$ for all $k,\ l,\ i,\ j$ and a row vector $q_i^{hl}=(q_{i1}^{hl},\ \cdots,\ q_{is}^{hl})$ is not equal to $q_i^{kl'}$ for all $h,\ k\neq h,\ l,\ l'$, we have $\delta_{hkll'}<1$ and by the theorem

$$\lim_{t\to\infty} I_t^1=H(\theta_1)\,.$$

5. Remarks.

The present paper gives the sufficient conditions for the players to obtain the total information on the parameters θ_1 and θ_2. But practically in the games, the player's main object is to obtain earlier the total information than the other player obtains or to obtain at each step more amount of information on the other player's parameter than the other player obtains. Furthermore, the system that the distribution of (X_t, Y_t) depends only on the initial distribution of (θ_1, θ_2) is not so flexible as a model of the game. Then we want to generalize it to the the system in which the player can choose the distribution of (X_t, Y_t) with taking account of the finite history of $X_1, Y_1, \cdots, X_{t-1}, Y_{t-1}$ and (θ_1, θ_2) so that he can obtain more amount of information on the other player's parameter than the other player can obtain. These problems have to be studied future.

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